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# On the imbedding of normed rings into the ring of operators in Hilbert space

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## § 1. Fundamental notions

This paper is devoted to the investigation of a class of normed rings. A set $R$ is called normed ring (cf. [3]) if

($\alpha$) $R$ is a linear normed complete space in the sense of Banach [1];

($\beta$) an (in general non-commutative) operation of multiplication is defined in $R$ with the ordinary properties ($\lambda$, $\mu$ are complex numbers)

$$x(\lambda y+\mu z)=\lambda xy+\mu xz, \quad x(yz)=(xy)z;$$

($\gamma$) for any two elements $x, y \in R$

$$\|xy\| \leqslant \|x\| \cdot \|y\|;$$

($\delta$) $R$ possesses a unit *i. e.* an element $e$ such that $ex=xe=x$ for all $x \in R$; moreover $|e|=1$.

A set $I_l$ of elements $x \in R$ will be called left ideal, if

1° $x \in I_l$, $y \in I_l$ imply $px+qy \in I_l$ for all $p, q \in R$;

2° $I_l \neq R$.

In a similar manner we define the right ideals $I_r$. A set $I$ will be called two-sided ideal, when it is a left and a right ideal. A normed ring is called simple if it does not contain any two-sided ideals.

Repeating the argument of [3], p. 5, we can state the following facts:

I. *An element $x \in R$ possesses a left (right) inverse element, if and only if it does not belong to any left (right) ideal.*

II. *The closure of a left (right) ideal is a left (right) ideal.*

A left (right) ideal will be called maximal, when it is not a rpoper subset of a left (right) ideal. The proposition II implies:

III. *Every maximal left (right) ideal is closed.*

Further we have:

IV. *Every left (right) ideal is contained in a maximal left (right) ideal.*

Hence by I and IV:

V. *An element $x \in R$ possesses a left (right) inverse element, if and only if it does not belong any maximal left (right) ideal.*

A normed ring $R$ will be called an $*$-ring if to every $x \in R$ there corresponds an element $x^* \in R$ satisfying the following conditions *:

1′. $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$;

2′. $x^{**} = x$;

3′. $(xy)^* = y^* x^*$;

4′. $\|x^* x\| = \|x^*\| \cdot \|x\|$;

5′. $\|x^*\| = \|x\|$ **;

6′. $x^* x + e$ possesses a two-sided inverse element for all $x \in R$.

From $xe = ex = e$ we obtain by 3′

$$e^* x^* = x^* e^* = e^*. \tag{1}$$

Since by 2′ $x^*$ runs through the whole $R$ when $x$ does so (1) means that $e^*$ is a unit in $R$. As $R$ possesses only one unit, we have

$$e^* = e. \tag{2}$$

Every closed subring $R$ of an $*$-ring is clearly also an $*$-ring, if $x \in R$ impliesx$^* \in R$. As an example of an $*$-ring we point out the set $B$ of all bounded operatos in Hilbert spacer *** $\mathfrak{H}$ where $x^*$ is the operator adjoint to $x$ and $\|x\|$ denotes the norm of $x$. In fact, 1′—3′, 5′—6′ express the well known properties of adjoint operators in Hilbert space, so that it remains to prove 4′. By 5′ ($|\xi| = |\eta| = 1$; $\xi, \eta \in \mathfrak{H}$)

$$\|x^* x\| = \sup_{\xi, \eta} (x^* x \xi, \eta) | \geqslant \sup_{\xi} | (x^* x \xi, \xi) | = \sup_{\xi} \|x \xi\|^2 = \|x\|^2 = \|x\| \cdot \|x^*\|;$$

on the other hand, by ($\gamma$)

$$\|x^* x\| \leqslant \|x^*\| \cdot \|x\|.$$

$B$ is thus an $*$-ring; hence every closed subring $R$ of $B$ is also an $*$-ring, if $x \in R$ implies $x^* \in R$.

The main purpose of this paper is the proof of the following.

Theorem 1. *Every normed $*$-ring can be isomorphically mapped onto a closed subring $R_1$ of the set $B$ of all bounded operators in a Hilbert space $\mathfrak{H}$ in such a manner that, if $x \in R$ and $X \in R_1$ correspond to each other, then $\|x\| = \|X\|$ and $x^*$, $X^*$ also correspond to each other by this mapping.*

## § 2. Some lemmas

In order to prove this theorem we establish some lemmas that are of independent interest.

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* Here and below $\lambda$, $\mu$ are complex numbers and $\bar{\lambda}$, $\bar{\mu}$ the conjugate numbers.

** The authors suppose the last two axioms to be corollaries of 1′—4′, but they have not succeeded in proof of this fact. We also note that the axioms 4′, 5′ may be replaced by the axiom: $\|x^* x\| = \|x\|^2$. For ($\gamma$) § 1 implies $\|x\|^2 = \|x^* x\| \leqslant \|x^*\| \cdot \|x\|$, hence $\|x\| \leqslant \|x^*\|$. Replacing $x$ by $x^*$ gives $\|x^*\| \leqslant \|x\|$, hence $\|x^*\| = \|x\|$.

*** Hilbert spaces considered in this paper are not supposed to be separable.

Lemma 1. *Let $R$ be a normed commutative ring and let an operation be de defined in $R$ putting in correspondence to every element $x \in R$ an element $x^*$under $\in R$ the following conditions:*

1′. $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$;
2′. $x^{**} = x$;
3′. $(xy)^* = x^* y^*$;
4′. $\| x^* x \| = \| x^* \| \cdot \| x \|$.

*Then $R$ can be isomorphically mapped onto the ring of all complex-valued continuous functions $x(M)$ over a bicompact topological space $\mathfrak{M}$ in such a manner that, if $x \in R$ and $x(M)$ correspond to each other, then*

$$\| x \| = \max_{M \in \mathfrak{M}} | x(M) |$$

*and the function corresponding to $x^*$ is the conjugate function $\overline{x (M)}$*

Proof. We first prove that in $R$

$$\| x^2 \| = \| x \|^2.$$

By 2′—4′

$$\| x^{*2} \| \cdot \| x^2 \| = \| x^{*2} x^2 \| = \| (x^* x)^* (x^* x) \| = \| (x^* x)^* \| \cdot \| x^* x \| = \| x^* x \|^2 = \\ = \| x^* \|^2 \cdot \| x \|^2; \qquad (1)$$

on the other hand, by $(\gamma)$ § 1

$$\| x^{*2} \| \leqslant \| x^* \|^2, \quad \| x^2 \| \leqslant \| x \|^2$$

hence (1) can take place if and only if

$$\| x^{*2} \| = \| x^* \|^2, \quad \| x^2 \| = \| x \|^2.$$

In virtue of Theorem 8′ and Theorem 10 in [3] $R$ is therefore isomorphic to a subring $C_1$ of the set $C(\mathfrak{M})$ of all complex-valued continuous functions $x(M)$ over the set $\mathfrak{M}$ of all maximal ideals $M$ in $R$. Moreover

$$\| x \| = \max_{M \in \mathfrak{M}} \| x(M) \|$$

if $x$ and $x(M)$ correspond to each other.

As G. Šilov [4] has proved, $\mathfrak{M}$ contains a unique minimal closed set $\mathfrak{M}_0$ on which every function $x(M) \in C_1$ attains the maximum of its modulus. Let $M_0$ be an element of $\mathfrak{M}_0$; we prove that $M_0^* = M_0$, where $M^*$ denotes the set of all $x^*$, when $x$ ranges through $M$. We first notice that the mapping $TM = M^*$ is a homeomorphism of $\mathfrak{M}$ into itself. In fact, every neighbourhood $U(M_1)$ is given by inequalities of the form

$$| x_k(M) - x_k(M_1) | < \varepsilon, \quad k = 1, 2, \ldots, n, \quad \varepsilon > 0, \qquad (2)$$

so that $U(M)$ is defined by $n, x_1, \ldots, x_n$ and $\varepsilon > 0$. On the other hand, by the definition of $x(M)$

$$x = x(M) e + m, \quad m \in M,$$

hence

$$x^* = \overline{x(M) e} = m^*,$$

*i. e.*,

$$x^*(M^*) = \overline{x(M)}. \tag{3}$$

We can therefore write (2) in the form

$$|x_k^*(M^*) - x_k^*(M_1^*)| = |\overline{x_k(M)} - \overline{x_k(M_1)}| = |x_k(M) - x_k(M_1)| < \varepsilon$$

We see that $T$ maps $U(M_1)$ onto a neighbourhood $V(M_1^*)$ defined by $n$, $x_1^*, \dots, x_n^*$, $\varepsilon$. Hence $T$ is a homeomorphism.

