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Almost periodic functions in finite-dimensional space with the spectrum in a cone

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We prove that an almost periodic function in finite-dimensional space extends to a holomorphic bounded function in a tube domain with a cone in the base if and only if the spectrum belongs to the conjugate cone. We also prove that an almost periodic function in finite-dimensional space has the bounded spectrum if and only if it extends to an entire function of exponential type.

A continuous function $f(z)$ on a strip

$$S_{a,b} = \{z = x + iy : x \in \mathbf{R}, \quad a \le y \le b\}, \ -\infty \le a \le b \le +\infty,$$

is called *almost periodic by Bohr* on this strip, if for any $\varepsilon > 0$ there exists $l = l(\varepsilon)$ such that every interval of the real axis of length l contains a number τ (ε-almost period for $f(z)$) with the property

$$\sup_{z \in S_{a,b}} |f(z+\tau) - f(z)| < \varepsilon. \tag{1}$$

In particular, when $a = b = 0$ we obtain the class of almost periodic functions on the real axis.

To each almost periodic function $f(z)$ assign the Fourier series

$$\sum_{n=0}^{\infty} a_n(y) e^{i\lambda_n x}, \quad \lambda_n \in \mathbf{R},$$

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where $a_n(y)$ are continuous functions of the variable $y \in [a, b]$.

In the case $a = b = 0$ all exponents λ_n are nonnegative if and only if the function $f(x)$ extends to the upper half-plane as a holomorphic bounded almost periodic function; the set of all exponents λ_n is bounded if and only if $f(x)$ extends to the plane $\mathbf{C}$ as an entire function of the exponential type $\sigma = \sup_n |\lambda_n|$, which is almost periodic in every horizontal strip of finite width (see [1, 4]).

A number of works connected with almost periodic functions of many variables on a *tube set* appeared recently (see [2, 6–9]). Recall that the set $T_K \subset \mathbf{C}^m$ is a *tube set* if

$$T_K = \{z = x + iy : x \in \mathbf{R}^m, y \in K\},$$

where $K \subset \mathbf{R}^m$ is *the base* of the tube set.

Definition. (See [6, 9]). *A continuous function $f(z)$, $z \in T_K$ is called almost periodic by Bohr on T_K, if for any $\varepsilon > 0$ there exists $l = l(\varepsilon)$, such that every m-dimensional cube on $\mathbf{R}^m$ with the side l contains at least one point τ ((T_K, ε)-almost period for $f(z)$) with the property*

$$\sup_{z \in T_K} |f(z + \tau) - f(z)| < \varepsilon. \tag{2}$$

Let f, g be locally integrable functions on every real plane

$$\{z = x + iy_0 : x \in \mathbf{R}^m\}, \quad y_0 \in K.$$

Definition. *Stepanoff distance of the order $p \geq 1$ between functions f and g is the value*

$$S_{p,T_K}(f, g) = \sup_{z \in T_K} \left(\int\limits_{[0,1]^m} |f(z + u) - g(z + u)|^p du \right)^{\frac{1}{p}}.$$

Using this definition, we can extend the concept of almost periodic functions by Stepanoff on a strip (see [4, p. 197]) to almost periodic functions on a tube set:

Definition. *A function $f(z)$, $z \in T_K$, is called almost periodic by Stepanoff on T_K, if for any $\varepsilon > 0$ there exists $l = l(\varepsilon)$ such that every m-dimensional cube with the side l contains at least one τ ((T_K, ε, p)-almost period by Stepanoff of the function $f(z)$) with the property*

$$S_{p,T_K}(f(z), f(z + \tau)) < \varepsilon. \tag{3}$$

The Fourier series for an almost periodic (by Bohr or by Stepanoff) function $f(z)$ on a set T_K is the series

$$\sum_{\lambda\in\mathbf{R}^m} a(\lambda,y)e^{i\langle x,\lambda\rangle}, \tag{4}$$

where $\langle x,\lambda\rangle$ is the scalar product on $\mathbf{R}^m$, and

$$a(\lambda,y)=\lim_{N\to\infty}\left(\frac{1}{2N}\right)^m \int\limits_{[-N,N]^m} f(x+x'+iy)e^{-i\langle x+x',\lambda\rangle}dx; \tag{5}$$

this limit exists uniformly in the parameter $x'\in\mathbf{R}^m$ and does not depend on this parameter (see [6, 8]).

