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К. N. Ilinski, A. S. Stepanenko, Q -деформированное суперпространство и q -расширенный суперсимметричный гамильтониан с произвольным суперпотенциалом, *Зап. научн. сем. ПОМИ*, 1996, том 235, 260–272

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18 февраля 2025 г., 03:40:49



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**q-DEFORMED SUPERSPACE AND q-EXTENDED
SUPERSYMMETRIC HAMILTONIAN
WITH ARBITRARY SUPERPOTENTIAL**

§1. INTRODUCTION

Recently deformations of standard mathematical objects have attracted a lot of attention ([1,2] and refs. in [2]). In particular, in [3] the quantum group representation theory and q -deformed spaces have been studied and in [4] the procedure of supergroup quantization has been presented.

The attempts to construct the description of dynamics on q -deformed spaces were undertaken in [5, 6, 7]. In [8] q -deformed quantum spaces were considered as a graded commutative algebra. On this basis the classical and quantum dynamics on q -deformed spaces were proposed. The particles appeared after the quantization satisfy the q -deformed commutation relations:

$$\begin{aligned} a_i a_i^+ \mp a_i^+ a_i &= 1 & a_i a_j^+ \mp q a_j^+ a_i &= 0 \\ a_i^+ a_j^+ \mp q^{-1} a_j^+ a_i^+ &= 0 & a_i a_j \mp q^{-1} a_j a_i &= 0 \end{aligned}$$

$$q^m = 1 \quad , \quad m \in N \quad , \quad 1 \leq i \leq j \leq n$$

$$a_i^{+2} = a_i^2 = 0 \quad \text{for } q\text{-deformed fermions,}$$

where the upper sign corresponds to the quantization on q -deformed usual spaces and lower one corresponds to the quantization on q -deformed superspaces. The particles can be naturally called q -bosons and q -fermions. It is interesting to consider the generalization of supersymmetric theory for the case when q -fermions are included into the theory instead of fermions.

As we can see later such supersymmetric theory will differ from the deformed supersymmetry (q -supersymmetry or q -SUSY) by V. P .

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Spiridonov [9] where the following relations for the supercharges Q^+, Q^- were used:

$$\begin{aligned}
 qQ^+Q^- + q^{-1}Q^-Q^+ &= H & Q^{+2} &= Q^{-2} = 0 \\
 qHQ^- - q^{-1}Q^-H &= qQ^+H - q^{-1}HQ^+ = 0.
 \end{aligned}$$

In our approach it is possible to discover the q -deformation only for the pair of different q -fermions and that is why this theory is q -extended supersymmetry (q -ESUSY). It is possible to coincide such theory in the framework of colour supersymmetry [10] (we are grateful N. Borisov for this remark).

In this paper we consider one of the possible generalization of SUSY. For this purpose we formulate a q -superspace formalism and construct the q -supertransformation group on this q -superspace. After this we start from a standard action for a scalar d -dimensional real superfield and build q -extended supersymmetric Lagrangian for component variables of the q -superfield. After the quantization procedure corresponding conserved quantities become q -supercharges. In difference from [11] where only general constructions of q -extended supersymmetry and simplest examples were considered in this paper we give exact expressions for q -supercharges and q -extended supersymmetric Hamiltonian with arbitrary superpotential.

§2. q -EXTENDED SUPERSYMMETRY AND q -DEFORMED SUPERSPACE FORMALISM

As in usual case [13, 14, 15] the most natural way to build supersymmetric Lagrangians is by employing the superspace formalism. For the sake of simplicity we will consider $(0+1)$ -dimensional case, but all constructions can be generalized for multidimensional one.

For the $(0+1)$ -dimensional case superspace reduces to the supertime. In the standard way besides time t complex anticommuting variables $\theta_i, \bar{\theta}_i, i = 1, 2$ are introduced. Instead of that we will consider the variables $\theta_i, \bar{\theta}_i \in HQ_m^{0|2}, i = 1, 2$, i.e.,

$$\begin{aligned}
 \theta_i \bar{\theta}_i &= -\bar{\theta}_i \theta_i & \theta_i^2 &= \bar{\theta}_i^2 = 0 & q^m &= 1, \quad m \in N \\
 \theta_1 \theta_2 &= -q^{-1} \theta_2 \theta_1 & \bar{\theta}_1 \theta_2 &= -q \theta_2 \bar{\theta}_1 \\
 \bar{\theta}_1 \bar{\theta}_2 &= -q^{-1} \bar{\theta}_2 \bar{\theta}_1 & \theta_1 \bar{\theta}_2 &= -q \bar{\theta}_2 \theta_1
 \end{aligned}$$

It was shown in [8] that the algebra $HQ_m^{0|2}$ can be described as a graded-commutative algebra. For that we began from the algebra

