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CONTINUOUS MEASURES WITH LARGE PARTIAL SUMS

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Abstract. It is proved that, in a weak sense, every measure in $M(\mathbb{T})$ supported by a sufficiently singular Cantor set has asymptotically large Fourier partial sums. It is also shown that every measure in $M(\mathbb{T})$ whose Fourier partial sums satisfy a mild growth condition has nontrivial null sets.

§1. Introduction

It is easily seen that if $f \in L^1(\mathbb{T})$, then the Fourier partial sums $S_N f = \sum_{-N}^{N} \hat{f}(n)e^{inx}$ satisfy the growth condition $\|S_N f\| = o(\log N)$, where $\|\cdot\|$ denotes the L^1-norm on $\mathbb{T}$ with respect to normalized (mass 1) Lebesgue measure. Since for an arbitrary $\mu \in M(\mathbb{T})$ we only have $\|S_N\mu\| = O(\log N)$, it seems reasonable to consider the class, say M_l, consisting of all measures μ in $M(\mathbb{T})$ such that $\|S_N\mu\| = o(\log N)$. In [4, Vol. I, p. 149, the proof of Theorem 5.20]) it was proved that if $\mu \in M(\mathbb{T})$ satisfies $\liminf_{N\to\infty} \|S_N(\mu)\|/\log N = 0$, then $\mu \in M_c(\mathbb{T})$, where $M_c(\mathbb{T})$ denotes the space of all continuous measures in $M(\mathbb{T})$. Consequently, $M_l \subset M_c(\mathbb{T})$. A similar result recently observed by M. E. Andersson is the following.

Theorem 1 (M. E. Andersson [2]). *Let $\mu \in M(\mathbb{T})$, and let μ_d denote the discrete part of μ. Then*

$$\liminf_{N\to\infty} \frac{\|S_N(\mu)\|}{\log(N)} \geqslant \frac{4}{\pi^2}\|\mu_d\|.$$

Moreover, if μ is discrete, then

$$\lim_{N\to\infty} \frac{\|S_N(\mu)\|}{\log(N)} = \frac{4}{\pi^2}\|\mu\|. \tag{1}$$

Here $\|\mu\|$ denotes the total variation of $\mu \in M(\mathbb{T})$. A question raised by M. E. Andersson reads as follows: Which of $M_l = M_c(\mathbb{T})$ or $M_l \neq M_c(\mathbb{T})$ is true?

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By Theorem 3, one of the main results of the present paper, it follows that if K is a sufficiently singular Cantor set (see the comments following Remark 3), then there exists a sequence $N_n \to \infty$ of positive integers (depending only on K) such that (1) is fulfilled along the sequence N_n for every $\mu \in M(K)$. In particular, applying Theorem 3 to some $0 \neq \nu \in M_c(K)$, we see that the answer to the above question is $M_l \neq M_c(\mathbb{T})$. Combining Theorem 2 (a result similar to Theorem 3) with the L-ideal property of M_l (see Lemma 4), we prove in Theorem 4 that if K belongs to a certain class of sets containing all sufficiently singular Cantor sets, then K is a null set for all the measures in M_l. This result sharpens the inclusion $M_l \subset M_c(\mathbb{T})$ discussed above (cf. Remark 3).

The proof of Theorem 2 is by approximation by discrete measures. Theorem 3 is deduced from Theorem 2 as a limit case. Compared to Theorem 1, the main new ingredient used in the proof of Theorem 2 is Lemma 2, which is a version of the standard asymptotic result

$$L_N = \|D_N\| = \frac{4}{\pi^2}\log(N) + O(1), \quad N \to \infty,$$

for the Lebesgue constants L_N. Here $D_N(x) = \sum_{-N}^{N} e^{inx}$ is the standard Dirichlet kernel. Lemma 2 is a special case of more general and precise asymptotic expansions proved by Lee Lorch (see Remark 1).