A set of all vectors $\lambda\in\mathbf{R}^m$ such that $a(\lambda,y)\not\equiv 0$ is called the spectrum of $f(z)$ and is denoted by $sp\,f$; this set is at most countable, therefore the series (4) can be written in the form

$$\sum_{n=0}^{\infty} a_n(y)e^{i\langle x,\lambda_n\rangle}.$$

Note that partial sums of the series (4), generally speaking, do not converge to the function $f(z)$. However the Bochner–Feyer sums *

$$\sigma_q(z)=\sum_{n=0}^{q-1} k_n^q a_n(y)e^{i\langle x,\lambda_n\rangle},\ 0\le k_n^q<1,\ k_n^q\to 1 \ as\ q\to\infty$$

converge to the function $f(z)$ uniformly for almost periodic functions by Bohr and with respect to the metric S_{p,T_K} for almost periodic functions by Stepanoff; in particular, if two functions have the same Fourier series, then these functions coincide identically. For holomorphic almost periodic functions the series (4) can be written in the form

$$\sum_{n=0}^{\infty} a_n e^{-\langle y,\lambda_n\rangle}e^{i\langle x,\lambda_n\rangle}=\sum_{n=0}^{\infty} a_n e^{i\langle z,\lambda_n\rangle},\ a_n\in, \tag{6}$$

(see [8]). Any series of the form (6) is called *Dirichlet series.*

By Γ we always denote a convex closed cone in $\mathbf{R}^m$; by $\widehat{\Gamma}$ we denote *the conjugate cone to* Γ*:*

$$\widehat{\Gamma}=\{t\in\mathbf{R}^m:\langle t,y\rangle\ge 0\ \forall y\in\Gamma\},$$

note that $\widehat{\widehat{\Gamma}}=\Gamma$. Also, $\mathring{\Gamma}$ is the interior of a cone Γ.

* For $n=1$ see [4], for $n>1$ see [8].

Theorem 1. *Let $f(x)$ be an almost periodic function by Bohr on $\mathbf{R}^m$ with Fourier series*

$$\sum_{n=0}^{\infty} a_n e^{i\langle x,\lambda_n\rangle}, \tag{7}$$

where all the exponents λ_n belong to a cone $\Gamma \subset \mathbf{R}^m$. Then $f(x)$ continuously extends to the tube set $T_{\widehat{\Gamma}}$ as an almost periodic by Bohr function $F(z)$ with Fourier series (6). The function $F(z)$ is holomorphic on the interior $T_{\widehat{\Gamma}}$, and for any $\Gamma' \subset\subset \widehat{\Gamma}$ uniformly w.r.t. $z \in T_{\Gamma'}$

$$\lim_{\|y\|\to\infty} F(z) = a_0, \tag{8}$$

where a_0 is the Fourier coefficient corresponding to the exponent $\lambda_0 = 0$ (if $0 \notin$ sp f, put $a_0 = 0$.) If sp $f \subset \overset{\circ}{\Gamma}$, then (8) is true uniformly w.r.t. $z \in T_{\widehat{\Gamma}}$.

Here the inclusion $\Gamma' \subset\subset \widehat{\Gamma}$ means that the intersection of Γ' with the unit sphere is contained in the interior of the intersection of $\widehat{\Gamma}$ with this sphere.

To prove this theorem, we use the following lemmas.

Lemma 1. *Suppose that a plurisubharmonic function $\varphi(z)$ on $\mathbf{C}^m$ is bounded from above on a set T_K, where $K \subset \mathbf{R}^m$ is a convex set. Then the function*

$$\psi(y) = \sup_{x\in\mathbf{R}^m} \varphi(x+iy)$$

is convex on K.

P r o o f o f L e m m a 1. Fix $y_1,\ y_2 \in K$. The plurisubharmonic on $\mathbf{C}^m$ function

$$\varphi_1(z) = \varphi(z) - \frac{\psi(y_2)-\psi(y_1)}{\|y_2-y_1\|^2}\ \langle \mathrm{Im}\, z, y_2-y_1\rangle$$

is bounded from above on the set $T_{[y_1,y_2]}$. Therefore the subharmonic function $\varphi_2(w) = \varphi_1((y_2-y_1)w + iy_1)$ is bounded on the strip $\{w = u+iv : u \in \mathbf{R},\ 0 \le v \le 1\}$. Hence, the value of φ_1 at any point of this strip does not exceed

$$\max\{\sup_{u\in\mathbf{R}} \varphi_2(u), \sup_{u\in\mathbf{R}} \varphi_2(u+i)\} \le \max\{\sup_{x\in\mathbf{R}^m} \varphi_1(x+iy_1), \sup_{x\in\mathbf{R}^m} \varphi_1(x+iy_2)\}.$$