$C(\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2)$ which is freely generated by $\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2$ and used a special graduation map. In this case the graduation map $\varphi : HQ_m^{0|2} \rightarrow Z_2 \oplus Z_m \oplus Z_m$ was defined by

$$\begin{aligned} \varphi(\theta_1^{k_1} \dots \bar{\theta}_2^{k_4}) &= k_1\varphi(\theta_1) + \dots + k_4\varphi(\bar{\theta}_2) \\ \varphi(\theta_1) &= (1, 1, 0); \quad \varphi(\theta_2) = (1, 0, 1); \quad \varphi(\bar{\theta}_i) = -\varphi(\theta_i) \end{aligned}$$

and the graded-commutator for the homogeneous elements A, B had the form:

$$\begin{aligned} [A, B]_q^{0|2} &= AB - (-1)^{\vartheta_2(\varphi(A), \varphi(B))} q^{-\vartheta_m(\varphi(A), \varphi(B))} BA \\ \vartheta_2(\varphi(A), \varphi(B)) &= \varphi(A)|_{Z_2} \varphi(B)|_{Z_2} \end{aligned}$$

and $\vartheta_m(\alpha, \beta)$ was defined by the following relations: $\vartheta_m : Z_2 \oplus Z_m \oplus Z_m \rightarrow Z$ being biadditive antisymmetric map,

$$\vartheta_m(\alpha, \beta) = \sum_{i,j} \alpha_i \beta_j \vartheta_m(\xi_i, \xi_j) \quad \alpha = \sum_i \alpha_i \xi_i \quad \beta = \sum_i \beta_i \xi_i$$

$$\vartheta_m(\xi_i, \xi_j) = 1 \quad i < j \quad \xi_1 = (l, 1, 0) \quad \xi_2 = (k, 0, 1) \quad \forall k, l = 0, 1$$

Our algebra $HQ_m^{0|2}$ can be described as a factor-algebra of $C(\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2)$ under the two-side ideal, generated by the elements $\{[C_1, C_2]_q^{0|2} | C_1, C_2 \in C(\theta_1 \dots \bar{\theta}_2)\}$ and in this sense $HQ_m^{0|2}$ is exactly the algebra with zero graded-commutators:

$$f, g \in HQ_m^{0|2} \quad [f, g]_q^{0|2} = 0.$$

We note that q is a particular case of ϵ -factor [12] (and refs. therein).

A dynamic q -supervariable is a quantity depending on $t, \theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2$. The coefficients of its expansion in $\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2$ have the commutation relations according to their graduations. In what follows we will use the following notations:

$$\begin{aligned} \theta &= \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} & \bar{\theta} &= (\bar{\theta}_1, \bar{\theta}_2) & \psi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} & \bar{\psi} &= (\bar{\psi}_1, \bar{\psi}_2) \\ \bar{\theta}\psi &\equiv \bar{\theta}_1\psi_1 + \bar{\theta}_2\psi_2 & \chi &= \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} & \bar{\chi} &= (\bar{\chi}_1, \bar{\chi}_2) & \partial_t &= \frac{\partial}{\partial t} \end{aligned}$$

The supersymmetry group is implemented in terms of $t, \theta, \bar{\theta}$ transformations in the following way:

$$Q_{\bar{\eta}} : \quad t \rightarrow t + \frac{i}{2} \bar{\eta} \theta \quad \bar{\theta} \rightarrow \bar{\theta} + \bar{\eta}$$

$$Q_\eta : t \rightarrow t - \frac{i}{2} \bar{\theta} \eta \quad \theta \rightarrow \theta + \eta$$

where $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, $\bar{\eta} = (\bar{\eta}_1, \bar{\eta}_2)$ are transformation parameters, such as $\varphi(\eta_i) = \varphi(\theta_i)$.

The generators of the supersymmetry transformations can be written in a differential form:

$$Q_\eta = \frac{\partial}{\partial \theta} + \frac{i}{2} \theta \partial_t \equiv \begin{pmatrix} Q_\eta^1 \\ Q_\eta^2 \end{pmatrix} \quad Q_{\bar{\eta}} = -\frac{\partial}{\partial \theta} - \frac{i}{2} \bar{\theta} \partial_t \equiv (Q_{\bar{\eta}}^1, Q_{\bar{\eta}}^2) \quad (1)$$

Now we note that

$$[Q_{\bar{\eta}}^\alpha, Q_{\eta^1 q}^\alpha]^{0|2} = -i \partial_t \quad \alpha = 1, 2$$

and

$$[Q_\eta^1, Q_{\eta^2 q}^{2|0}]^2 = Q_\eta^1 Q_\eta^2 + q^{-1} Q_\eta^2 Q_\eta^1 = 0 \quad Q_{\bar{\eta}}^{\alpha 2} = Q_{\bar{\eta}}^2 = 0$$

$$[Q_{\bar{\eta}}^1, Q_{\bar{\eta}^2 q}^{2|0}]^2 = [Q_{\bar{\eta}}^\alpha, Q_{\bar{\eta}^1 q}^{\alpha'}]^{0|2} = 0 \quad \alpha' \neq \alpha. \quad (2)$$

Above relations give the connections between the supersymmetry generators and the Hamiltonian of the system. So, graded-commutators $[Q_{\bar{\eta}}^\alpha, Q_{\eta^1 q}^\alpha]^{0|2}$, $\alpha = 1, 2$ are the generators of time translations (up to a factor $-i$).

