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ON FINDING THE EXACT VALUES OF THE CONSTANT IN A $(1, q_2)$-GENERALIZED TRIANGLE INEQUALITY FOR BOX-QUASIMETRICS ON 2-STEP CARNOT GROUPS WITH 1-DIMENSIONAL CENTER

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Abstract. For 2-step Carnot groups with 1-dimensional center, a method for defining the exact values of the constant q_2 in a $(1, q_2)$-generalized triangle inequality for their Box-quasimetrics is developed. The exact values of the constant q_2 are defined for 4-, 5-, and 6-dimensional 2-step Carnot groups with 3-dimensional horisontal subbundle.

Keywords: (q_1, q_2)-quasimetric spase, Carnot group, exact value, Box-quasimetric.

Introduction

We refer as a (q_1, q_2)-quasimetric space [1]–[9] to a pair (X, d), where X is some set, $d : X \times X \to \mathbb{R}^+ \cup 0$ is some function such that the *identity axiom*

$$d(x, y) = 0 \Leftrightarrow x = y$$

holds for it and (q_1, q_2) is a *generalized triangle inequality*, that is,

$$d(x, y) \leq q_1 d(x, z) + q_2 d(z, y) \quad \forall x, y, z \in X.$$

The expression $d(x, y)$ denotes a (q_1, q_2)-quasi-distance exactly *from the point x to the point y*. If $q_1 = q_2 = 1$, then (X, d) is a quasimetric space [10]. If for a

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(q_1, q_2)-quasimetric space (X, d) the following condition holds

$$d(x, y) \leq q_0 d(y, x) \quad \forall x, y \in X,$$

where the constant q_0 does not depend on the choice of the points x, y, then we refer to a (q_1, q_2)-quasimetric space (X, d) as a q_0-*symmetric* one; for the case when $q_0 = 1$, we use the notion of *symmetric* (q_1, q_2)-*quasimetric space*. An important special case of symmetric (q_1, q_2)-quasimetric spaces are the symmetric $(1, q_2)$-quasimetric spaces [1]; these include Carnot groups and more general equiregular Carnot–Carathéodory spaces $(M, \rho_{\mathrm{Box}_M})$, equiped by Box-quasimetrics ρ_{Box_M} [6]–[14]. Moreover, in the general case, the constant q_2 does not equal 1 [18]. Box-quasimetrics were introduced in work [15]. The $(1, q_2)$-generalized triangle inequality plays a crucial role in obtaining the «divergence» estimates of the equiregular Carnot–Carathéodory space $(M, \rho_{\mathrm{Box}_M})$ from its nilpotent tangential cone, see, for example, [16, 17].

For a (q_1, q_2)-quasimetric space (X, d) we denote by $R = R(d)$ the set of points $(q_1', q_2') \in \mathbb{R}^2$, such that for ρ, the (q_1', q_2')-generalized triangle inequality holds. Directly from the definition of the set R, follows the

Property 0.1 ([1, 2]). 1^0 *The set* $R = R(d)$ *is convex and closed, and, moreover,* $R \subseteq \{(x, y) \in \mathbb{R}^2 \mid x \geq 1,\, y \geq 1\}$; 2^0 *the condition* $(1, 1) \in R$ *is equivalent to the fact that* d *is a quasimetric*; 3^0 *if* (q_1, q_2)-*quasimetric is symmetric, then the set* R *is symmetric with respect to the bisector of the right upper coordinate angle of the Euclidean plane.*

If $(q_1', q_2') \in R$ and $\tilde{q}_i \geq q_i'$, $i = 1, 2$, then $(\tilde{q}_1, \tilde{q}_2) \in R$. Drawing supporting lines through the boundary points of the closed convex set R, we obtain that R has extreme points. (Recall that a point $x_0 \in A$ is called an *extreme point* of a set A, if there are no points $x_1, x_2 \in A$, such that $x_0 \in (x_1, x_2)$, that is, $x_0 = tx_1 + (1-t)x_2$ for some $0 < t < 1$.) It is easy to see that every extreme point of the set R is a Pareto optimal point of the set R (in the sense of minimisation of components). The point $(q_1^0, q_2^0) \in R$ is called *best*, if for every $(q_1', q_2') \in R$ we have that $q_i' \geq q_i^0$, $i = 1, 2$. From the definition of the best point, directly follows the next

Property 0.2 ([1, 2]). *If a best point exists, then it is unique*; *if such point exists, then* $R = R(d) = \{(x, y) \in \mathbb{R}^2 \mid x \geq a, \quad y \geq b\}$ *for some* $a, b \geq 1$.