Now suppose $M_0 \neq M_0^*$; there exist then two neighbourhoods $U(M_0)$, $U(M_0^*)$ without common element. By the continuity of $T$, there exists in $x(M_0)$ such a neighbourhood $V(M_0)$ that its $T$-map $(V(M_0))^*$ lies in $U(M_0^*)$. Thus $V(M_0)$ and $(V(M_0)^*$ have no element in common.

By the definition of $\mathfrak{M}_0$ there exists such a function $x_0(M) \in C_1$ that $|x_0(M)|$ does not attain its maximum on $\mathfrak{M}_0 - V(M_0)$. Hence it attains this maximum on $V(M_0)$. By (3) we have

$$|x^*(M)| = |\overline{x^*(M)}| = |x^{**}(M^*)| = |x(M^*)|. \tag{4}$$

Hence $|x_0^*(M)|$ does not attain its maximum on $T\mathfrak{M}_0 - (V(M_0))^*$, but it attains this maximum on $(V(M_0))^*$. On the other hand, it follows from (4) that $T\mathfrak{M}_0$ is also a minimal closed set, on which every function $x(M) \in C_1$ attains the maximum of its modulus. By the uniqueness of such a set we have $T\mathfrak{M}_0 = \mathfrak{M}_0$. Hence, by the construction of $x_0(M)$,

$$\max_{M \in \mathfrak{M}_0} |x_0^*(M) x_0(M)| = \max_{M \in \mathfrak{M}_0} (|x_0^*(M)| \cdot |x_0(M)|) < \max_{M \in \mathfrak{M}_0} |x_0^*(M)| \cdot \max_{M \in \mathfrak{M}} |x_0(M)|,$$

*i. e.*,

$$\|x_0^* x_0\| < \|x^*\| \cdot \|x_0\|.$$

This inequality contradicts 4′, and thus $M^* = M$ for all $M \in \mathfrak{M}_0$.

Now consider all $x(M) \in C_1$ for $M \in \mathfrak{M}_0$ only. By the definition of $\mathfrak{M}_0$ this contraction of the domain of $x(M)$ does not change $\max |x(M)|$, so that after this contraction $R$ and $C_1$ remain isomorphic and $\|x\| = \max_{M \in \mathfrak{M}_0} |x(M)|$. Moreover by (3)

$$x^*(M) = \overline{x(M^*)} = \overline{x(M)} \qquad \text{for } M \in \mathfrak{M}_0,$$

so that $x(M) \in C_1$ implies $\overline{x(M)} \in C_1$. By Theorem 6 in [3] $C_1$ is the set of all continuous functions over $C_1$; on the other hand, $C_1$ and $R$ are continuously isomorphic, hence $\mathfrak{M}_0 = \mathfrak{M}$, $C_1 = C(\mathfrak{M})$, q. e. d.

The lemma just proved implies 5′, 6′ § 1. In fact,

$$\|x^*\| = \max_{M \in \mathfrak{M}} |\overline{x(M)}| = \max_{M \in \mathfrak{M}} |x(M)| = \|x\|.$$

Further $\frac{1}{|x(M)|^2 + 1}$ is a continuous function over $\mathfrak{M}$, consequently $(x^*x + e)^{-2}$ exists.

2. We shall now deduce some corollaries from Lemma 1.

Let $R$ be an arbitrary normed $*$-ring; an element $h \in R$ will be called Hermitian, if $h^* = h$. Every element $x \in R$ can be uniquely represented in the form $x = h_1 + ih_2$, where $h_1, h_2 \in R$ are Hermitian. In fact, if $x = h_1 + ih_2$ is any such representation, then $x^* = h_1 - ih_2$, whence

$$h_1 = \frac{x + x^*}{2}, \quad h_2 = \frac{x - x^*}{2i}. \tag{5}$$

Such representation thus is unique. Conversely $h_1$, $h_2$ in (5) are Hermitian and $x = h_1 + ih_2$.

Now let $h \in R$ be an Hermitian element and $R_1$ the minimal closed subring of $R$ containing $h$. This ring $R_1$ is evidently commutative, hence it satisfies the conditions of Lemma 1. Thus $R_1$ is isomorphic to the set $C(\mathfrak{M})$. Denote by $h(M)$ the function corresponding to $h$; then $h = h^*$ means that $h(M)$ is real. Hence for any non-real $\lambda$

$$\frac{1}{h(M) - \lambda}$$

is a function belonging to $C(M)$, *i. e.* the two-sided inverse $(h - \lambda e)^{-1}$ exists. If we call the spectrum of $x$ the set of all $\lambda$'s, for which the two-sided $(x - \lambda e)^{-1}$ does not exist, we have the following result:

Corollary 1. *The spectrum of every Hermitian element is real.*

Further we have

$$\left| \frac{1}{1 + ih(M)} \right| \leqslant 1,$$

whence

$$\max_{M \in \mathfrak{M}} \left| \frac{1}{1 + ih(M)} \right| \leqslant 1,$$

*i. e.*

$$| (e + ih)^{-1} | \leqslant 1. \tag{5}$$

An Hermitian element $h$ will be called positive if its spectrum consists of non-negative numbers*. Evidentely in this case $h(M) \geqslant 0$, hence $\sqrt{h(M)}$ belongs to $C(\mathfrak{M})$ and is also $\geqslant 0$. Denote by $g$ the element of $R_1$ corresponding to $\sqrt{h(M)}$; then $g$ is positive and $g^2 = h$. Thus, we have

Corollary 2. *Every positive element $h$ can be represented in the form $h = g^2$, where $g$ is also positive.*

We denote $g$ by $\sqrt{h}$ or by $h^{1/2}$.

Corollary 3. *If $h_1$ is positive and*** $h_1^{-1}$ *exists, then* $(h_1 + ih_2)^{-1}$ *exists also for any Hermitian* $h_2$.

In fact,

$$h_1 + ih_2 = h_1^{1/2} (e + ih_1^{-1/2} h_2 h_1^{-1/2}) h_1^{1/2};$$

but, $h_1^{-1/2} h_2 h_1^{-1/2}$ being Hermitian, $(e + ih_1^{-1/2} h_2 h_1^{-1/2})^{-1}$ exists, hence $(h_1 + ih_2)^{-1}$ exists also***.

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* The spectrum of $h$ will then be said to be positive.

** Here and below $x^{-1}$ always means the two-sided inverse element.

*** So far we have only used the axioms 1'—5' (p. 198).

If in 6′ § 1 we replace $x$ by $\frac{1}{\sqrt{\varepsilon}}x$ $(\varepsilon > 0)$ we obtain that $\left(\frac{1}{\varepsilon}x^*x+x\right)^{-1}$ exists; hence $(x^*x+\varepsilon e)^{-1}=\frac{1}{\varepsilon}\left(\frac{1}{\varepsilon}x^*x+e\right)^{-1}$ exists also, *i. e.* $x^*x$ *is positive for all* $x \in R$.

Corollary 4. *If $h_1$, $h_2$ are positive and $h_1^{-1}$ exists, then the spectrum of $h_1h_2$ is positive.*

Put $x=h_2^{1/2}h_1^{1/2}$; then

$$h_1^{1/2}h_2h_1^{1/2}=x^*x$$

is positive, *i. e.* the spectrum of $h_1^{1/2}h_2h_1^{1/2}$ is positive. On the other hand, the correspondence $y=h_1^{1/2}xh_1^{1/2}$ being a ring-automorphism, it does not change the spectrum. Hence, the spectrum of

$$h_1h_2=h_1^{1/2}\,(h_1^{1/2}h_2h_1^{1/2})\,h_1^{-1/2}$$

is also positive.