Representation of the supersymmetric group can be implemented, for example, by means of d -dimensional scalar real q -superfield $\phi = (\phi^1 \dots \phi^d)$. Its expansion in $\theta, \bar{\theta}$ looks like

$$\begin{aligned} \phi^k(t) = & x^k(t) + \frac{1}{\sqrt{2}} \bar{\theta} \psi^k(t) + \frac{1}{\sqrt{2}} \bar{\psi}^k(t) \theta + \\ & + \bar{L}^k(t) \theta_1 \theta_2 + \bar{\theta}_2 \bar{\theta}_1 L^k(t) + \text{Tr}(\hat{\theta} \mathbf{F}^k(t)) + \frac{1}{\sqrt{2}} \bar{\chi}^k(t) \tilde{\theta} + \\ & + \frac{1}{\sqrt{2}} \tilde{\theta} \tilde{\chi}^k(t) + H^k(t) \bar{\theta}_1 \theta_1 \bar{\theta}_2 \theta_2, \quad k = 1 \dots d \end{aligned} \quad (3)$$

where

$$\hat{\theta} = \begin{pmatrix} \bar{\theta}_1 \theta_1 & \bar{\theta}_2 \theta_1 \\ \bar{\theta}_1 \theta_2 & \bar{\theta}_2 \theta_2 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad \tilde{\theta} = \begin{pmatrix} \theta_1 \bar{\theta}_2 \theta_2 \\ \theta_2 \bar{\theta}_1 \theta_1 \end{pmatrix}.$$

Q-supersymmetry transformations are realized in component variables as follows:

for $\bar{\eta}Q\bar{\eta}$

$$\delta_1 x^k = \frac{1}{\sqrt{2}} \bar{\eta} \psi^k, \quad \delta_1 H^k = -\frac{i}{2\sqrt{2}} \bar{\eta} \chi^k$$

$$\delta_1 \bar{\psi}^k = \sqrt{2} \left\{ \frac{i}{2} \bar{\eta} \dot{x}^k + (\bar{\eta}_1 F_{11}^k + \bar{\eta}_2 q^{-1} F_{21}^k, \bar{\eta}_2 F_{22}^k + \bar{\eta}_1 q F_{12}^k) \right\},$$

$$\delta_1 \psi^k = \sqrt{2} \begin{pmatrix} -q\bar{\eta}_2 \\ \bar{\eta}_1 \end{pmatrix} L^k$$

$$\delta_1 \bar{\chi}^k = \sqrt{2} \left\{ \bar{\eta} H^k + \frac{i}{2} (\bar{\eta}_1 \dot{F}_{22}^k - \bar{\eta}_2 q^{-1} \dot{F}_{21}^k, \bar{\eta}_2 \dot{F}_{11}^k - \bar{\eta}_1 q \dot{F}_{12}^k) \right\},$$

$$\delta_1 \chi^k = -\frac{i}{\sqrt{2}} \begin{pmatrix} -q\bar{\eta}_2 \\ \bar{\eta}_1 \end{pmatrix} \dot{L}^k$$

$$\delta_1 L^k = 0, \quad \delta_1 \bar{L}^k = \frac{1}{\sqrt{2}} \left(\bar{\eta}_2 \left(\frac{i}{2} \dot{\psi}_1^k + \dot{\chi}_1^k \right) + \left(\frac{i}{2} \dot{\psi}_2^k + \dot{\chi}_2^k \right) \bar{\eta}_1 \right)$$

$$\begin{aligned} \delta_1 \mathbf{F}^k &= \delta_1 \begin{pmatrix} F_{11}^k & F_{12}^k \\ F_{21}^k & F_{22}^k \end{pmatrix} = \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{\eta}_1 \left(-\frac{i}{2} \dot{\psi}_1^k \right) + \bar{\eta}_2 \dot{\chi}_2^k & -\bar{\eta}_2 \left(\frac{i}{2} \dot{\psi}_1^k + \dot{\chi}_1^k \right) \\ -\bar{\eta}_1 \left(\frac{i}{2} \dot{\psi}_2^k + \dot{\chi}_2^k \right) & \bar{\eta}_2 \left(-\frac{i}{2} \dot{\psi}_2^k \right) + \bar{\eta}_1 \dot{\chi}_1^k \end{pmatrix} \end{aligned} \quad (4)$$

and for $Q_\eta\eta$ it is possible to obtain transformation laws using (1) and (4).