See the examples of (q_1, q_2)-quasimetric spaces with the best points (q_1^0, q_2^0) such that $q_1^0 + q_2^0 > 2$ in [1], [4]–[6].

In work [18], the exact values of the constant q_2 for the $(1, q_2)$-generalized triangle inequality of Box-quasimetrics on the canonical Heisenberg groups $\mathbb{H}^n_\alpha$, $n \in \mathbb{N}$, and the canonical Engel group $\mathbb{E}_{\alpha,\beta}$ were obtained. In this work, we consider a problem of finding the exact values of the constant q_2 for Box-quasimetrics $\rho_{\mathrm{Box}_{\mathbb{D}_n}}$, where $\mathbb{D}_n$ is a canonical 2-step Carnot group with a one-dimensional centre, whose topological dimension equals $n+1$. Here, the term «exact value» implies such value of the constant q_2 that for every number q_2', $q_2' < q_2$, the $(1, q_2')$-generalized triangle inequality does not hold.

We refer as a *canonical finite-dimensional Lie group* [19] to an analytical Lie group K, whose exponential mapping is the identity. Therefore, K matches to some Euclidean space $\mathbb{R}^N$ with the coordinate system $(x_1, \ldots, x_N)$, induced by the coordinate frame $(O, e_1, \ldots, e_N)$. Hence, we can match any element $u \in K$ with its coordinate notation; in particular, the unit element of the group K is the point $O = (0,\dots,0)$ (the origin of the Euclidean space $\mathbb{R}^N$), and for every $u = (x_1,\dots,x_N)$, we have $u^{-1} = (-x_1,\dots,-x_N)$. A group operation «·» on K (in other words, the left translation $L_u^K u' = u\cdot u'$ of the element $u' \in K$ by the element $u \in K$) is defined with the help of the Campbell–Hausdorff formula [20] and the corresponding commutator table given on the orthogonal basis $\{e_i\}_{i=1,\dots,N}$ of the Euclidean space $\mathbb{R}^N$.

A Lie algebra is called graduated [21] if it decomposes into a direct sum of vector subspaces $V = \bigoplus_{i=1}^{r} V_i$, and, moreover, $[V_i, V_k] \subset V_{i+k}$, if $i+k \le r$, and $[V_i, V_k] = 0$, if $i+k > r$. Note that a graduated algebra is always nilpotent of degree r. An r-step *stratified* Lie algebra V [22] is a Lie algebra nilpotent of degree r, that has a *stratification*, that is,

$$V = \bigoplus_{i=1}^{r} V_i, \quad V_{i+1} = [V_1, V_i], \quad [V_1, V_r] = \{0\}.$$

An *r-step Carnot algebra* [22] is a graduated Lie algebra V, which has a stratification; a simply connected Lie group G, corresponding to an r-step Carnot algebra V, is called an *r-step Carnot group*. Let

$$N = \sum_{i=1}^{r} n_i, \quad n_i = \dim V_i, \tag{0.1}$$

and the basis of the left-invariant vector fields $\{X_1,\dots,X_N\}$ of the Carnot group G is ordered such that the values of the first n_1 of them form at every point $v \in G$ the basis of the subspace $V_1(v)$, the values of the next n_2 of them form at every point $v \in G$ the basis of the subspace $V_2(v)$, and so on. We assign to every vector field X_k a natural number $j = \deg X_k$, defined by the inclusion $X_k \in V_j$. A Box-quasimetric is defined as

$$\rho_{\mathrm{Box}_G}(u,w) = \max\{|a_i|^{\frac{1}{\deg X_i}} \mid i = 1,\dots,N\}, \quad w = \exp\Big(\sum_{i=1}^{N} a_i X_i\Big)(u). \tag{0.2}$$

The definition implies that ρ_{Box_G} satisfies the identity and symmetry axioms. Homogeneous dilatations on the Carnot group G are defined with the help of the operator δ_ε, $\varepsilon \ge 0$, acting by the rule

$$\delta_\varepsilon : (x_1,\dots,x_N) \mapsto (\varepsilon^{\deg X_1} x_1,\dots,\varepsilon^{\deg X_N} x_N).$$