Corollary 5. *The sum of two positive elements is positive.*

Let $h_1$, $h_2$ be positive; we have to prove, that $h_1+h_2$ is also positive, *i. e.* that $(h_1+h_2+\varepsilon h)^{-1}$ exists for all $\varepsilon>0$. As $(h_1+\varepsilon e)^{-1}$ exists, we have

$$h_1+h_2+\varepsilon e=(h_1+\varepsilon e)\,[e+(h_1+\varepsilon e)^{-1}h_2].$$

By Corollary 4 the spectrum of $(h_1+\varepsilon e)^{-1}h_2$ is positive, hence $[e+(h_1+\varepsilon e)^{-1}h_2]^{-1}$ exists. Thus $(h_1+h_2+\varepsilon e)^{-1}$ exists also.

3. A set $R$ will be called algebraic normed ring, if all the axioms $(\alpha)$ — $(\delta)$ § 1 are satisfied for $R$ with the eventual exception of completeness of $R$. Hence an algebraic normed ring is a normed ring if and only if it is complete.

Lemma 2. *Let $R$ be a commutative ring satisfying the conditions of Lemma* 1, *and let $R_1$ be an algebraic normed ring containing no generalized nilpotent elements**. *If $R$ and $R_1$ are algebraically isomorphic, then $R_1$ is complete, and every algebraic isomorphism between $R$ and $R_1$ is continuous.*

Proof. Let $\tilde{R}_1$ be the minimal complete ring containing $R_1$ and $N$ a maximal ideal in $\tilde{R}_1$. This ideal determines a homomorphism of $\tilde{R}$ into the field $K$ of all complex numbers, hence a homomorphism of $R_1$ into $K$. Let now $R_1$ and $R$ be isomorphic. Then we have also a homomorphism of $R$ into $K$, which determines a maximal ideal $M$ in $R$. Thus to every maximal ideal $N$ in $\tilde{R}_1$ there corresponds a maximal ideal $M$ in $R$. Moreover if $x$ and $y$ are elements of $R$ and $R_1$ which correspond to each other by the given isomorphism $\varphi$ between $R$ and $R_1$ [*i. e.* $x=\varphi(y)$], then by the definition of $M$

$$x(M)=y(N). \tag{6}$$

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* An element $x \neq 0$ is called generalized nilpotent element if $\lim\limits_{n\to\infty}\sqrt[n]{\|x^n\|}=0$ (cf. [3] § 6).

This correspondence between maximal ideals in $\tilde{R}$ and $R$, that we denote by $M=\psi(N)$, is one-to-one. In fact, if

$$\psi(N_1)=\psi(N_2)=M,$$

then $N_1$, $N_2$ determine the same homomorphism of $R_1$ into $K$, hence by continuity also the same homomorphism of $\tilde{R}_1$ into $K$. But this can be only if $N_1=N_2$.

Now prove that $M=\psi(N)$ is continuous, so that, $\mathfrak{N}$ being bicompact, this correspondence is a homeomorphism.

Any neighbourhood $U(M_0)$ of a maximal ideal $M_0\in\mathfrak{M}$ can be represented as the set of all $M\in\mathfrak{M}$ satisfying the inequalities

$$|x_k(M)-x_k(M_0)|<\varepsilon, \quad k=1, 2, \ldots, n; \quad \varepsilon>0, \quad x_k\in R, \tag{7}$$

where $\varepsilon>0$, $n$, $x_k$, $k=1, \ldots, n$, are fixed for this $U(M_0)$. Let $M_0=\psi(N_0)$, $x_k=\varphi(y_k)$, $k=1, 2, \ldots, n$; denote by $V(N_0)$ the neighbourhood of $N_0$ determined by

$$|y_k(N)-y_k(N_0)|<\varepsilon, \quad k=1, 2, \ldots, n. \tag{8}$$

But in virtue of (6) for $M=\psi(N_0)$ we have

$$y_k(N)=x_k(M), \quad y_k(N)=x_k(M_0),$$

so that (8) implies (7), *i. e.* $M=\psi(N)\in U(M_0)$, as soon as $N\in V(N_0)$. This means that $\psi$ is continuous.

Denote by $\mathfrak{M}_1$ the $\psi$-map of $\mathfrak{N}$; as a topological map of the bicompact $\mathfrak{N}$ it is also bicompact and therefore closed in $\mathfrak{M}$. Suppose $\mathfrak{M}_1\neq\mathfrak{M}$; there exists then a function $x(M)\in C(\mathfrak{M})$, satisfying the conditions

$$x(M)\not\equiv 0, \quad x(M)=0 \quad \text{for all } M\in\mathfrak{M}_1. \tag{9}$$

Let $x$ be the corresponding element of $R$; then (9) can be written in the form

$$x\neq 0, \quad x\in M \quad \text{for all } M\in\mathfrak{M}_1. \tag{10}$$

Denote by $y$ the corresponding element in $R_1$, so that $x=\varphi(y)$. Then (10) imply

$$y\neq 0, \quad y\in N \quad \text{for all } y\in\mathfrak{N}.$$

By Theorem 8 in [3] this means that $y$ is a generalized nilpotent element in $R_1$.

The contradiction so obtained shows that $\mathfrak{M}_1=\mathfrak{M}$, so that $\psi$ is a homeomorphism of $\mathfrak{N}$ into $\mathfrak{M}$. Hence using (6) we can consider $R$ as the ring $C(\mathfrak{N})$ of all continuous functions over $\mathfrak{N}$; by (6) we then have $R=R_1$. On the other hand, $R\supseteq\tilde{R}_1\supseteq R_1$, hence $R_1=\tilde{R}_2$. As $R_1$ is complete, by Theorem 17 in [3] every isomorphism between $R$ and $R_1$ is continuous.

Corollary 6. *Let $R$, $R_1$ be two normed $*$-rings* and let $y=\varphi(x)$ be an isomorphic mapping of $R$ into $R_1$ satisfying the condition $y^*=\varphi(x^*)$. Then $\|y\|=\|x\|$, so that the $\varphi$-map of $R$ is closed in $R_1$.*

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* In this corollary condition 6' will not be used.

Proof. Let $h$ be an Hermitian element in $R$ and $g=\varphi(h)$ the corresponding element in $R_1$. Let further $R'$ be the minimal closed subring of $R$ containing $h$, and $R'_1=\varphi(R')$—the $\varphi$-map of $R'$. Evidentely $R', R'_1$ are commutative and $R'$ satisfies all the conditions of Lemma 1. Using the same argument as on p. 199 we get: $\|y^2\|=\|y\|^2$ for all $y\in R'_1$. Thus $R'_1$ contains no generalized nilpotent elements. By Lemma 2 $R'_1$ is complete and $\varphi$ is a continuous isomorphism between $R'$ and $R'_1$. If $\mathfrak{M}$, $\mathfrak{N}$ are the sets of all maximal ideals in $R'$, $R'_1$ respectively, then for $y=\varphi(x)$, $x\in R'$, $y\in R'_1$

$$\|y\|=\max_{N\in\mathfrak{N}}|y(N)|=\max_{M\in\mathfrak{M}}|x(M)|=\|x\|,$$

in particular,

$$\|h\|=\|g\|=\|\varphi(h)\|. \tag{11}$$

We see that (11) holds for any Hermitian $h$. Now let $x$ be any element of $R$ and $y=\varphi(x)$ the corresponding element of $R_1$. Then $x^*x$ is Hermitian and $y^*y=\varphi(x^*x)$. Hence by (11)

$$\|x^*x\|=\|y^*y\|,$$

*i. e.*, by 4′, 5′ § 1

$$\|x\|^2=\|y\|^2, \quad \|x\|=\|y\|.$$

### 3. Proof of Theorem 1

Let us now proceed to the proof of Theorem 1. First we construct the Hilbert space. Let $M$ be a maximal left ideal in $R$; we divide $R$ into equivalence-classes modulo $M$, the totality of which will be denoted by $\mathfrak{H}'$. These equivalence-classes will be later considered as elements of a Hilbert space. If $a$ is an element of $R$, $\xi$—an equivalence-class from $\mathfrak{H}'$, then $x_1, x_2\in\xi$ imply $x_1-x_2\in M$, hence $a(x_1-x_2)\in M$, *i. e.* $ax_1$, $ax_2$ belong to the same equivalence-class modulo $M$. We denote this equivalence-class by $A\xi$ and write $A=\varphi(a)$. It is clear that $A$ is a linear transformation in $\mathfrak{H}'$ and that the correspondence $\varphi$ between $a$ and $A$ so obtained is a homomorphism.