Therefore, for any $z = x+iy \in T_{[y_1,y_2]}$,

$$\varphi_1(z) \le$$

$$\max\{\psi(y_1) - \frac{\psi(y_2)-\psi(y_1)}{\|y_2-y_1\|^2}\langle y_1, y_2-y_1\rangle, \psi(y_2) - \frac{\psi(y_2)-\psi(y_1)}{\|y_2-y_1\|^2}\langle y_2, y_2-y_1\rangle\}$$

$$= \frac{\|y_2\|^2 - \langle y_1, y_2\rangle}{\|y_2 - y_1\|^2}\,\psi(y_1) + \frac{\|y_1\|^2 - \langle y_1, y_2\rangle}{\|y_2 - y_1\|^2}\,\psi(y_2).$$

Hence for any $y \in [y_1, y_2]$ we have

$$\begin{aligned}\psi(y) \quad \le \quad & \frac{\|y_2\|^2 - \langle y_1, y_2\rangle - \langle y, y_2 - y_1\rangle}{\|y_2 - y_1\|^2}\,\psi(y_1) \\ & + \frac{\|y_1\|^2 - \langle y_1, y_2\rangle + \langle y, y_2 - y_1\rangle}{\|y_2 - y_1\|^2}\,\psi(y_2).\end{aligned}$$

If $y = \lambda y_1 + (1-\lambda) y_2$, $\lambda \in (0,1)$, then we obtain the inequality

$$\psi(\lambda y_1 + (1-\lambda) y_2) \le \lambda\psi(y_1) + (1-\lambda)\psi(y_2).$$

Therefore, the function $\psi(y)$ is convex on K. ■

Lemma 2. *Let $\psi(y)$ be a convex bounded function on a cone Γ. Then $\psi(y) \le \psi(0)$ for all $y \in \Gamma$.*

P r o o f o f L e m m a 2. Since $\psi(y)$ is convex, we have

$$\psi(y) \le \left(1 - \frac{1}{t}\right)\psi(0) + \frac{1}{t}\psi(ty), \;\; t > 1.$$

Taking $t \to \infty$, we obtain $\psi(y) \le \psi(0)$. ■

P r o o f o f T h e o r e m 1. Let $\sigma_q(x)$, $q = 0, 1, 2, \ldots$ be the Bochner–Feyer sums for the series (7). Obviously, these functions are also defined for $z \in \mathbf{C}^m$. Assume that

$$\varphi_{q,l}(z) = \log(|\sigma_q(z) - \sigma_l(z)|).$$

For any fixed q and l, $q > l$, the function $\varphi_{q,l}(z)$ is plurisubharmonic on $\mathbf{C}^m$. Moreover, for $z \in T_{\widehat{\Gamma}}$ we have $\langle y, \lambda_n\rangle \ge 0$ and

$$|\sigma_q(z) - \sigma_l(z)| \le |\sigma_q(z)| + |\sigma_l(z)|$$

$$\sum_{n=0}^{q-1} |a_n| e^{-\langle y, \lambda_n\rangle} + \sum_{n=0}^{l-1} |a_n| e^{-\langle y, \lambda_n\rangle} \le 2\sum_{n=0}^{q-1} |a_n|.$$

Consider the function

$$\psi_{q,l}(y) = \sup_{x \in \mathbf{R}^m} \log(|\sigma_q(z) - \sigma_l(z)|).$$

Using lemma 1, we obtain that $\psi_{q,l}(y)$ is convex in $\widehat{\Gamma}$. Therefore, by lemma 2, we have

$$\sup_{z \in T_{\widehat{\Gamma}}} (|\sigma_q(z) - \sigma_l(z)|) \le \sup_{x \in \mathbf{R}^m} (|\sigma_q(x) - \sigma_l(x)|). \tag{9}$$

Further, the function $f(x)$ is almost periodic, therefore the Bochner–Feyer sums converge uniformly on $\mathbf{R}^m$, and for $q, l \geq N(\varepsilon)$

$$\sup_{x\in\mathbf{R}^m}(|\sigma_q(x)-\sigma_l(x)|)\leq\varepsilon.$$

Hence, $\sup\limits_{x\in\mathbf{R}^m}(|\sigma_q(z)-\sigma_l(z)|)\leq\varepsilon$ for all $z\in T_{\widehat{\Gamma}}$, $q, l\geq N(\varepsilon)$.

Thus the Bochner–Feyer sums uniformly converge on $T_{\widehat{\Gamma}}$, their limit is an almost periodic function by Bohr and holomorphic on the interior of $T_{\widehat{\Gamma}}$ with the Dirichlet series (6).