The Box-quasimetric ρ_{Box_G} of the Carnot group G is invariant with respect to the left translations and the action of the operator of dilatations, see [11]–[14], that is,

$$\rho_{\mathrm{Box}_G}\big(L_u^G v, L_u^G w\big) = \rho_{\mathrm{Box}_G}(v,w), \quad \rho_{\mathrm{Box}_G}\big(\delta_\varepsilon v, \delta_\varepsilon w\big) = \varepsilon \rho_{\mathrm{Box}_G}(v,w) \quad \forall u,v,w \in G.$$

A canonical 2-step group $\mathbb{D}_n$ with a one-dimensional centre is defined in the standard Euclidean space $\mathbb{R}^{n+1}$ with the coordinate system $(x_1,\dots,x_n,t)$ and the coordinate frame $(O', e_1,\dots,e_n,e_{n+1})$ with the help of the following commutator table

$$[e_i, e_j] = \alpha_{ij} e_{n+1}, \quad \sum_{i,j=1}^{n} \alpha_{ij}^2 \ne 0, \tag{0.3}$$

the rest of possible commutators $e_1,\dots,e_{n+1}$ equal 0. Suppose that

$$x = (x_1,\dots,x_n,t), \quad x = (x_1',\dots,x_n',t').$$

Using the Campbell–Hausdorff formula [20], with the help of (0.3), we obtain

$$(0.4)\quad L_x^{\mathbb{D}_n} x' = x \cdot x' = \Big(x_1 + x_1', \dots, x_n + x_n',\ t + t' + \sum_{i,j=1,\dots,n,\, i<j} \frac{\alpha_{ij}}{2}(x_i x_j' - x_j x_i')\Big).$$

The values of basis left-invariant vector fields $X_1, \dots, X_n, T$ of the group $\mathbb{D}_n$ at every point $u = (x_1, \dots, x_n, t)$ are defined as

$$(X_1, \dots, X_n, T)(u) = \frac{\partial L_u^{\mathbb{D}_n}(x_1', \dots, x_n', t')}{\partial(x_1', \dots, x_n', t')}\Big|_{(x_1', \dots, x_n', t') = (0, \dots, 0)}.$$

If in (0.3) we put $n = 2m$, $m \in \mathbb{N}$, $\sum\limits_{i=1}^{m-1} \alpha_{2i,2i+1}^2 = 0$ and $\alpha_{2j-1,2j} = \alpha \neq 0$, $j = 1, \dots, m$, then we obtain a commutator table that defines the canonical Heisenberg group $\mathbb{H}_\alpha^m$ [18]. In particular, $\mathbb{D}_2 = \mathbb{H}_\alpha^1$.

For every point $u \in \mathbb{D}_n$, consider the mapping $\theta_u : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, acting by the rule

$$\theta_u(x_1, \dots, x_n, t) = \exp\big(x_1 X_1 + \dots x_n X_n + tT\big)(u).$$

According to (0.2), $(1, q_2)$-quasimetric $\rho_{\mathrm{Box}_{\mathbb{D}_n}}$ is defined by the rule

$$\rho_{\mathrm{Box}_{\mathbb{D}_n}}(u, v) = \max\big\{|a_1|, \dots, |a_n|, |b|^{\frac{1}{2}} \ \big| \ \theta_u(a_1, \dots, a_n, b) = v\big\}.$$

To find the exact value of the constant q_2 for the Carnot group $\mathbb{D}_3$, we more or less follow the methods of work [18], where calculation of the exact value was based on the following simple observation: the expression

$$x_1 x_2' - x_2 x_1', \quad x_1, x_2', x_2, x_1' \in [-1, 1],$$

reaches its maximal value, equal 2, when $x_1 = 1$, $x_2' = 1$, $x_2 = 1$, $x_1' = -1$. However, when we turn to the general group $\mathbb{D}_n$, it becomes complicated to «guess» the values of $x_1, \dots, x_n, x_1', \dots, x_n' \in [-1, 1]$, for which the expression

$$\sum_{i,j=1,\dots,n,\, i<j} \frac{\alpha_{ij}}{2}(x_i x_j' - x_j x_i')$$

from (0.4) has the maximal value. In § 1, we provide the proof of the fact that both minimal and maximal values of a bi-linear function in $\mathbb{R}^n$, considered on a standard unit n-dimensional cube

$$Q(n) = \{(x_1, \dots, x_n) \mid \max_{i=1,\dots,n} |x_i| \leq 1\},$$

are reached on pairs of vectors, each of which has coordinates of some vertex of the cube $Q(n)$ (Theorem 1.1). This provides us with a method of defining the exact value of the constant q_2 for an arbitrary group $\mathbb{D}_n$, which will help to obtain the exact value of the constant q_2 for the group $\mathbb{D}_3$, see (2.2) and Corollary 2.2. In § 3, we find the exact value of the constant q_2 for some 2-step groups related to $\mathbb{D}_3$.