The covariant derivatives D and \bar{D} have the form

$$D = \frac{\partial}{\partial \theta} - \frac{i}{2} \theta \partial_t \equiv \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \quad \bar{D} = -\frac{\partial}{\partial \theta} + \frac{i}{2} \bar{\theta} \partial_t \equiv (\bar{D}_1, \bar{D}_2)$$

satisfying the usual conditions

$$D\delta_i\phi = \delta_i D\phi, \quad \bar{D}\delta_i\phi = \delta_i \bar{D}\phi$$

where $\delta_i\phi$, $i = 1, 2$ are the variances of the q -supervariables under a supersymmetric transformations.

The action for the q -extended supersymmetric quantum mechanics can be written in the following way

$$S = \int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 (\bar{D}\phi^k D\phi^k - 2V(\phi)) = \int dt \mathcal{L}$$

where $V(\cdot)$ is an arbitrary function (superpotential). We imply the integration over $\theta_i, \bar{\theta}_i$ in the following sense

$$\int 1 d\theta_i = \int 1 d\bar{\theta}_i = \int \theta_j d\bar{\theta}_i = \int \bar{\theta}_j d\theta_i = 0 \quad \forall i, j = 1, 2$$

$$\int \theta_i d\theta_k = \int \bar{\theta}_i d\bar{\theta}_k = \delta_{ik}, \quad \phi(\theta_i) = \phi(d\theta_i), \quad \phi(\bar{\theta}_i) = \phi(d\bar{\theta}_i)$$

As a result the Lagrangian takes the form

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} \dot{x}^k \left(\dot{F}_{11}^k + \dot{F}_{22}^k \right) - 2H^k \left(F_{11}^k + F_{22}^k \right) + \frac{1}{4} \dot{\bar{\psi}}^k \dot{\psi}^k + \bar{\chi}^k \chi^k + \\ & + \frac{i}{4} \left(\bar{\psi}^k \chi^k - \bar{\chi}^k \psi^k + \bar{\chi}^k \dot{\psi}^k - \dot{\bar{\chi}}^k \psi^k \right) + 2i \bar{L}^k \dot{L}^k - 2 \frac{\partial V}{\partial x^k} H^k + \\ & + 2 \frac{\partial^2 V}{\partial x^k \partial x^l} \left(\frac{1}{2} \bar{\psi}^l \chi^k + \frac{1}{2} \bar{\chi}^l \psi^k + F_{12}^k F_{21}^l - F_{11}^k F_{22}^l - \bar{L}^l L^k \right) - \\ & - \frac{\partial^3 V}{\partial x^l \partial x^m \partial x^n} \left(F_{12}^l \bar{\psi}_1^m \psi_2^n + F_{21}^l \bar{\psi}_2^m \psi_1^n - F_{11}^l \bar{\psi}_2^m \psi_2^n - F_{22}^l \bar{\psi}_1^m \psi_1^n + \right. \\ & \left. + \bar{L}^l \psi_1^m \psi_2^n + L^l \bar{\psi}_2^m \bar{\psi}_1^n \right) - \frac{1}{2} \frac{\partial^4 V}{\partial x^l \partial x^m \partial x^n \partial x^k} \bar{\psi}_1^l \bar{\psi}_2^m \psi_2^n \psi_1^k \end{aligned} \quad (6)$$

where full derivatives have been neglected. The auxiliary variables $F_{ij}^k, \chi^k, \bar{\chi}^k, H^k$ may be excluded in the usual way if matrix $\partial_{lk} V = K$ is invertible. Omitting spatial symbols to simplify the expressions we can put down the expressions for auxiliary fields

$$\begin{aligned} \chi_i(t) &= (-2\psi_i(t)V''[x(t)] + \dot{\psi}_i(t)i)/2 \\ \bar{\chi}_i(t) &= -(2\bar{\psi}_i(t)V''[x(t)] + \dot{\bar{\psi}}_i(t)i)/2 \\ F_{i,j \neq i}(t) &= K^{-1}V'''[x(t)]\bar{\psi}_j(t)\psi_i(t)/2 \\ F_{ii}(t) &= (-1)^i \frac{1}{4} ((-1)^k \bar{\psi}_k(t)\psi_k(t)K^{-1}V''''[x(t)]) - \frac{1}{2}V'[x(t)] \\ H(t) &= \frac{1}{4}(\bar{\psi}_i(t)\psi_i(t)V''''[x(t)]) + \frac{1}{2}V'[x(t)]V''[x(t)] + \frac{1}{4}\ddot{x}(t). \end{aligned}$$