Further, passing to the limit in (9) as $q\to\infty$, we get

$$\sup_{z\in T_{\widehat{\Gamma}}}(|F(z)-\sigma_l(z)|)\leq\sup_{x\in\mathbf{R}^m}(|f(x)-\sigma_l(x)|).$$

Choose l such that the right hand side of this inequality is less than ε. We have for $\Gamma'\subset\subset\widehat{\Gamma}$

$$\sup_{z\in T_{\Gamma'}}|F(z)-a_0|\leq\sup_{z\in T_{\Gamma'}}|F(z)-\sigma_l(z)|+\sup_{z\in T_{\Gamma'}}|\sigma_l(z)-a_0|\leq\varepsilon+\sup_{z\in T_{\Gamma'}}|\sigma_l(z)-a_0|.$$

Note that for any fixed $\lambda_n\in\Gamma\setminus\{0\}$ the value $\langle y,\lambda_n\rangle$ tends to $+\infty$ as $\|y\|\to\infty$, $y\in\Gamma'$, therefore, we have

$$|\sigma_l(z)-a_0|=|\sum_{j=1}^{l-1}k_j^l a_j e^{i\langle x,\lambda_j\rangle}e^{-\langle y,\lambda_j\rangle}|\leq\sum_{j=1}^{l-1}|a_j|e^{-\langle y,\lambda_j\rangle}\to 0$$

as $\|y\|\to\infty$ on Γ'. Hence, uniformly w.r.t. $z\in T_{\Gamma'}$

$$\overline{\lim_{\|y\|\to\infty}}|F(z)-a_0|\leq\varepsilon. \tag{10}$$

This is true for arbitrary $\varepsilon>0$, then (8) follows.

If $sp\,f\subset\overset{\circ}{\Gamma}$, then for any $\lambda_n\in sp\,f$, $\langle y,\lambda_n\rangle\to+\infty$ as $\|y\|\to\infty$ uniformly w.r.t. $y\in\widehat{\Gamma}$, therefore (10) is true uniformly w.r.t. $z\in T_{\widehat{\Gamma}}$, and (8) is also true. The theorem has been proved. ■

Theorem 2. *Let $f(x)$ be an almost periodic function by Stepanoff on $\mathbf{R}^m$ with the Fourier series (7). Let all the exponents λ_n belong to a cone $\Gamma\subset\mathbf{R}^m$. Then there exists an almost periodic by Stepanoff function $F(z)$ in the tube set $T_{\widehat{\Gamma}}$ with the Fourier series (6) such that $F(x)=f(x)$. The function $F(z)$ is holomorphic almost periodic by Bohr on any domain $T_{\widehat{\Gamma}+b}$, $b\in\overset{\circ}{\widehat{\Gamma}}$. Besides, for any cone $\Gamma'\subset\subset\widehat{\Gamma}$ we have uniformly w.r.t. $z\in T_{\Gamma'}$*

$$\lim_{\|y\|\to\infty}F(z)=a_0, \tag{11}$$

where a_0 is the Fourier coefficient for the exponent $\lambda = 0$. If $sp\, f \subset \overset{\circ}{\Gamma}$, then (11) is true uniformly w.r.t. $z \in T_{\widehat{\Gamma}+b}$ for any $b \in \overset{\circ}{\widehat{\Gamma}}$.

P r o o f. To prove the first part of the theorem, we need to replace $\varphi_{q,l}(z)$ by

$$\widehat{\varphi}_{q,l}(z) = \log \left(\int\limits_{[0,1]^m} |\sigma_q(z+u) - \sigma_l(z+u)|^p du \right)^{\frac{1}{p}}.$$

Arguing as in the proof of theorem 1, we obtain that the Bochner–Feyer sums $\sigma_q(z)$ converge in the Stepanoff metric uniformly w.r.t. $z \in T_{\widehat{\Gamma}}$ to an almost periodic function by Stepanoff $F(z)$ with Fourier series (6).

Let $b \in \overset{\circ}{\widehat{\Gamma}}$. The module of the function $\sigma_q(z) - \sigma_l(z)$ is estimated from above by the mean value on the corresponding ball contained in $T_{\widehat{\Gamma}}$. Using the Hölder inequality, we have

$$\sup_{x\in\mathbf{R}^m} |\sigma_q(x+bi) - \sigma_l(x+bi)| \le C \sup_{z\in T_{\widehat{\Gamma}}} \left(\int\limits_{[0,1]^m} |\sigma_q(z+u) - \sigma_l(z+u)|^p du \right)^{\frac{1}{p}},$$

where the constant C depends only on b and $\widehat{\Gamma}$.