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1. On extremes of bi-linear functions

Let Q_i be vertices of a cube $Q(n)$; such are all the points, whose coordinates consist only of the numbers ± 1. Consider the bi-linear function

$$\begin{aligned} P(x,y) &= P(x_1,\dots,x_n,y_1,\dots,y_n) \\ &= \left\langle \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle \\ &= \sum_{i,j=1}^{n} A_{ij}x_i y_j, \quad \sum_{i,j=1}^{n} A_{ij}^2 \neq 0, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is a standard dot product in the Euclidean space $\mathbb{R}^n$

Theorem 1.1. *There exist vertices Q_i, Q_j, Q_k, Q_l of the cube $Q(n)$, such that*

$$\max_{x,y\in Q(n)} P(x,y) = P(Q_i,Q_j), \qquad \min_{x,y\in Q(n)} P(x,y) = P(Q_k,Q_l).$$

Proof. Elementary geometric considerations show that pairs of points x, y, on which the minimum and maximum of the function $P(x,y)$ are reached, belong to the boundary $\partial Q(n)$. Suppose that the maximum of the function $P(x,y)$ is reached on the pair

$$(x^0,y^0) = (x_1^0,\dots,x_n^0,y_1^0,\dots,y_n^0).$$

We will prove Theorem 1.1 for the maximum of the function $P(x,y)$ (the arguments for the minimum are similar). Assume that

$$\sum_{j=1}^{n} A_{1j}y_j^0 \neq 0,$$

and, moreover, $x_1^0 \neq \pm 1$. Then, shifting the coordinate x_1 on some interval $(-\varepsilon + x_1^0, \varepsilon + x_1^0)$, we obtain the values of the function $P(x,y)$ that are both larger and smaller than $P(x^0,y^0)$, which contradicts the fact that on the pair (x^0,y^0) the maximum of the function $P(x,y)$ is reached. Therefore, $x_1^0 = \pm 1$. If

$$\sum_{j=1}^{n} A_{1j}y_j^0 = 0,$$

then we consider the situation where

$$\sum_{j=1}^{n} A_{2j}y_j^0 \neq 0,$$

and so on. From the mentioned above, it follows that for every k such that

$$\sum_{j=1}^{n} A_{kj}y_j^0 \neq 0,$$

we obtain that $x_k^0 = \pm 1$, and for k such that

$$\sum_{j=1}^{n} A_{kj}y_j^0 = 0,$$

for x_k^0 one can take any number from $[-1, 1]$, in particular, ± 1. Hence, we obtain that we can always consider as x^0 some vertex of the cube $Q(n)$.

We have

$$P(x^0, y^0) = \sum_{i,j=1}^{n} A_{ij}\delta_{ij}y_j^0,$$

where $\delta_{ij} = \pm 1$.

Now we fix some vertex $Q_k = (\delta_{k1}, \dots, \delta_{kn})$ and consider the linear mapping

$$P(Q_k, y) = \sum_{i,j=1}^{n} A_{ij}\delta_{ki}y_j : \mathbb{R}^n \to \mathbb{R}.$$

We denote by A the matrix build of $\{A_{ij}\}$. If

$$(\delta_{k1}, \dots, \delta_{kn}) \in kerA,$$

then $P(Q_k, y) \equiv 0$. But since the matrix A is nonzero, then $\dim kerA < n$; on the other hand, it is easy to see that the vector set $\{(\delta_{k1}, \dots, \delta_{kn})\}$ forms a complete system in $\mathbb{R}^n$. Suppose that

$$(\delta_{k1}, \dots, \delta_{kn}) \notin kerA.$$

In this case, the equations

$$\sum_{i,j=1}^{n} A_{ij}\delta_{ki}y_j + D = 0 \tag{1.1}$$

for all possible $D \in \mathbb{R}$ are the equations of parallel hyper-planes Π_D. The maximum of the expression