After substitutions of auxiliary fields Lagrangian has the form:

$$\begin{aligned} \mathcal{L} = & - \left(8\bar{L}(t)L(t)V''[x(t)] - 8\bar{L}(t)\dot{L}(t)i + 4\bar{L}(t)\psi_1(t)\psi_2(t)V''''[x(t)] - \right. \\ & - 4\bar{\psi}_1(t)\bar{\psi}_2(t)L(t)V''''[x(t)]q + 3\bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)\psi_2(t)K^{-1}(V''''[x(t)])^2q - \\ & - 2\bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)\psi_2(t)V^{(\nu)}[x(t)]q + 2\bar{\psi}_i(t)\psi_i(t)V'[x(t)]V''''[x(t)] + \\ & + 4\bar{\psi}_i(t)\psi_i(t)(V''[x(t)])^2 - 2\bar{\psi}_i(t)\dot{\psi}_i(t)V''[x(t)]i + 2\dot{\bar{\psi}}_i(t)\psi_i(t)V''[x(t)]i + \\ & \left. + 2(V'[x(t)])^2V''[x(t)] - 2V''[x(t)](\dot{x}(t))^2 \right) / 4. \end{aligned}$$

Let us denote the action as follows

$$S = \int dt d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \mathcal{L} = \int dt \hat{\mathcal{L}}.$$

We can find conserved Noether's currents directly for $\hat{\mathcal{L}}$, then integrate over $\theta, \bar{\theta}$ and obtain conserved supercharges. These expressions contain not only "physical" fields but also auxiliary fields. After substitutions of constrains supercharges have the following form in terms of "physical" fields:

$$\begin{aligned} Q_{\eta_1} \eta_1 &= - (4\bar{L}(t)\psi_2(t)V''[x(t)]iq^{-1} - \bar{\psi}_1(t)\bar{\psi}_2(t)\psi_2(t)V''''[x(t)]i - \\ & - 2\bar{\psi}_1(t)V'[x(t)]V''[x(t)]i + 2\bar{\psi}_1(t)V''[x(t)]\dot{x}(t))\eta_1 / 2\sqrt{2} \\ Q_{\eta_2} \eta_2 &= (4\bar{L}(t)\psi_1(t)V''[x(t)]i - \bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)V''''[x(t)]iq + \\ & + 2\bar{\psi}_2(t)V'[x(t)]V''[x(t)]i - 2\bar{\psi}_2(t)V''[x(t)]\dot{x}(t))\eta_2 / 2\sqrt{2} \\ Q_{\bar{\eta}_1} \bar{\eta}_1 &= - (4\bar{\psi}_2(t)L(t)V''[x(t)]iq + \bar{\psi}_2(t)\psi_1(t)\psi_2(t)V''''[x(t)]iq - \\ & - 2\psi_1(t)V'[x(t)]V''[x(t)]i - 2\psi_1(t)V''[x(t)]\dot{x}(t))\bar{\eta}_1 / 2\sqrt{2} \\ Q_{\bar{\eta}_2} \bar{\eta}_2 &= (4\bar{\psi}_1(t)L(t)V''[x(t)]i + \bar{\psi}_1(t)\psi_1(t)\psi_2(t)V''''[x(t)]i + \\ & + 2\psi_2(t)V'[x(t)]V''[x(t)]i + 2\psi_2(t)V''[x(t)]\dot{x}(t))\bar{\eta}_2 / 2\sqrt{2}. \end{aligned}$$

§3. CLASSICAL DYNAMICS AND QUANTIZATION PROCEDURE FOR THE COMPONENT VARIABLES

Here we construct classical and quantum dynamics in close analogy with [8]. It is important however that the new fields L, \bar{L} arise which are not q -fields in the sense [8].

To sake of simplicity let us make the change of variables:

$$\bar{L}, L \rightarrow \frac{\bar{L}}{\sqrt{2}}, \frac{L}{\sqrt{2}}, \quad \bar{\psi} \rightarrow \bar{\psi}K^{-1/2}, \psi K^{-1/2}.$$

Then we have for Lagrangian of the system can be rewritten in the form:

$$\begin{aligned} \mathcal{L} = & - \left(4\bar{L}(t)L(t)V''[x(t)] - 2\bar{L}(t)\dot{L}(t)i + 2\dot{\bar{L}}(t)L(t)i + \right. \\ & + 2\sqrt{2}\bar{L}(t)\psi_1(t)\psi_2(t)K^{-1}V'''[x(t)] - \\ & - 2\sqrt{2}\bar{\psi}_1(t)\bar{\psi}_2(t)L(t)K^{-1}V''''[x(t)]q + \\ & + 3\bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)\psi_2(t)K^{-3}(V''''[x(t)])^2q + \\ & + 2\bar{\psi}_i(t)\psi_i(t)K^{-1}V'[x(t)]V''''[x(t)] - \\ & - 2\bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)\psi_2(t)K^{-2}V^{(V)}[x(t)]q + \\ & + 4\bar{\psi}_i(t)\psi_i(t)K - 2\bar{\psi}_i(t)\dot{\psi}_i(t)i + \\ & \left. + 2\dot{\bar{\psi}}_i(t)\psi_i(t)i + 2(V'[x(t)])^2K - 2K\dot{x}^2(t) \right) / 4. \end{aligned}$$