We shall now define a scalar product in $\mathfrak{H}'$. To this purpose we first construct in $R$ a functional $f(x)$ satisfying the following conditions:

1* $f(\lambda x)=\lambda f(x)$;

2* $f(x+y)=f(x)+f(y)$;

3* $f(x^*)=\overline{f(x)}$;

4* $f(x)\geqslant 0$ if $x$ is Hermitian and positive;

5* $f(x)=1$;

6* $f(x)=0$ for $x\in M$.

We denote by $H$ the set of all Hermitian elements in $R$ and by $P$—the set of all elements $h$ of $H$ representable in the form $h=m_1+m^*_2$ $m_1$, $m_2\in M$, *i. e.* of those belonging to the linear sum $M+M^*$. All elements of $P$ can also be written in the form $h=m+m^*$, $m\in M$.

In fact, $h = m_1 + m_2^*$; $m_1, m_2 \in M$ implies $h = (m_1 + m_2)^* = m_2 + m_1^*$, hence

$$h = \frac{m_1 + m_2}{2} + \left(\frac{m_1 + m_2}{2}\right)^* = m + m^*,$$

where $m = \frac{m_1 + m_3}{2}$.

For no positive element $h \in P$ can $h^{-1}$ exist. In fact, if

$$h = m + m^*$$

is a positive element of $P$ and $h^{-1}$ exists, put $m = h_1 + ih_2$; $h_1, h_2 \in H$. Then $h_1 = \frac{1}{2} h$ hence $h_1$ is positive and $h_1^{-1}$ exists. By corollary 3 this implies that $m^{-1}$ exists also, which is impossible.

The same is true for the closure $\overline{P}$ of $P$. If $h \in \overline{P}$ is a positive element and $h^{-1}$ exists, we choose an element $g$ in $P$ in such a way that

$$\|g - h\| < \|h^{-1}\|^{-1}$$

and write

$$g = h + (g - h) = h^{-1/2}\,[e + h^{-1/2}(g - h)\,h^{-1/2}]\,h^{1/2}.$$

As

$$\|h^{-1/2}(g - h)\,h^{1/2}\| \leqslant \|h^{-1/2}\|^2 \cdot \|g - h\| = \|h^{-1}\| \cdot \|g - h\| < 1,$$

the element

$$e + h^{-1/2}(g - h)\,h^{-1/2}$$

is positive and $[e + h^{-1/2}(g - h)\,h^{-1/2}]^{-1}$ exists. Put

$$e + h^{-1/2}(g - h)\,h^{-1/2} = h_1^2,$$

where $h_1$ is Hermitian and positive. We obtain that

$$g \in P, \quad g = h^{1/2} h_1^2 h^{1/2} = (h_1 h^{1/2})^* (h_1 h^{1/2})$$

is positive and

$$g^{-1} = h^{-1/2} h_1^{-2} h_1^{-1/2}$$

exists. The obtained contradiction proves our statement also for $\overline{P}$.

From this fact we easily deduce that $\overline{P}$ lies at the distance 1 from $e$. Otherwise we would have

$$h = e + x, \quad h \in \overline{P}, \quad x \in H, \quad \|x\| < 1,$$

whence follows that $h$ is positive and $h^{-1}$ exists, which is impossible. It follows by a well known theorem of Banach ([1], p. 57, Lemma) that there exists a linear functional defined on the set

$$E = \{\lambda e + x,\ x \in \overline{P},\ \lambda \text{ real}\}$$

such that

$$f(e) = x, \ f(x) = 0 \quad \text{for } x \in \overline{P}.$$

We prove that it satisfies also 4*. Evidently,

$$f(\lambda e + x) = \lambda \quad \text{for } x \in \overline{P}.$$

Let us prove that $\lambda \geqslant 0$ if $\lambda e + x$ is positive. If $\lambda$ were negative, $\lambda = -\varepsilon$, $\varepsilon > 0$ for a positive $h = \lambda e + x$, $x \in P$, then

$$x = h - \lambda e = h + \varepsilon e$$

should be a positive element and $x^{-1}$ should exist. This contradiction proves our statement. Now consider the set $\mathfrak{K}$ of all positive elements of $H$. It possesses the following properties:

$1^{**}$ $x \in \mathfrak{K}$, $\lambda \geqslant 0$ imply $\lambda x \in \mathfrak{K}$;
$2^{**}$ $x \in \mathfrak{K}$, $x \neq 0$ imply $-\overline{x} \in \mathfrak{K}$;
$3^{**}$ $x, y \in \mathfrak{K}$ implies $x + y \in \mathfrak{K}$ (Corollary 5).

According to the terminologie of M. Krein [5] $\mathfrak{K}$ is a cone. It contains the sphere $\|x - e\| < 1$, hence contains inner points. By a theorem due to M. Krein [5] $f(x)$ can be extended over the whole $H$ in such a manner that it remains linear and non-negative on $\mathfrak{K}$.

For an arbitrary element $x = h_1 + ih_2$ of $R$ we put

$$f(x) = f(h_1) + if(h_2).$$

Evidently $f(x)$ satisfies $1^* - 6^*$.

Now, for any $\xi, \eta \in \mathfrak{H}'$ we put

$$(\xi, \eta) = f(y^* x),$$

where $x \in \xi$, $y \in \eta$. The value of $(\xi, \eta)$ does not depend on the choice of the elements $x \in \xi$, $y \in \eta$. In fact, if, *e. g.*, $x_1$ is any other element of $\xi$, then $x_1 - x \in M$, hence, $y^* x_1 - y^* x \in M$ and by $6^*$

$$f(y^* x_1 - y^* x) = 0, \quad f(y^* x_1) = f(y^* x).$$

We now prove that $(\xi, \eta)$ possesses all the properties of the scalar product. By $1^* - 3^*$

$$(\eta, \xi) = f(x^* y) = f((y^* x)^*) = \overline{f(y^* x)} = \overline{(\xi, \eta)}; \quad x \in \xi, \quad y \in \eta;$$
$$(\xi_1 + \xi_2, \eta) = f(y^*(x_1 + x_2)) = f(y^* x_1) + f(y^* x_2) = (\xi_1, \eta) + (\xi_2, \eta);$$
$$x_1 \in \xi_1, \quad x_2 \in \xi_2, \quad y \in \eta;$$
$$(\lambda \xi, \eta) = f(y^* \lambda x) = \lambda f(y^* x) = \lambda (\xi, \eta); \quad x \in \xi, \quad y \in \eta.$$

Further, $x^* x^*$ being positive, we have by $4^*$

$$(\xi, \xi) = f(x^* x) \geqslant 0; \quad x \in \xi.$$

Thus it remains to prove that $(\xi, \xi) = 0$ implies $\xi = M$ ($M$ plays here the rôle of the null-element). From the properties of $(\xi, \eta)$ already established follows Schwarz inequality

$$|(\xi, \eta)|^2 \leqslant (\xi, \xi)(\eta, \eta).$$

Consequently $(\xi_0, \xi_0) = 0$ implies $(\xi, \eta) = 0$ for any $\eta \in \mathfrak{H}'$, *i. e.*, $f(y^* x_0) = 0$ for any $y \in R$. $y$ being arbitrary, we may replace $y$ by $y^*$ that gives

$$f(yx^*) = 0 \quad \text{for all } y \in R, \ x_0 \in \xi_0. \tag{1}$$

Denote by $I$ the set of all elements $x \in R$ satisfying the equality

$$g(yx) = 0 \quad \text{for all } y \in R.$$

By (1)

$$\xi_0 \subseteq I. \tag{2}$$

We shall prove that $I$ is a left ideal containing $M$. In fact,

$$x_1,\ x_2 \in I; \quad p, q \in R,$$

imply

$$f(y(px_1+qx_2)) = f((yp)\,x_1) + f((yq)\,x_2) = 0$$

for all $y \in R$, *i. e.* $px_1+qx_2 \in I$. Further, $I \neq R$, because for $y=e$, $x=e$

$$f(yx) = f(x) = 1.$$

Hence $x \in I$ and we see that $I$ is a left ideal. If $x \in M$, then $yx \in M$ for all $y \in R$, and by 6*

$$f(yx) = 0,$$

*i. e.* $x \in I$. Thus $M \subseteq I$. $M$ being maximal, this implies $M = I$, whence by (2) $\xi_0 = M$. Thus $(\xi, \eta)$ possesses all the properties of the scalar product.