$$\sum_{i,j=1}^{n} A_{ij}\delta_{ki}y_j$$

on the cube $Q(n)$ will be reached in the case when the hyper-plane (1.1) has a non-empty intersection with the boundary $\partial Q(n)$, but at the same time has no common points with the interior of the cube $Q(n)$. But in this case, there always exists a vertex $Q_l \in (\Pi_D \cap \partial Q(n))$. Then

$$\max_{x,y \in Q(n)} P(x, y) = \max_{k,l} P(Q_k, Q_l).$$

□

Consider an arbitrary parallelepiped $\bar{P} \subset \mathbb{R}^n$ with edges parallel to the coordinate axes and the center of symmetry at the origin.

Theorem 1.2. *There exist vertices P_i, P_j, P_k, P_l of the parallelepiped $\bar{P}$ such that*

$$\max_{x,y \in \bar{P}} P(x, y) = P(P_i, P_j), \quad \min_{x,y \in \bar{P}} P(x, y) = P(P_k, P_l).$$

Proof. Theorem 1.2 is proved similarly to Theorem 1.1.

□

2. A method of finding the exact constants in the $(1, q_2)$-generalized triangle inequality for 2-step Carnot groups with a one-dimensional centre

We will provide some considerations that follow from work [18], which will help us to define the exact value of the constant q_2 in the $(1, q_2)$-generalized triangle inequality for Box-quasimetrics of the Carnot groups $\mathbb{D}_n$.

Consider an arbitrary canonical Carnot group G. To define the exact value of the constant q_2 in the $(1, q_2)$-generalized triangle inequality, for every triple of points $u, v, w \in G$, we must find a number $q = q(u, v, w)$ such that $\rho_{Box_G}(u, w) = \rho_{Box_G}(u, v) + q\rho_{Box_G}(v, w)$, and then $q_2 = \sup\limits_{u,v,w\in G} q(u, v, w)$. But, taking into account the fact that the $(1, q_2)$-quasimetrics $\rho_{Box_{\mathbb{D}_n}}$ is invariant with respect to the left translations and action of dilatations, it suffices to consider only such triples of points u, v, w, where

$$u = 0, \quad \rho_{Box_G}(0, v) = 1, \quad w = v \cdot \delta_\varepsilon w', \quad \rho_{Box_G}(0, w') = 1, \quad \varepsilon > 0,$$

and therefore, search for $q = q(0, v, w)$ from the equality

$$\rho_{Box_G}(0, w) = \rho_{Box_G}(0, v) + q\rho_{Box_G}(v, w) = 1 + q\varepsilon.$$

We denote $S_G(0, 1) = \{x \in G \mid \rho_{Box_G}(0, x) = 1\} = \partial Q(N)$, see (0.1).

We transfer our considerations on the Carnot group $\mathbb{D}_n$. Let

$$v = (x_1, \dots, x_n, t) \in S_{\mathbb{D}_n}(0, 1), \quad w' = (x'_1, \dots, x'_n, t') \in S_{\mathbb{D}_n}(0, 1).$$

Using (0.4), we obtain

$$v \cdot \delta_\varepsilon w' = \Big(x_1 + \varepsilon x'_1, \dots, x_n + \varepsilon x'_n,\ t + \varepsilon^2 t' + \varepsilon \sum_{i,j=1,\dots,n,\, i<j} \frac{\alpha_{ij}}{2}(x_i x'_j - x_j x'_i)\Big).$$

We denote

$$M_{\mathbb{D}_n} = \sup_{(x_1,\dots,x_n),\,(x'_1,\dots,x'_n)\in Q(n)} \Big| \sum_{i,j=1,\dots,n,\, i<j} \frac{\alpha_{ij}}{2}(x_i x'_j - x_j x'_i)\Big|. \tag{2.1}$$

Note that by Theorem 1.1, we can consider as $(x_1, \dots, x_n)$, $(x'_1, \dots, x'_n)$ in (2.1), the vertices of the cube $Q(n)$.

We have

$$\Big|t + \varepsilon^2 t' + \varepsilon \sum_{i,j=1,\dots,n,\, i<j} \frac{\alpha_{ij}}{2}(x_i x'_j - x_j x'_i)\Big| \le 1 + M_{\mathbb{D}_n}\varepsilon + \varepsilon^2.$$

Then

$$q_2 = \begin{cases} 1, & M_{\mathbb{D}_n} \le 2, \\ \frac{M_{\mathbb{D}_n}}{2}, & M_{\mathbb{D}_n} > 2. \end{cases} \tag{2.2}$$

It is formula (2.2) that provides the exact value of the constant q_2 for the Carnot group $\mathbb{D}_n$.