The principle of extreme action for trajectories is

$$\delta S = 0$$

$$\begin{aligned} S[x, p, \psi, \bar{\psi}, L, \bar{L}] = & \int dt \left\{ \frac{i}{2} \sum_{j=1}^2 [\bar{\psi}_j(t)\dot{\psi}_j(t) - \dot{\bar{\psi}}_j(t)\psi_j(t)] + \right. \\ & \left. + \frac{i}{2} [\bar{L}(t)\dot{L}(t) - \dot{\bar{L}}(t)L(t)] + p(t)\dot{x}(t) - \mathcal{H}[x, p, \psi, \bar{\psi}, L, \bar{L}] \right\} \equiv \int dt \mathcal{L} \end{aligned}$$

where

$$\dot{\psi}_j^k = \frac{d\psi_j^k}{dt}, \quad \dot{\bar{\psi}}_j^k = \frac{d\bar{\psi}_j^k}{dt}, \quad \dot{L}^k = \frac{dL^k}{dt}, \quad \dot{\bar{L}}^k = \frac{d\bar{L}^k}{dt}, \quad \varphi(\mathcal{H}) = 0.$$

Derivatives are calculated as in the Grassmann algebra case.

The equations for extreme trajectories have the form:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \psi_j} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\psi}_j}, \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}_j} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}_j}, \quad \frac{\partial \mathcal{L}}{\partial L} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{L}}, \\ \frac{\partial \mathcal{L}}{\partial \bar{L}} &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\bar{L}}}, \quad \frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}, \quad \frac{\partial \mathcal{L}}{\partial p} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{p}}. \end{aligned}$$

Here $\frac{\partial}{\partial L^k}, \frac{\partial}{\partial \bar{L}^k}, \frac{\partial}{\partial \psi_j^k}, \frac{\partial}{\partial \bar{\psi}_j^k}$ are defined as left derivatives in close analogy with the Grassmann case. Note that they are differentiations of graduated algebra of functions, i.e.,

$$\frac{\partial}{\partial \psi_j^k} f g = \left(\frac{\partial}{\partial \psi_j^k} f \right) g + (-1)^{-\theta_2(\varphi(\psi_j^k), \varphi(f))} q^{-\theta_m(\varphi(\bar{\psi}_j^k), \varphi(f))} f \left(\frac{\partial}{\partial \psi_j^k} g \right)$$

for homogeneous elements f, g and similar for $\frac{\partial}{\partial L}, \frac{\partial}{\partial L}, \frac{\partial}{\partial \psi}$.

Putting $p = \dot{x}K$ we can rewrite the motion equations in Hamiltonian form:

$$\dot{\chi}_k = (\Omega^{-1})_{kj} \frac{\partial \mathcal{H}}{\partial \chi_j}$$

where

$$\begin{pmatrix} \psi_1 \\ \bar{\psi}_1 \\ \psi_2 \\ \bar{\psi}_2 \\ L \\ \bar{L} \\ x \\ p \end{pmatrix}, \quad \Omega \equiv \begin{pmatrix} \Omega_1 & 0 & 0 & 0 \\ 0 & \Omega_2 & 0 & 0 \\ 0 & 0 & \Omega_3 & 0 \\ 0 & 0 & 0 & \Omega_4 \end{pmatrix},$$

$$\Omega_{1,2} \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \Omega_3 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Omega_4 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence, the inverse matrix Ω^{-1} is

$$\Omega^{-1} = \begin{pmatrix} \Omega_1^{-1} & 0 & 0 & 0 \\ 0 & \Omega_2^{-1} & 0 & 0 \\ 0 & 0 & \Omega_3^{-1} & 0 \\ 0 & 0 & 0 & \Omega_4^{-1} \end{pmatrix},$$

$$\Omega_{1,2}^{-1} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \Omega_3^{-1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Omega_4^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is possible to introduce the Poisson structure in close analogy with the Grassmann case:

$$\{f, g\}_{cl} = - \sum_{s, s'} (\Omega^{-1})_{s, s'} \frac{\partial}{\partial \chi_{s'}} \frac{\partial}{\partial \chi_s} f(\chi) g(\chi') \Big|_{\chi = \chi'}$$