Applying Lemmas 1 and 2 to the functions

$$\tilde{\psi}_{q,l,b}(y) = \sup_{x\in\mathbf{R}^m} \log |\sigma_q(z+bi) - \sigma_l(z+bi)|,$$

we get that the Bochner–Fourier sums converge uniformly on $T_{\widehat{\Gamma}+b}$ to $F(z)$, thus $F(z)$ is holomorphic almost periodic by Bohr in $T_{\widehat{\Gamma}+b}$ for any $b \in \overset{\circ}{\widehat{\Gamma}}$.

Then the other statements of the theorem follow from Theorem 1. ∎

Now we prove the inverse statements to Theorems 1 and 2.

Theorem 3. *Suppose that an almost periodic by Bohr function $f(x)$ continuously extends to the interior of T_Γ as a holomorphic function $F(z)$. If $F(z)$ is bounded on any set $T_{\Gamma'}$, Γ' being a the cone in $\mathbf{R}^m$, $\Gamma' \subset\subset \Gamma$, then $F(z)$ is an almost periodic function by Bohr on T_Γ and the spectrum of $F(z)$ is contained in $\widehat{\Gamma}$.*

P r o o f. Take $\lambda \notin \widehat{\Gamma}$. Then there exists $y_0 \in \overset{\circ}{\Gamma}$ such that $\langle y_0, \lambda\rangle < 0$.

Choose a neighbourhood $U \subset \overset{\circ}{\Gamma}$ of y_0 such that $\langle y, \lambda\rangle \le \frac{1}{2}\langle y_0, \lambda\rangle$ for all $y \in U$. Let A be any nondegenerate operator in $\mathbf{R}^m$ such that A maps all the vectors $e_1 = (1, 0, \dots, 0), \dots, e_m = (0, 0, \dots, 1)$ into U.

The function $F(A\zeta)$ is holomorphic and bounded on the set

$$\{\zeta = \xi + i\eta \in \mathbf{C}^m : \xi \in \mathbf{R}^m,\ \eta^j > 0, j = 1, \dots, m\}$$

because $A\{\eta : \eta^j \ge 0\} \subset \Gamma$.

If for each coordinates $\zeta^1, \dots, \zeta^m$ we change the integration over the segments $-N \le \xi^j \le N$, $\eta^j = 0$ to the integration over the half-circles $\zeta^j = Ne^{i\theta^j}$, $0 \le \theta^j \le \pi$, $j = 1, \dots, m$ we obtain the equality

$$\left(\frac{1}{2N}\right)^m \int\limits_{[-N,N]^m} F(A\xi) e^{-i\langle A\xi, \lambda\rangle} d\xi$$

$$= \left(\frac{i}{2}\right)^m \int\limits_{[0,\pi]^m} F(ANe^{i\theta}) \prod_{j=1}^m e^{i\theta^j - iNe^{i\theta^j}\langle Ae_j, \lambda\rangle} d\theta, \tag{12}$$

where $\theta = (\theta^1, \dots, \theta^m)$, $e^{i\theta} = (e^{i\theta^1}, \dots, e^{i\theta^m})$.

Since $\langle Ae_j, \lambda\rangle < 0$ for $j = 1, \dots, m$, we see that the integrand in the right-hand side of (12) is uniformly bounded for all $N > 1$. By Lebesgue theorem (12) tends to zero as $N \to \infty$.

Thus

$$\lim_{N\to\infty} \frac{1}{(2N)^m} \int\limits_{A([-N,N]^m)} f(x) e^{-i\langle x, \lambda\rangle} dx = 0. \tag{13}$$

Cover the set $A([-N^2, N^2]^m)$ by cubes $L_j = x'_j + [-N, N]^m$ such that the interiors of these cubes are not intersected. We may assume that the number of the cubes intersecting the boundary of the set $A([-N^2, N^2]^m)$ is $O(N^{m-1})$ as $N \to \infty$. Taking into account boundedness of the function $f(x)e^{-i\langle x,\lambda\rangle}$ on $\mathbf{R}^m$ and equality (13), we have

$$\frac{1}{(2N)^{2m}} \int\limits_{\cup L_j} f(x) e^{-i\langle x, \lambda\rangle}\, dx$$

$$= \frac{1}{(2N)^{2m}} \left(\int\limits_{A([-N^2,N^2]^m)} f(x) e^{-i\langle x, \lambda\rangle}\, dx + O(N^{2m-1}) \right) = o(1) \tag{14}$$

Since (5) we see that uniformly w.r.t. $x' \in \mathbf{R}^m$ as $N \to \infty$

$$\frac{1}{(2N)^m} \int\limits_{x'+[-N,N]^m} f(x) e^{-i\langle x,\lambda\rangle} dx = a(\lambda, f) + o(1). \tag{15}$$