We apply the method described above to find the exact value of the constant q_2 for the Carnot group $\mathbb{D}_3$, that is defined with the help of the following commutator table:

$$\begin{cases} [e_1, e_2] = a_{12}e_4, \\ [e_2, e_3] = a_{23}e_4, \\ [e_3, e_1] = a_{31}e_4. \end{cases} \tag{2.3}$$

The group operation, see (0.4), on $\mathbb{D}_3$ is written in the form

$$(2.4)\quad (x,y,z,t)(x',y',z',t') = \Big(x+x', y+y', z+z', \\ t+t'+\frac{1}{2}\big(a_{12}(xy'-x'y)+a_{23}(yz'-y'z)+a_{31}(zx'-z'x)\big)\Big).$$

We introduce notations $(x,y,z)=u$, $(x',y',z')=v$, $a=(a_{12},a_{23},a_{31})$; then

$$a_{12}(xy'-x'y)+a_{23}(yz'-y'z)+a_{31}(zx'-z'x) = \langle a, u\otimes v\rangle.$$

Lemma 2.1. *Let u, v, $u\neq v$, be non-collinear vectors whose coordinates coincide with the coordinates of the vertices of the cube $Q(3)$. We denote*

$$u\otimes v = (q_1,q_2,q_3).$$

Then the triples of numbers q_1,q_2,q_3 satisfy the following conditions: one of the numbers equals 0, two others have the values ± 2.

Proof. It is easy to obtain the proof of Lemma 2.1 by direct calculations. □

Corollary 2.2.

$$M_{\mathbb{D}_3} = \max\{|a_{12}\pm a_{23}|,\ |a_{31}\pm a_{12}|,\ |a_{23}\pm a_{31}|\}.$$

3. Some corollaries

Consider the 2-step Carnot group G_1, which is defined by the following commutator table:

$$\begin{cases}[e_1,e_2]=a_{12}e_4,\\ [e_2,e_3]=a_{23}e_5,\\ [e_3,e_1]=a_{31}e_6.\end{cases}$$

The group operation, see (0.4), on G_1 is written in the form

$$(x,y,z,t_4,t_5,t_6)(x',y',z',t_4',t_5',t_6') = \Big(x+x', y+y', z+z', \\ t_4+t_4'+\frac{a_{12}}{2}(xy'-x'y),\ t_5+t_5'+\frac{a_{23}}{2}(yz'-y'z),\ t_6+t_6'+\frac{a_{31}}{2}(zx'-z'x)\Big).$$

We denote $a=\max\{|a_{12}|,|a_{23}|,|a_{31}|\}$. The following theorem follows from the results of work [18].

Theorem 3.1. *For a canonical Carnot group G_1, the exact value of the constant q_2 is defined by the formula*

$$q_2 = \begin{cases}1, & a\le 2,\\ \frac{a}{2}, & a>2.\end{cases}$$

Consider a 2-step Carnot group G_2, which is defined by the following commutator table:

$$\begin{cases}[e_1,e_2]=a_{12}e_4,\\ [e_2,e_3]=a_{23}e_4,\\ [e_3,e_1]=a_{31}e_5.\end{cases}$$

The group operation, see (0.4), on G_2 is written in the form

$$(x,y,z,t_4,t_5)(x',y',z',t_4',t_5') = \Big(x+x', y+y', z+z',$$
$$t_4+t_4'+\frac{a_{12}}{2}(xy'-x'y)+\frac{a_{23}}{2}(yz'-y'z),\, t_3+t_3'+\frac{a_{31}}{2}(zx'-z'x)\Big).$$

We denote

$$b=\max\{|a_{31}|,\ |a_{12}\pm a_{23}|\}.$$

Using the results of § 2 and the results of work [18], we obtain the following theorem.