If now $\mathfrak{H}'$ is not complete with respect to $(\xi, \eta)$ we extend it to the minimal complete space $\mathfrak{H}$ containing $\mathfrak{H}'$. Clearly $\mathfrak{H}$ is a Hilbert space. We have seen that to every $a \in R$ there corresponds a linear operator $A = \varphi(a)$ in $\mathfrak{H}'$. This operator is bounded with respect to the norm $|\xi| = (\xi, \xi)^{1/2}$. In order to prove this we first note that

$$\left\| \frac{a^*a}{\|a\|^2+\varepsilon} \right\| < 1,$$

whence

$$[(\|a\|^2+\varepsilon)\,e - a^*a]^{-1} = \frac{1}{\|a\|^2+\varepsilon}\left[e - \frac{a^*a}{\|a\|^2+\varepsilon}\right]^{-}$$

exists for any $\varepsilon > 0$. This means that $\|a\|^2 e - a^*a$ is positive. We put $\|a\|^2 e - a^*a = g^2$, where $g$ is positive Hermitian. Then for any $x \in R$

$$x^*(\|a\|^2 e - a^*a)\,x = x^*g^2x = (gx)^*(gx)$$

is positive and by 4*

$$f(x^*(\|a\|^2 e - a^*a)\,x) \geqslant 0,$$

*i. e.*

$$f(x^*a^*ax) \leqslant \|a\|^2 f(x^*x).$$

If $x \in \xi$, the last inequality can be written in the form

$$(A\xi, A\xi) \leqslant \|a\|^2 (\xi, \xi),$$

*i. e.* $A$ is bounded in $\mathfrak{H}'$. It can therefore be uniquely extended to a bounded operator in $\mathfrak{H}$. This operator we also denote by $A$ and write also $A = \varphi(a)$. Evidently $\|A\| \leqslant \|a\|$, where $\|A\|$ is the norm of the operator, and $\varphi$ remains to be a homomorphism. If $B = \varphi(a^*)$, then for $\xi, \eta \in \mathfrak{H}'$, $x \in \xi$, $y \in \eta$

$$(A\xi, \eta) = f(y^*ax) = f((a^*y)^*x) = (\xi, B\eta).$$

Thus we have

$$(A\xi, \eta) = (\xi, B\eta). \tag{3}$$

In virtue of the continuity of the scalar product and by the definition of $A$, $B$ in $\mathfrak{H}$ (3) is also valid for any $\xi, \eta \in \mathfrak{H}$, *i. e.*

$$B = A^*, \quad \varphi(a^*) = [\varphi(a)]^*.$$

If now $\varphi$ is an isomorphism, then all the conditions of Corollary 6 are satisfied; we have only to take for $R_1$ the set of all bounded operators in $\mathfrak{H}$. By this corollary $\|a\| = \|A\|$ and our theorem is proved. If $R$ is simple, then $\varphi$ is an isomorphism. In fact, let $I$ be the set of all $x \in R$ such that $\varphi(x) = 0$. This set $I$ is a two-sided ideal in $R$, because $x, y \in I$; $a, b \in R$ imply

$$\varphi(ax + by) = \varphi(a)\varphi(x) + \varphi(b)\varphi(y) = 0,$$
$$\varphi(xa + yb) = \varphi(x)\varphi(a) + \varphi(y)\varphi(b) = 0,$$

and $\varphi(e) = 1$, where 1 is the operator defined by $1 \cdot \xi = \xi$. Thus, $R$ being simple, $I = (0)$, *i. e.* $\varphi$ is an isomorphism and our theorem is proved in the case of a simple ring $R$.

Let now $R$ be any $*$-ring. For every maximal ideal we construct as above the corresponding Hilbert space and denote it now by $\mathfrak{H}_M$. Then we take the direct sum

$$\mathfrak{H} = \sum_M \mathfrak{H}_M$$

of all these spaces, *i. e.*, the set of all complexes $\xi = \{\xi_M\}$, $\xi_M \in \mathfrak{H}_M$, with $\sum_M |\xi_M|^2 < +\infty$* where the operations are defined by

$$\lambda\xi = \{\lambda\xi_M\}, \qquad \xi + \eta = \{\xi_M + \eta_M\}, \qquad (\xi, \eta) = \sum_M (\xi_M, \eta_M)$$

for

$$\xi = \{\xi_M\}, \qquad \eta = \{\eta_M\}.$$

To every element $a \in R$ there corresponds a bounded operator $A_M$; in $\mathfrak{H}_M$. Write $A_M = \varphi_M(a)$. As we have shown

$$\|A_M\| \leqslant \|a\|. \tag{4}$$

Put

$$A\{\xi_M\} = \{A_M\xi_M\},$$

where $A_M = \varphi_M(a)$. $A$ is obviously a linear operator. In virtue of (4)

$$|A\{\xi_M\}|^2 = \sum_M |A_M\xi_M|^2 \leqslant \|a\|^2 \sum_M |\xi_M|^2 = \|a\|^2 |\{\xi_M\}|^2,$$

*i. e.* $A$ is bounded. We write $A = \varphi(a)$; evidently $\varphi(a^*) = [\varphi(a)]^*$.

Suppose that $\varphi$ is an isomorphism. Then by Corollary 6 $\|a\| = \|\varphi(a)\| = \|A\|$ and our theorem is proved. It remains to show that $\varphi$ is really an isomorphism.

Consider the set $I$ of all $a \in R$ such that $\varphi(a) = 0$. It is sufficient to show that $I = (0)$. $a \in I$ is equivalent to $\varphi(a) = 0$ or $A_M = \varphi_M(a) = 0$

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* Whence evidently follows that in every such complex only enumerably many $\xi$'s do not vanish.

for all $M$. This means further: $A_M \xi_M = M$ for all $\xi_M \in \mathfrak{H}_M$ and all $M$, *i. e.* $ax \in M$ for all $x \in R$ and all $M$. In particular, for $x = e$ we get: $a \in M$ for all $M$, *i. e.* $a \in \Pi M$, where $\Pi M$ is the intersection of all $M$. But $\Pi M = 0$. In fact as an intersection of left ideals it is also a left ideal, hence $a \in \Pi M$ implies $a^* a \in \Pi M$. If $\lambda \neq 0$ belongs to the spectrum of $a^* a$, then $a^* a - \lambda e$ belongs to some of the $M$, say $M_0$. On the other hand, $a^* a$, as an element of $\Pi M$, belongs to $M_0$ too; hence $\lambda e \in M_0$, which is impossible. Thus we see that the spectrum of $a^* a$ consists only of $\lambda = 0$, and we obtain $a^* a = 0$, $a = 0$ that completes the proof of the theorem.

It follows from the theorem just proved that all rings discussed by S. W. P. Steen [10] can be considered as rings of operators in Hilbert space. Consequently, the results of Steen follow immediately from the corresponding results of F. J. Murray and J. v. Neumann ([6], [7], [9]).

The residue ring of an $*$-ring is an $*$-ring itself. Hence follows that if $R$ is a closed $*$-subring of the ring of operators in Hilbert space, then any its residue ring can be imbedded into the ring of operators in Hilbert space.