Such defined Poisson brackets satisfy the following relations for homogeneous elements f, g, h :

$$\begin{aligned} \{f, g\}_{cl} &= (-1)^{\theta_2(\varphi(f), \varphi(g))} q^{\theta_m(\varphi(f), \varphi(g))} \{g, f\}_{cl} \\ \{fg, h\}_{cl} &= f\{g, h\}_{cl} + (-1)^{\theta_2(\varphi(g), \varphi(h))} q^{\theta_m(\varphi(g), \varphi(h))} \{f, h\}g \\ &\quad + \{f, \{g, h\}_{cl}\}_{cl} + \\ &\quad + (-1)^{\theta_2(\varphi(g), \varphi(h)) + \theta_2(\varphi(f), \varphi(h))} q^{\theta_m(\varphi(g), \varphi(h)) + \theta_m(\varphi(f), \varphi(h))} \{h, \{f, g\}_{cl}\}_{cl} + \\ &\quad + (-1)^{\theta_2(\varphi(f), \varphi(g)) + \theta_2(\varphi(f), \varphi(h))} q^{\theta_m(\varphi(f), \varphi(g)) + \theta_m(\varphi(f), \varphi(h))} \{g, \{h, f\}_{cl}\}_{cl} = 0 \end{aligned} \quad (7)$$

which correspond to generalization of antisymmetry, differentiation property and Jacobi identity for ordinary Poisson brackets.

The equations of motion can be written by means of these Poisson brackets as

$$\dot{\chi}_j = \{H, \chi_j\}_{cl}.$$

It should be noted that classical dynamics complies with the graduation structure (i.e., evolution does not change graduation of χ) due to $\varphi(H) = 0$.

The commutation relations for canonical variables related to $\{ \}_{cl}$ have the form:

$$\{\psi_i, \bar{\psi}_i\}_{cl} = i, \quad \{L, \bar{L}\}_{cl} = i, \quad \{x, p\}_{cl} = -1 \quad (8)$$

and others are zeroth.

To canonically quantize our theory one has

- 1 to construct the operators of canonical variables in Hilbert space, which satisfy the commutation relations (8)
- 2 to define the quantum Poisson brackets $\{ \}_h$ which have the same algebraic properties as the classical ones.

In this case item 2 leads to following definition of quantum Poisson brackets:

$$\{A, B\}_h = \frac{i}{h}[A, B]_q^{0|2}.$$

The commutation relations for the field operators follow from the above:

$$(L^+)^+ = L, \quad (\psi_i^+)^+ = \psi_i, \quad q^* = q^{-1},$$

$$(\psi_i^+)^2 = (\psi_i)^2 = 0, \quad \psi_1\psi_2 + q^{-1}\psi_2\psi_1 = 0, \quad \psi_1\psi_2^+ + q\psi_2^+\psi_1 = 0,$$

$$L\psi_1 - q\psi_1L = 0, \quad L\psi_2 - q^{-1}\psi_2L = 0,$$

$$L\psi_2^+ - q\psi_2^+L = 0, \quad L\psi_1^+ - q^{-1}\psi_1^+L = 0,$$

$$LL^+ - L^+L = h, \quad \psi_i\psi_i^+ + \psi_i^+\psi_i = h, \quad [x, p] = ih \quad (9)$$

and relations which can be obtained by Hermitian conjugation.

After quantization q -supercharges have the form:

$$Q_1 = - (2\sqrt{2}\bar{L}\psi_2 K^{1/2} i q^{-1} - \bar{\psi}_1 \bar{\psi}_2 \psi_2 K^{-3/2} V''' i + \bar{\psi}_1 K^{-3/2} V''' h i + \\ + 2\bar{\psi}_1 K^{-1/2} p - 2\bar{\psi}_1 K^{1/2} V' i) / 2\sqrt{2}$$

$$Q_2 = (2\sqrt{2}\bar{L}\psi_1 K^{1/2} i - \bar{\psi}_1 \bar{\psi}_2 \psi_1 K^{-3/2} V''' i q - \bar{\psi}_2 K^{-3/2} V''' h i - \\ - 2\bar{\psi}_2 K^{-1/2} p + 2\bar{\psi}_2 K^{1/2} V' i) / 2\sqrt{2}$$

$$Q_1^+ = (2\sqrt{2}\bar{\psi}_2 L K^{1/2} i q + \bar{\psi}_2 \psi_1 \psi_2 K^{-3/2} V''' i q - \\ - 2\psi_1 K^{-1/2} p - 2\psi_1 K^{1/2} V' i) / 2\sqrt{2}$$

$$Q_2^+ = - (2\sqrt{2}\bar{\psi}_1 L K^{1/2} i + \bar{\psi}_1 \psi_1 \psi_2 K^{-3/2} V''' i + \\ + 2\psi_2 K^{-1/2} p + 2\psi_2 K^{1/2} V' i) / 2\sqrt{2}.$$