On the other hand, the number of the cubes L_j equals $O(N^m)$ as $N \to \infty$, then the equality $a(\lambda, f) = 0$ follows from (14) and (15). This yields the inclusion $sp\, f \subset \widehat{\Gamma}$. Using theorem 1 we complete the proof of our theorem. ■

Theorem 4. *If $F(z)$ is bounded on each set $T_{\Gamma'}$, $\Gamma' \subset\subset \Gamma$, and the nontangential limit value of $F(z)$ as $y \to 0$ is an almost periodic function by Stepanoff on $\mathbf{R}^m$, then $F(z)$ extends to T_Γ as an almost periodic function by Stepanoff and the spectrum of $F(z)$ is contained in $\widehat{\Gamma}$.*

P r o o f. The proof of this theorem is the same as of theorem 3, but we have to use theorem 2 instead of theorem 1. ■

To formulate further results we need the concept of P-indicator. (See, for example, [5, p. 275].)

Definition. *P-indicator of an entire function $F(z)$ on $\mathbf{C}^m$ is the function*

$$h_F(y) = \sup_{x \in \mathbf{R}^m} \overline{\lim_{r\to\infty}} \frac{1}{r} \log |F(x + iry)|.$$

Theorem 5. *(For $m = 1$ see [3, 4].) Let $f(x)$, $x \in \mathbf{R}^m$ be an almost periodic function by Stepanoff with the Fourier series (7), and let $\|\lambda_n\| \le C < \infty$ for all n. Then $f(x)$ extends to $\mathbf{C}^m$ as an entire function $F(z)$ of exponential type, which is almost periodic by Bohr on any tube domain in $\mathbf{C}^m$ with bounded base; $F(z)$ has the Dirichlet series (6), and P-indicator $h_F(y)$ satisfies the equation $h_F(y) = H_{sp\, f}(-y)$, where $H_{sp\, f}(\mu) := \sup_{x \in sp\, f} \langle x, \mu\rangle$ is the support function of the set $sp\, f$.*

P r o o f. Take $\mu \in \mathbf{R}^m$ such that $\|\mu\| = 1$. Put

$$f_\mu(x) = f(x) e^{-i[H_{sp\, f}(\mu)+\varepsilon]\langle x,\mu\rangle}.$$

The Fourier series $\sum_{n=0}^{\infty} a_n e^{i\langle x, \lambda_n - (H_{sp\, f}(\mu)+\varepsilon)\mu\rangle}$ corresponds to the function $f_\mu(x)$, hence

$$sp\, f_\mu \subset \{x \in \mathbf{R}^m : \langle x, \mu\rangle \le -\varepsilon\}.$$

Since $sp\, f_\mu$ is bounded, we obtain for some $\delta > 0$

$$sp\, f_\mu(x) \subset \Gamma_{\delta,-\mu} = \{\lambda \in \mathbf{R}^m : \langle \lambda, -\mu \rangle \geq \delta \|\lambda\|\}.$$

Theorem 2 yields that $f_\mu(x)$ extends to the interior of the domain $T_{\widehat{\Gamma}_{\delta,-\mu}}$, where

$$\widehat{\Gamma}_{\delta,-\mu} = \{y : \langle y, -\mu \rangle \geq \sqrt{1-\delta^2} \|y\|\}$$

is the conjugate cone to $\Gamma_{\delta,-\mu}$, as an almost periodic function by Bohr $F_\mu(z)$. This function is holomorphic on any domain $T_{\widehat{\Gamma}_{\delta,-\mu}+b}$, $b \in \overset{\circ}{\widehat{\Gamma}}_{\delta,-\mu}$ with the Dirichlet series

$$\sum_{n=0}^{\infty} a_n e^{i\langle z, \lambda_n - [H_{sp\, f}(\mu)+\varepsilon]\mu \rangle},$$

and $F_\mu(z) \to 0$ as $\|y\| \to \infty$ uniformly w.r.t. $z \in T_{\Gamma'}$ for any cone $\Gamma' \subset\subset \widehat{\Gamma}_{\delta,-\mu}$. Using (5), we get

$$\left| a_n e^{-\langle y, \lambda_n - [H_{sp\, f}(\mu)+\varepsilon]\mu \rangle} \right| \leq \sup_{x \in \mathbf{R}^m} |F_\mu(x+iy)|, \ y \in \Gamma'. \tag{16}$$