## § 4. Weakly closed rings

We have proved that every $*$-ring can be considered as a subring of the ring $B$ of all bounded operators in a Hilbert space $\mathfrak{H}$. Let us now consider the case, when this subring is weakly closed (cf. [7]). If $\mathfrak{S}$ is a subset of $B$, denote by $\mathfrak{S}^p$ the set of all projections from $\mathfrak{S}$. We then have:

Lemma 3. *If $I$ is a left ideal in $R$, then $I^p = (0)$ implies $I = (0)$.*

Proof. Let $A \in I$, $A \neq 0$. Since $R$ is weakly closed, $A$ can be represented in the form

$$A = UH,$$

where $H$ is positive Hermitian, $U$—partially isometric and $U, H \in R$ (cf. [8], Theorem 7 and [6] I, Lemma 4.4.2). Moreover $U^* U H = H$, whence

$$H = U^* A \in I.$$

Since $H = 0$ implies $A = 0$, we must have $H \neq 0$. Let $E(\lambda)$, $-\infty < \lambda < +\infty$, be the resolution of the identity corresponding to $H$; $H$ being positive, $E(\lambda) = 0$ for $\lambda < 0$. Moreover the equality $E(\varepsilon) = 1$ does not hold for all $\varepsilon > 0$, because otherwise we would have $H = 0$. Thus for some $\varepsilon > 0$

$$E(\varepsilon) \neq 1. \tag{1}$$

As $1 - E(\varepsilon) \in R$, we have also $H_\varepsilon = [1 - E(\varepsilon)] H \in I$. The operator $B = [1 - E(\varepsilon)] H + E(\varepsilon)$ is an element of $R$ and possesses a bounded inverse. Hence

$$B^{-1} H_\varepsilon \in I. \tag{2}$$

On the other hand, it is obvious that $B^{-1}=H_\varepsilon^{-1}$ on $[1-E(\varepsilon)]\mathfrak{H}$ and $B^{-1}=1$ on $E(\varepsilon)\mathfrak{H}$. Hence $B^{-1}H_\varepsilon=1$ on $[1-E(\varepsilon)]\mathfrak{H}$ and $B^{-2}H_\varepsilon=0$ on $E(\varepsilon)\mathfrak{H}$, *i. e.* $B^{-1}H_\varepsilon=1-E(\varepsilon)$, so that by (1) and (2) $1-E(\varepsilon)\in I$, $1-E(\varepsilon)\neq 0$ contrary to the hypothesis: $I^p=(0)$.

Lemma 4. *Let $R$ be a factor*, $I$ a two-sided ideal in $R$ and $P$ an element of $I^P$. Then every projection $Q$ equivalent to $P$* (mod $R$) *is also an element of $I^P$.*

Proof. By the definition of equivalence modulo $R$ a partially isometric operator $U\in R$ exists, satisfying

$$U^*U=P, \qquad UU^*=Q.$$

Hence, $I$ being a two-sided ideal,

$$Q=Q^2=UU^*UU^*=UPU^*\in I.$$

Corollary 7. *If $R$ is a factor in the separable Hilbert space and $I$ is a two-sided ideal in $R$, then $I$ does not contain infinite projections.*

Proof. Let $P$ be an infinite projection from $I$; then $P$ is equivalent to 1 (mod $R$) (cf. [6] I, Lemma 7. 2. 1), and by Lemma 4, $1\in I$ which is impossible.

Corollary 8. *Every factor of class* III *in the separable Hilbert space is a simple ring.*

Proof. Let $I$ be a two-sided ideal in $R$. Since all projections in $R$ different from zero are all infinite, Corollary 7 gives $I^P=(0)$, whence by Lemma 3, $I=(0)$.

Corollary 9. *Every factor $R$ of finite class is simple.*

Proof. Let $D(P)$ be the relative dimension in $R$ (cf. [6] I, Theorem VII) and $I\neq(0)$ a two-sided ideal in $R$. Then there exists an element $P_0\in I^p$, $P_0\neq 0$, and $D(P_0)>0$. Choose an integer $n$ such that $\frac{1}{n}\leqslant D(P_0)$. In virtue of the well known properties of $D(P)$ a projection** $Q\leqslant P_0$ exists satisfying the equality $D(Q)=\frac{1}{n}$. Moreover

$$Q=QP_0\in I.$$

On the other hand, there exist $n$ mutually orthogonal projections $Q_1, \dots, Q_n$ equivalent to $Q$ such that $Q_1+Q_2+\dots+Q_n=1$. By Lemma 4 all $Q_j$ as well as their sum 1 belong to $I$ that contradicts to the definition of the ideal.

Lemma 5. *Every simple weakly closed ring is a factor.*

Proof. Denote by $R'$ the set of all elements of $B$ that commute with every element of $R$. If $Z$ is the centrum of $R$, we have:

$$Z=R\cap R',$$

whence $Z$ is also a weakly closed ring.

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* A weakly closed ring is called a factor (cf. [6] I, p. 138). if its centrum consists only of elements $\lambda 1$.

** $Q\leqslant P$ means $Q\mathfrak{H}\subseteqq P\mathfrak{H}$, *i. e.* $QP=PQ=Q$.

Let $P$ be an element of $Z^P$. Then the set

$$I = \{AP,\ A \in R\}$$

is a two-sided ideal, provided that it does not coincide with $R$. In fact, if $A_1, A_2, B_1, B_2 \in R$ then

$$B_1(A_1P) + B_2(A_2P) = (B_1A_1 + B_2A_2)P \in I,$$
$$(A_1P)B_1 + (A_2P)B_2 = (A_1B_1 + A_2B_2)P \in I.$$

But, $R$ being simple, we have either $I = (0)$, or $I = R$. In the first case $AP = 0$ for all $A \in P$, hence $P = 1 \cdot P = 0$. In the second case all elements of $R$ have the form $AP$, in particular, $1 = AP$ for some $A$. Then $1 \cdot (1 - P) = AP(1 - P) = 0$, $P = 1$. We see that, $Z^P$ consists only of 0 and 1, consequently $Z = \{\lambda 1\}$ (cf. [7] Theorem 2).

Theorem 2. *Every factor $R$ of class $\mathrm{I}_\infty$ or $\mathrm{II}_\infty$ in the separable Hilbert space is not simple. It possesses only one* non-trivial *two-sided ideal closed in the uniform topology, which coincides with the smallest two-sided ideal $I_0$ containing all finite projections and closed in the uniform topology.*

Proof. Let $I$ be the set of all such $A \in R$ that $\overline{A\mathfrak{H}}$ is finite*; $I$ is a two-sided ideal. In fact**

$$\overline{(A+B)\mathfrak{H}} \subseteq \overline{\overline{A\mathfrak{H}} + \overline{B\mathfrak{H}}},$$

hence $A \in I$, $B \in I$ imply $A + B \in I$. We have further

$$\overline{AB\mathfrak{H}} \subseteq \overline{A\mathfrak{H}}.$$

hence $A \in I$, $B \in R$ imply $AB \in I$. The equality $D(\overline{A\mathfrak{H}}) = D(\overline{A^*\mathfrak{H}})$ (cf. [6] I, Lemma 6.2.1) shows that $A \in I$ implies $A^* \in I$. Consequently, if $B \in \mathfrak{B}$, $A \in I$, then

$$A^* \in I, \quad A^*B^* \in I, \quad BA = (A^*B^*)^* \in I.$$

Since, on the other hand, $1 \bar{\in} I$ the set $I$ is a two-sided ideal $\neq (0)$. We see that $R$ is not simple.

Let now $I_1$ be an arbitrary non-trivial two-sided ideal closed in the uniform topology. According to Lemma 3 $I_1^P$ contains elements different from zero; by Corollary 7 all these elements must be finite. Suppose that $P \in I_1^P$, $P \neq 0$ and $Q$ is any projection from $R$ satisfying the condition $D(Q) \leqslant D(P)$. Then***

$$Q \sim P' \leqslant P \quad (\mathrm{mod}\ R).$$

Since $P' = PP' \in I_1$, Lemma 4 gives: $Q \in I_1$. Now let $Q_0$ be any finite projection from $R$; then $Q_0$ can be represented in the form

$$Q_0 = P_1 + P_2 + \dots + P_n + Q,$$

where $P_i$, $Q$, $i = 1, 2, \dots, n$, are mutually orthogonal, $P_i \sim P$ (mod $R$) and $D(Q) < D(P)$.

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* $\overline{A\mathfrak{H}}$ denotes the closure of $A\mathfrak{H}$.