It is possible to check that such q -supercharges obey the q -extended supersymmetry algebra

$$\{Q_i, Q_i^+\} = H, \quad (Q_i)^2 = (Q_i^+)^2 = 0, \quad (Q_i^+)^+ = Q_i,$$

$$Q_1 Q_2 = -q^{-1} Q_2 Q_1, \quad Q_1 Q_2^+ = -q Q_2^+ Q_1, \quad i = 1, 2$$

with following quantum q - extended supersymmetric Hamiltonian

$$\mathcal{H} = h \left(8\bar{L}L K + 4\sqrt{2}\bar{L}\psi_1 \psi_2 K^{-1} V''' - 4\sqrt{2}\bar{\psi}_1 \bar{\psi}_2 L K^{-1} V''' q + \right. \\ + 6\bar{\psi}_1 \bar{\psi}_2 \psi_1 \psi_2 K^{-3} (V''')^2 q - 4\bar{\psi}_1 \bar{\psi}_2 \psi_1 \psi_2 K^{-2} V^{(\nu\nu)} q + \\ + 3\bar{\psi}_1 \psi_1 K^{-3} (V''')^2 h - 2\bar{\psi}_1 \psi_1 K^{-2} V^{(\nu\nu)} h + 4\bar{\psi}_1 \psi_1 K^{-1} V' V''' + \\ + 8\bar{\psi}_1 \psi_1 K + 3\bar{\psi}_2 \psi_2 K^{-3} (V''')^2 h - 2\bar{\psi}_2 \psi_2 K^{-2} V^{(\nu\nu)} h + \\ + 4\bar{\psi}_2 \psi_2 K^{-1} V' V''' + 8\bar{\psi}_2 \psi_2 K - 3K^{-3} (V''')^2 h^2 + \\ + 4K^{-2} V''' p h i + 2K^{-2} V^{(\nu\nu)} h^2 - 4K^{-1} V' V''' h + \\ \left. + 4K^{-1} p^2 + 4K(V')^2 - 4K h \right) / 8.$$

Let us introduce the creation (annihilation) operators a_i^+, a_i, l^+, l which correspond to the fields $\psi, \bar{\psi}, L, \bar{L}$:

$$\psi_i^k = \sqrt{h} a_i^k \quad L^k = \sqrt{h} l^k \\ \psi_i^{k+} = \sqrt{h} a_i^{k+} \quad L^{k+} = \sqrt{h} l^{k+}$$

It is easy to obtain their commutation relations from (9).

Now we would like to discuss creation and annihilation operator representations. Let us consider the relations:

$$\begin{aligned}
 a_1^k &= b_1^k, & (a_1^{k+})^+ &= a_1^k \\
 a_2^k &= e^{i\varphi} \sum_{i=1}^d (b_i^i)^+ b_i^i b_2^k, & (a_2^{k+})^+ &= a_2^k, & q &= e^{-i\varphi} \\
 l^k &= e^{i\varphi} \sum_{i=1}^d (b_i^i)^+ b_i^i - b_2^i b_2^i d^k, & (l^{k+})^+ &= l^k
 \end{aligned} \tag{10}$$

where b_i^{k+} , b_i^k , d^{k+} , d^k are creation (annihilation) operators of bosons and fermions respectively with usual commutation relations:

$$\begin{aligned}
 [b_i^k, d^l] &= [b_i^{k+}, d^l] = [b_i^k, d^{l+}] = [b_i^{k+}, d^{l+}] = 0 \\
 \{b_i^k, b_j^{l+}\} &= \delta_{ij} K_{kl}^{-1}, & [d^k, d^{l+}] &= M_{kl}^{-1} \\
 \{b_i^k, b_j^l\} &= \{b_i^{k+}, b_j^{l+}\} = 0, & [d^k, d^l] &= [d^{k+}, d^{l+}] = 0.
 \end{aligned}$$

Using matrix representation for fermion operators we can put down matrix representation for q -fermion operators and operators l^k, l^{k+} can be implemented by means of matrix differential operators. Hence q -supercharges and q -extended supersymmetric Hamiltonian can be represented by means of matrix differential operators as it was done for usual supercharges and supersymmetric Hamiltonian in [14].

§4. CONCLUSION

In this paper we have studied the generalization of the supersymmetry, which was generated by changing fermions by q -fermions. The latters can be obtained after quantization of the fields, which take values in q -deformed spaces.

For this purpose we formulated the q -superspace formalism and constructed the q -supertransformation group. On this basis q -extended supersymmetric Lagrangian was built and corresponding q -supercharges were put down. We considered the rule for the quantization such systems and fulfilled this procedure for our case. We note that all q -deformed objects (q -supercharges and q -extended supersymmetric Hamiltonian) contain arbitrary functional parameter – superpotential V . It is especially interesting for investigations of topological properties of q -extended supersymmetric Hamiltonians.

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Поступило 12 мая 1993г.

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