Theorem 3.2. *For a canonical Carnot group G_2, the exact value of the constant q_2 is defined by the formula*

$$q_2=\begin{cases}1, & b\le 2,\\ \frac{b}{2}, & b>2.\end{cases}$$

References

[1] Arutyunov A.V., Greshnov A.V., *(q_1,q_2)-quasimetric spaces. Covering mappings and coincidence points*, Izvestiya RAN: Ser. Mat. **82**:2 (2018), 3–32. MR3780044

[2] Arutyunov A.V., Greshnov A.V., *Theory of (q_1,q_2)-quasimetric spaces and coincidence points*, Dokl. Math. **9**4:1 (2016), 434–437. Zbl 1352.54030

[3] Arutyunov A.V., Greshnov A.V., *Coincidence points of multi-valued mappings in (q_1,q_2)-quasimetric spaces*, Dokl. Math. **96**:2 (2017), 438–441. MR3791184

[4] Arutyunov A.V., Greshnov A.V. , Lokoutsievskii L.V., Storozhuk K.V., *Topological and geometrical properties of spaces with symmetric and nonsymmetric f-quasimetrics*, Topology Appl., **221** (2017), 178–194. MR3624455

[5] Sengupta R., About fixed points of contraction mappings acting in (q_1,q_2)-quasi-metric spaces, Eurasian Math. Jour, **8**:3 (2017), 70–76. MR3719796

[6] Greshnov A.V., *(q_1,q_2)-quasimetrics bi-Lipschitz equivalent to 1-quasimetrics*, Siberian Adv. Math., **27**:4 (2017), 253–262. MR3676094

[7] Greshnov A.V., *Regularization of distance functions and separation axioms on (q_1,q_2)-quasimetric spaces*, Siberian Electronic Mathematical Reports, **14** (2017), 765–773. MR3693752

[8] Greshnov A.V., *Some problems of regularity of f-quasimetrics*, Siberian Electronic Mathematical Reports, **15** (2018), 355–361. MR3796418

[9] Greshnov A.V., Zhukov R.I., *Completeness theorem in (q_1,q_2)-quasimetric spaces*, Siberian Electronic Mathematical Reports, **16** (2018), 2090–2097. Zbl 1433.30125

[10] Wilson W. A., *On quasi-metric spaces*, American J. of Math., **53**:3 (1931), 675–684. MR1506845

[11] Vodopyanov S. K., *Geometry of Carnot-Carathéodory spaces and differentiability of mappings*, Contemporary Mathematics, **424** (2007), 247–301. MR2316341

[12] Basalaev S.G., Vodopyanov S.K., *Approximate differentiability of mappings of Carnot–Carathéodory spaces*, Eurasian Math. J., **4**:2 (2013), 10–48. MR3118880

[13] Greshnov A.V., *On the generalized triangle inequality for quasimetrics induced by noncommuting vector fields*, Siberian Adv. Math., **22**:2 (2012), 95–114. MR2858658

[14] Greshnov A.V., *Proof of Gromov's theorem on homogeneous nilpotent approximation for vector fields of class C^1*, Siberian Adv. Math., **23**:3 (2013), 180–191. MR3074455

[15] Nagel A., Stein E.M. , Wainger S., *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math., **155**:1-2 (1985), 103–47. MR0793239

[16] Greshnov A.V., *Local approximation of uniformly regular Carnot -Carathéodory quasispaces by their tangent*, Siberian Math. J., **48**:2 (2007), 229–248. MR2330060

[17] Greshnov A.V., *Metrics and tangent cones of uniformly regular Carnot–Carathéodory spaces*, Siberian Math. J., **47**:2 (2006), 209–238. MR2227976

[18] Greshnov A.V., Tryamkin M.V., *Exact values of constants in the generalized triangle inequality for some* $(1, q_2)$*-quasimetrics on canonical Carnot groups*, Math. Notes, **98**:4 (2015), 694–698. MR343851

[19] Ovsyannikov L.V., *Group Analysis of Differential Equations*, Nauka, Moscow, 1978; Academic Press, New York, 1982. MR0511921

[20] Postnikov M. M., *Lie Groups and Lie Algebras. Lectures in Geometry. Semester V*, Nauka, Moscow, 1982; Mir,Moscow, 1986. MR0685757

[21] Rothchild L. P., Stein E. S., *Hypoelliptic differential operators and nilpotent groups*, Acta Math., **137** (1976), 247–320. MR0436223

[22] Bonfiglioli A., Lanconelli E., Uguzzoni F., *Stratified Lie groups and potential theory for their sub-Laplacian*, Springer–Verlag, Berlin–Heidelberg, 2007. MR2363343

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