Put

$$F(z) := F_\mu(z) e^{i[H_{sp\, f}(\mu)+\varepsilon]\langle z, \mu \rangle}.$$

$F(z)$ is almost periodic on $T_{\Gamma'}$ with Dirichlet series (6). Therefore it follows from (16) that

$$|a_n| \leq \sup_{x \in \mathbf{R}^m} |F(x+iy)| e^{\langle y, \lambda_n \rangle}. \tag{17}$$

On the other hand, the function $F_\mu(z)$ is bounded on $T_{\Gamma'}$, hence

$$|F(z)| \leq C(\Gamma') e^{-[H_{sp\, f}(\mu)+\varepsilon]\langle y, \mu \rangle}, \ z \in T_{\Gamma'} \tag{18}$$

Cover the space $\mathbf{R}^m$ by the interiors of a finite number of cones $\Gamma'_1, \dots, \Gamma'_N$. There exist holomorphic on the interior of Γ'_k almost periodic functions $F_k(z)$, $k = 1, \dots, N$, with identical Dirichlet series (6). Using the uniqueness theorem, we obtain that these functions coincide on the intersections of the cones and thus define a holomorphic function $F(z)$ on $\mathbf{C}^m \setminus \mathbf{R}^m$. The Bochner–Feyer sums for $F(z)$ converge to this function uniformly on any set

$$\{z = x + iy : x \in \mathbf{R}^m, \ \|y\| = r > 0\}.$$

Hence, these sums converge on the tube domain $T_{\{\|y\|<r\}}$. Thus $F(z)$ extends to $\mathbf{C}^m$ as the holomorphic function, which is almost periodic on any tube set with

a bounded base. Owing to the uniqueness of expansion into Fourier series, we have $F(x) = f(x)$.

Let us prove that $h_F(y) = H_{sp\,f}(-y)$. From inequality (18) with $\mu = -y$ it follows that

$$h_F(y) \le \overline{\lim_{r\to\infty}} \frac{1}{r}[H_{sp\,f}(-y) + \varepsilon]\langle ry, y\rangle = H_{sp\,f}(-y) + \varepsilon.$$

The functions $h_F(y)$ and $H_{sp\,f}(y)$ are positively homogenious, hence the inequality

$$h_F(y) \le H_{sp\,f}(-y)$$

is true for all $y \in \mathbf{R}^m$.

Further, fix $x, y \in \mathbf{R}^m$. The holomorphic on $\mathbf{C}$ function $\varphi(w) = F(x + wy)$ is bounded on the axis $\operatorname{Im} w = 0$. Then the estimate

$$|\varphi(w)| \le Ce^{a|\operatorname{Im} w|}$$

for some $a > 0$ and all $w \in \mathbf{C}$ follows from (18). Using the definition of P-indicator, we get

$$\overline{\lim_{v\to+\infty}} \frac{1}{v} \log|\varphi(iv)| \le h_F(y).$$

Therefore the function $\varphi(w)e^{i(h_F(y)+\varepsilon)w}$ is bounded on the positive part of the imaginary axis. Applying the Fragmen–Lindelof principle to the quadrants $\operatorname{Re} w \ge 0$, $\operatorname{Im} w \ge 0$ and $\operatorname{Re} w \le 0$, $\operatorname{Im} w \ge 0$, we get boundedness of this function on the upper half-plane. Applying the Fragmen–Lindelof principle to the half-plane $\operatorname{Im} w \ge 0$, we get the inequality

$$|\varphi(w)| \le \left(\sup_{\operatorname{Im} w=0} |\varphi(w)|\right) e^{h_F(y)\operatorname{Im} w} \qquad (\operatorname{Im} w > 0).$$

Hence, for all $z \in \mathbf{C}^m$, we have

$$|F(z)| \le \sup_{x\in\mathbf{R}^m} |F(x)|e^{h_F(y)}.$$

Now using formula (17) for coefficients of the Dirichlet series of the function $F(z)$, we get the estimate

$$|a_n| \le \sup_{x\in\mathbf{R}^m} |f(x)|e^{h_F(y)+\langle y,\lambda_n\rangle}. \tag{19}$$

Suppose $\langle y_0, \lambda_n\rangle + h_F(y_0) < 0$ for some $y_0 \in \mathbf{R}^m$. Put $y = ty_0$ in (19) and let $t \to \infty$. We obtain $a_n = 0$. This is impossible because $\lambda_n \in sp\,f$.

Thus for all $y \in \mathbf{R}^m$ and $\lambda_n \in sp\,f$ we have $h_F(y) + \langle y, \lambda_n\rangle \ge 0$, hence

$$H_{spf}(-y) = \sup_{\lambda_n\in sp\,f} \langle -y, \lambda_n\rangle \le h_F(y).$$

This completes the proof of the theorem. ■

The following theorem is inverse to the previous one.