** $\mathfrak{M} + \mathfrak{N}$ denotes the set of all $\xi + \eta$, $\xi \in \mathfrak{M}$, $\eta \in \mathfrak{N}$.

*** $Q \sim P'$ (mod $R$) denotes that $Q$ and $P'$ are equivalent with respect to $R$.

By Lemma 4 $P_i \in I$; moreover, as we have just proved, $Q \in I$, hence $Q_0 \in I$. Thus, $I_1$ contains all finite projections and therefore

$$I_0 \subseteq I_1. \tag{3}$$

Further we notice that $A \in I_1$ implies $A^* \in I_1$. In fact, if $A \in I_1$, $A = UH$, where $H$ is positive Hermitian and $U$—partially isometric, then $H = U^* A \in I_1$, whence $A^* = HU^* \in I_1$. Therefore if we write

$$A = H_1 + iH_2, \quad H_1 = \frac{A + A^*}{2}, \quad H_2 = \frac{A - A^*}{2i},$$

we see that $A \in I_1$ implies $H_1, H_2 \in I_1$. Let now $E_1(\lambda)$ be the resolution of the identity corresponding to $H_1$. Repeating the argument used on p. 209—210 we obtain that $E_1(\Delta) \in I_1$ for any closed interval $\Delta$, which does not contain zero. By Corollary 7 every such $E_1(\Delta)$ is finite, hence $E_1(\Delta) \in I_0$. Since $H_1$ can be represented as the limit

$$H_1 = \lim \sum_{k=1}^{n} \lambda_k E_1(\Delta_k), \quad 0 \bar{\in} \Delta_k, \quad k = 1, \ldots, n$$

in the sense of the uniform topology, we have $H_1 \in I_0$. Analogously $H_2 \in I_1$, so that $A = H_1 + iH_2 \in I_0$. We have proved that $A \in I_1$ implies $A \in I_0$, *i. e.*, $I_1 \subseteq I_0$. Combining this result with (3) we finally get: $I_0 = I_1$, that completes the proof of the theorem.

Remark. Consider the quotient-ring $\frac{R}{I_0}$ of equivalence-classes modulo $I_0$. It is evidently simple and is also an, $*$-ring. By Theorem 1 it can therefore be considered as a ring of operators in a Hilbert space. If, in particular, $R = B$, then the finiteness of a projection $P$ means that $P\mathfrak{H}$ is finite-dimensional in the usual sense. Consequently $I_0$ coincides with the set of all completely-continuous operators, so that Theorems 1 and 2 contain, in particular, the results of J. W. Calkin [2].

Combining Corollaries 8, 9, Lemma 5 and Theorem 2 we obtain:

Theorem 3. *The only simple weakly closed rings in the separable Hilbert space are the factors of the classes* $\mathrm{I}_n$, $\mathrm{II}_1$ *and* III.

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# О включении нормированного кольца в кольцо операторов в гильбертовом пространстве

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(*Резюме*)

## § 1. Основные понятия

Совокупность $R$ элементов $x$, $y$, ... называется алгебраическим нормированным кольцом, если $R$—(вообще, неполное) линейное с комплексными коэффициентами, нормированное пространство в смысле Банаха [1], если в $R$ введена операция умножения, удовлетворяющая условиям *:

$$x(\lambda y + \mu z) = \lambda xy + \mu xz, \qquad x(yz) = (xy)z,$$

$$\| xy \| \leqslant \| x \| \| y \|,$$

и если в $R$ существует единица, т. е. элемент $e$ такой, что $ex = xe = x$ для всех $x \in R$. Алгебраическое нормированное кольцо $R$ будем называть нормированным кольцом (см. [3]), если $R$ полно.

Совокупность $I_e$ элементов $R$ называется левым идеалом, если $I_e \neq R$ и если из $x$, $y \in I_e$; $p$, $q \in R$ следует $px + qy \in I_e$; аналогично определяется правый идеал. Идеал одновременно левый и правый называется двусторонним. Кольцо, не имеющее двусторонних идеалов $\neq 0$, называется простым. Левый идеал называется максимальным, если он не является правильной частью другого левого идеала. Максимальный идеал всегда замкнут; всякий левый идеал содержится в некотором максимальном левом идеале. Элемент $x$ не имеет левого обратного тогда и только тогда, когда он содержится в некотором левом максимальном идеале. Аналогичные определения и предложения можно установить и для правых идеалов.

Алгебраическое нормированное кольцо $R$ называется алгебраическим $*$-кольцом, если в $R$ имеется операция, которая каждому $x \in R$ ставит в соответствие некоторый элемент $x^* \in R$ так, что

1′ $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$,

2′ $x^{**} = x$,

3′ $(xy)^* = y^* x^*$,

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* $\lambda$, $\mu$, ... всегда будут обозначать комплексные числа, а $\bar{\lambda}$, $\bar{\mu}$, ... комплексно-сопряженнные числа.

4′ $\|x^*x\| = \|x^*\|\,\|x\|$,
5′ $\|x^*\| = \|x\|$ *,
6′ $x^*x+e$ имеет двусторонний обратный для всех $x \in R$.

Из этих аксиом следует, что $e^*=e$. Если кольцо еще полно, то оно называется нормированным $*$-кольцом. Примером нормированного $*$-кольца является кольцо $B$ всех ограниченных операторов в гильбертовом пространстве $\mathfrak{H}$ **, причем в качестве нормы можно взять норму оператора, а в качестве $x^*$ — оператор, сопряженный к $x$. Основной целью этой статьи является доказательство следующей теоремы:

Теорема 1. *Всякое нормированное $*$-кольцо $R$ можно изоморфно отобразить на замкнутое подкольцо $R$ совокупности $B$ всех ограниченных операторов в гильбертовом пространстве $\mathfrak{H}$ и притом так, что норма элемента $x \in R$ равна в норме соответствующего оператора, а $x^*$ соответствует сопряженному оператору.*

## § 2. Некоторые леммы

Доказательство этой теоремы основано на следующих леммах:

Лемма 1. *Пусть $R$—коммутативное нормированное кольцо и пусть в $R$ задана операция $x \to x^*$, удовлетворяющая условиям 1′—4′ § 1. Тогда $R$ можно изоморфно отобразить на кольцо всех комплексных непрерывных функций $x(M)$ на некотором бикомпактном топологическом пространстве $\mathfrak{M}$ и притом так, что норма элемента $x \in R$ равна максимуму модуля соответствующей функции $x(M)$, а $x^*$ переходит в комплексно сопряженную функцию $\overline{x(M)}$.*

Будем называть спектром элемента $x$ совокупность тех значений $\lambda$, для которых не существует***

$$(x-\lambda e)^{-1}.$$

Элемент $h$ назовем эрмитовским, если $h^*=h$. Всякий элемент $x \in R$ можно, и притом единственным образом, представить в виде $x=h_1+ih_2$, где $h_1$, $h_2$ — эрмитовские элементы. Очевидно,

$$h_1=\frac{x+x^*}{2}, \qquad h_2=\frac{x-x^*}{2}.$$

Рассматривая минимальное нормированное кольцо, содержащее эрмитовский элемент $h$ и применяя лемму 1, получаем:

Следствие 1. Спектр эрмитовского элемента действителен.

Эрмитовский элемент назовем позитивным, если его спектр не содержит отрицательных чисел. Из 6′ § 1 следует, что $x^*x$ всегда позитивный элемент.

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* Авторы предполагают, что последние две аксиомы являются следствиями первых четырех; однако, им не удалось это доказать.

** Гильбертовы пространства, рассматриваемые в этой работе, не обязательно сепарабельны.

*** $x^{-1}$ всегда обозначает двусторонний обратный элемент.

Следствие 2. Всякий позитивный элемент $h$ можно представить в виде $h=g^2$, где $g$ — также позитивный элемент. Мы будем писать $g=h^{\frac{1}{2}}$.

Следствие 3. Если $h_1$—позитивный элемент и если существует $h_1^{-1}$, то существует также $(h_1+ih_2)^{-1}$ при любом эрмитовском $h_2$.