Theorem 6. *(For $m = 1$ see [3, 4].) Let $F(z)$ be an entire function on $\mathbf{C}^m$, $|F(z)| \le Ce^{b\|z\|}$, let $F(x)$, $x \in \mathbf{R}^m$ be an almost periodic function by Stepanoff with the Fourier series (7). Then $F(z)$ is an almost periodic function by Bohr on any tube domain $T_D \subset \mathbf{C}^m$ with the bounded base, $F(z)$ has the Dirichlet series (6), and $sp\, F \subset \{\lambda : \|\lambda\| \le b\}$.*

P r o o f. It follows from theorem 5, that it suffices to prove the inclusion

$$sp\, F \subset \{\lambda : \|\lambda\| \le b\}.$$

Let the function $F(x)$ be bounded on $\mathbf{R}^m$. Arguing as in theorem 5, we see that for all $z \in \mathbf{C}^m$

$$|F(z)| \le \sup_{x \in \mathbf{R}^m} |F(x)| e^{h_F(y)},$$

where $h_F(y)$ is P-indicator for $F(z)$. Further, for all $x \in \mathbf{R}^m$ we have

$$h_F(y) = \sup_{x \in \mathbf{R}^m} \overline{\lim_{r\to\infty}} \frac{1}{r} \log |F(x + iry)| \le \sup_{x \in \mathbf{R}^m} \overline{\lim_{r\to\infty}} \frac{1}{r} (\log C + b\|x + iry\|) \le b\|y\|,$$

therefore for all $z \in \mathbf{C}^m$

$$|F(z)| \le Ce^{b\|y\|}.$$

Take $\varepsilon > 0$, $\mu \in \mathbf{R}^m$, $\|\mu\| = 1$. Consider the function

$$F_\mu(z) = F(z) e^{-i\langle z,\mu\rangle(b+\varepsilon)}.$$

Since $|F_\mu(z)| \le Ce^{b\|y\| + (b+\varepsilon)\langle y,\mu\rangle}$ uniformly w.r.t. $x \in \mathbf{R}^m$, then $F_\mu(z)$ is uniformly bounded for $z \in T_{\Gamma_{-\mu}}$, where $\Gamma_{-\mu}$ is the cone $\{y : \langle y, -\mu\rangle \ge (1 - \frac{\varepsilon}{b+\varepsilon})\|y\|\}$. Using theorem 4, we obtain that the spectrum F_μ is contained in $\widehat{\Gamma}_{-\mu}$ and

$$sp\, F = sp\, F_\mu + (b + \varepsilon)\mu \subset \widehat{\Gamma}_{-\mu} + (b + \varepsilon)\mu.$$

Finally, using the inclusion

$$\bigcap_{\mu:\, \|\mu\|=1} (\widehat{\Gamma}_{-\mu} + (b + \varepsilon)\mu) \subset \{\lambda : \|\lambda\| \le b + \varepsilon\}$$

and the arbitrarity of choice of ε we get the assertion of the theorem in the case of bounded on $\mathbf{R}^m$ function $F(z)$.

Now let the function $F(z)$ be unbounded on $\mathbf{R}^m$. Put for some $N > 0$

$$g(z) = \frac{1}{N^m} \int\limits_{[0,N]^m} F(z + t)dt.$$

The function $g(z)$ satisfies the estimate on $\mathbf{C}^m$

$$|g(z)| \le Ce^{bmN}e^{b\|z\|}.$$

As in the case $m = 1$ (see [4]), we can prove that $g(x)$ is an almost periodic function by Bohr and is bounded on $\mathbf{R}^m$. The function $g(x)$ has the Fourier series

$$\sum_{n=0}^{\infty} a_n \frac{e^{i\lambda_n^1 N} - 1}{N\lambda_n^1} \cdots \frac{e^{i\lambda_n^m N} - 1}{N\lambda_n^m} e^{i\langle x, \lambda_n \rangle},$$

where λ_n^j are coordinates of the vector λ_n (if $\lambda_n^j = 0$, the corresponding multiplier should be replaced by 1).

Using countability of $sp\,F$, we can choose N in such a way that none of the numbers $\lambda_n^j N$ coincides with $2\pi k$, $k \in \mathbf{Z} \setminus \{0\}$. In this case $sp\,g = sp\,F$. Applying the proved above statement to the function $g(z)$, we obtain the inclusion

$$sp\,F \subset \{\lambda : \|\lambda\| \le b\}. \qquad \blacksquare$$

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