Это следствие получается из следствия 1 и равенства

$$h_1+ih_2=h_1^{\frac{1}{2}}(e+ih_1^{-\frac{1}{2}}h_2h_1^{-\frac{1}{2}})\,h^{\frac{1}{2}}.$$

Следствие 4. Если $h_1$, $h_2$ позитивны и если существует $h_1^{-1}$, то спектр $h_1h_2$ состоит из неотрицательных чисел.

Полагая $x=h_2^{\frac{1}{2}}h_1^{\frac{1}{2}}$, получаем $h_1h_2=h_1^{\frac{1}{2}}x^*xh_1^{\frac{1}{2}}$, откуда и следует утверждение.

Следствие 5. Сумма двух позитивых элементов позитивна.

Это утверждение следует из равенства

$$h_1+h_2+\varepsilon e=(h_1+\varepsilon e)\,[e+(h_1+\varepsilon e)^{-1}h_2]$$

и следствия 3.

Лемма 2. *Пусть $R$—коммутативное нормированное кольцо, удовлетворяющее всем условиям леммы* 1, *а $R_1$—алгебраическое нормированное кольцо, не имеющее обобщенных нильстепенных элементов* (см. [3] § 6). *Если тогда $R$ и $R_1$ алгебраически изоморфны, то $R_1$ полно и всякий изоморфизм между $R$ и $R_1$ непрерывен.*

Следствие 6. Пусть $R$, $R_1$—два нормированных $*$-кольца и пусть $y=\varphi(x)$—изоморфное отображение $R$ в $R_1$ такое, что $\varphi(x^*)=[\varphi(x)]^*$. Тогда также $\|\varphi(x)\|=\|x\|$, следовательно, $\varphi$-образ $R$ замкнут в $R_1$.

## § 3. Идея доказательства теоремы 1

Пусть $M$ левый максимальный идеал в $R$; разобьем $R$ на классы вычетов по модулю $M$ и обозначим через $\mathfrak{H}'$ полученное фактор-пространство. Умножению слева на $a\in R$ соответствует линейная операция $A$ над элементами $\mathfrak{H}'$. Будем писать $A=\varphi(a)$; очевидно $\varphi$ — гомоморфизм.

Обозначим через $H$ совокупность всех эрмитовских элементов $R$, через $P$ совокупность элементов $H$ вида $m_1+m_1^*$, $m_1$, $m_2\in M$, через $\overline{P}$ замыкание $P$, через $\mathfrak{K}$—совокупность всех позитивных элементов $H$.

Пользуясь следствием 3, выводим, что $\overline{P}$ находится на расстоянии 1 от $e$, следовательно, на совокупности $E=\{\lambda e+x,\ x\in\overline{P}\}$ ($\lambda$—веществ.) существует линейный функционал $f(\lambda e+x)=\lambda$. Он $\geqslant 0$ на элементах $\mathfrak{K}$. Так как, в силу следствия 5, $\mathfrak{K}$ образует конус (см. [5]), то $f(x)$ можно продолжить на все $\mathfrak{N}$ так, что он остается линейным и $\geqslant 0$ на $\mathfrak{K}$.

Положим для $x = h_1 + ih_2$; $h_1$, $h_2 \in \mathfrak{N}$; $f(x) = f(h_1) + if(h_2)$. Тогда

$$f(\lambda x) = \lambda f(x), \quad f(x+y) = f(x) + f(y), \quad f(x^*) = \overline{f(x)},$$

$f(x) \geqslant 0$, если $x \in \mathfrak{K}$, $f(e) = 1$, $f(x) = 0$ если $x \in M$.

Положим далее для $\xi$, $\eta \in \mathfrak{H}'$ и $x \in \xi$, $y \in \eta$

$$(\xi, \eta) = f(y^* x);$$

$(\xi, \eta)$ не зависит от выбора $x \in \xi$, $y \in \eta$ и обладает всеми свойствами скалярного произведения. Топологическое дополнение $\mathfrak{H}$ пространства $\mathfrak{H}'$ относительно $(\xi, \eta)$ есть гильбертовское пространство. Так как элемент $x^*(\| a \|^2 e - a^* a) x$ позитивен, то $f(x^* \| a \|^2 x) \geqslant f(x^* a^* a x)$, т. е.

$$f(x^* a^* a x) \leqslant \| a \|^2 f(x^* x).$$

Если $x \in \xi$, $A = \varphi(a)$, это неравенство перепишется в виде

$$(A\xi, A\xi) \leqslant \| a \|^2 (\xi, \xi),$$

т. е. $A$ — ограниченный оператор в $\mathfrak{H}'$, а значит, и в $\mathfrak{H}$. Очевидно, $\varphi(a^*) = [\varphi(a)]^*$; если поэтому $\varphi$ — изоморфизм, то, согласно следствию 6, $\| \varphi(a) \| = \| a \|$ и теорема доказана. Но $\varphi$ наверное изоморфизм, если $R$ — простое кольцо; значит, теорема доказана в случае простого кольца.

В случае произвольного кольца $R$ обозначаем прежние $\mathfrak{H}$ и $A = \varphi(a)$ через $\mathfrak{H}_M$ и $A_M = \varphi_M(a)$ и строим прямую сумму пространств

$$\mathfrak{H} = \sum_M \mathfrak{H}_M,$$

распространенную на все максимальные левые идеалы кольца $R$. Для $\xi = \{\xi_M\}$, $\xi \in \mathfrak{H}$, полагаем

$$A\xi = \{A_M \xi_M\}, \qquad A = \varphi(a).$$

Легко проверить, что $\varphi$ — изоморфизм и $\varphi(a^*) = [\varphi(a)]^*$, следовательно $\| a \| = \| \varphi(a) \|$ и теорема доказана.

Из доказанной теоремы следует, что все кольца, рассмотренные S. W. P. Steen'ом [10], можно рассматривать как кольца операторов в гильбертовом пространстве, следовательно, результаты Steen'а непосредственно следуют из соответствующих результатов F. J. Murray и J. v. Neumann'а ([6], [7], [9]).

## § 4. Слабо замкнутые кольца

Рассмотрим теперь слабо замкнутые (ср. [7]) подкольца $R$ кольца $B$. Будем при этом обозначать через $\mathfrak{S}^p$ ($\mathfrak{S} \subset B$) совокупность всех операторов проектирования из $\mathfrak{S}$.

Лемма 3. *Если $I$ — двусторонний идеал в $R$, то из $I^p = (0)$ следует $I = (0)$.*

Лемма 4. *Если $R$—фактор* (ср. [6] I, стр. 138), *$I$ — двусторонний идеал в $R$, $P \in I^p$, $Q \sim P$ (... $R$), то также $Q \in I^p$.*

Следствие 7. Если $R$ — фактор в сепарабельном гильбертовом пространстве, $I$ — двусторонний идеал в $R$, то $I$ не содержит бесконечных операторов проектирования.

Следствие 8. Всякий фактор класса III в сепарабельном гильбертовом пространстве есть простое кольцо.

Следствие 9. Всякий фактор конечного класса есть простое кольцо.

Лемма 5. *Всякое простое слабо замкнутое кольцо — фактор.*

Теорема 2. *Всякое кольцо классов* $\mathrm{I}_\infty$, $\mathrm{II}_\infty$ *в сепарабельном гильбертовом пространстве не просто. Оно имеет только один* $\neq (0)$, *замкнутый в смысле нормы оператора двусторонний идеал, который совпадает с минимальным таким идеалом, содержащим все конечные операторы проектирования кольца.*

Комбинируя эти результаты, получаем:

Теорема 3. *Единственные слабо замкнутые простые кольца в сепарабельном гильбертовом пространстве суть факторы классов* $\mathrm{I}_n$, $\mathrm{II}_1$ и III.

Рассмотрим фактор-кольцо $R/I_0$; оно, очевидно, простое кольцо и, кроме того, $*$-кольцо. Поэтому к нему применима теорема 1. Если в частности, $R = B$, то конечность $P$ означает конечномерность $P\mathfrak{H}$, следовательно, $I_0$ есть совокупность всех вполне непрерывных операторов. Таким образом, как частный случай получается результат J. W. Calkin'a [2] относительно $B/I_0$.

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