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§2. Results and Proofs

For the proof of our main results we need some lemmas.

Lemma 1. *Let $K \subset \mathbb{T}$ be a compact set such that there exist Borel sets $E_{n,j}$, $1 \leqslant j \leqslant j_n$, $n \geqslant 1$, with $K \subset \bigcup_{j=1}^{j_n} E_{n,j}$, $E_{n,j} \cap E_{n,k} = \varnothing$ for $j \neq k$, and $\operatorname{diam}(E_{n,j}) \to 0$ as $n \to \infty$ uniformly in j. If $\nu \in M(K)$, then*

$$\lim_{n\to\infty} \sum_{j=1}^{j_n} |\nu(E_{n,j})| = \|\nu\|.$$

Above, $\operatorname{diam}(E)$ denotes the diameter of the set E.

Proof. It is obvious that $\sum_{j=1}^{j_n} |\nu(E_{n,j})| \leqslant \|\nu\|$. Set

$$\mu_n = \sum_{j=1}^{j_n} \nu(E_{n,j})\delta_{x_{n,j}}, \quad x_{n,j} \in E_{n,j}.$$

It is easy to see that $\mu_n \to \nu$ weakly*. Hence,

$$\|\nu\| \leqslant \liminf_{n\to\infty} \|\mu_n\| = \liminf_{n\to\infty} \sum_{j=1}^{j_n} |\nu(E_{n,j})|. \quad \bullet$$

Lemma 2. *Let* $\{\varepsilon_N\}_{N=1}^{\infty}$ *be a sequence of positive numbers such that* $1/\varepsilon_N = o(N)$. *Then, as* $N \to \infty$, *we have*

$$\frac{1}{2\pi} \int_{-\varepsilon_N}^{\varepsilon_N} |D_N(x)|dx = \frac{4}{\pi^2} \log(\varepsilon_N N) + O(1).$$

Remark 1. Lemma 2 is a special case of more general and precise results by Lee Lorch. See [3, the theorem on pp. 246–247]. In particular we point out that the case of interest in Lemma 2 is when $\varepsilon_N \to 0$.

Lemma 3. *There is a constant* C *such that*

$$\frac{1}{2\pi} \int_{\mathbb{T}} |D_N(x-y) - D_N(x)|dx \leqslant C|y|N\log(N).$$

Proof. Assume for simplicity that $y > 0$. Since $D_N(x-y) - D_N(x) = -\int_{-y}^{0} D'_N(x+t)dt$, we have $\|D_N(\cdot - y) - D_N\| \leqslant \int_{-y}^{0} \|D'_N(\cdot + t)\|dt = y\|D'_N\|$. By Bernstein's inequality (see [4, Vol. II, p. 11, Theorem 3.16]), $\|D_N(\cdot - y) - D_N\| \leqslant yNL_N$. $\bullet$

Theorem 2. *Let* $K \subset \mathbb{T}$ *be a compact set such that there exist closed intervals* $I_{n,j}$, $1 \leqslant j \leqslant j_n$, $n \geqslant 1$, *with* $K \subset \bigcup_{j=1}^{j_n} I_{n,j}$ *and* $I_{n,j} \cap I_{n,k} = \varnothing$ *for* $j \neq k$. *Denote by* $\lambda(n)$ *and* $\delta(n)$ *the quantities*

$$\lambda(n) = \max_j |I_{n,j}|, \quad \delta(n) = \min_{j\neq k} \operatorname{dist}(I_{n,j}, I_{n,k})$$

and suppose that $\lambda(n) \to 0$. *Assume that*

$$\lim_{n\to\infty} \frac{\lambda(n)}{\delta(n)^{2+\alpha}} = 0 \quad \textit{for some } \alpha > 0. \tag{2}$$

Let N_n, $N_n \to \infty$, be a sequence of positive integers such that

$$\lim_{n\to\infty} N_n\lambda(n) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} N_n\delta(n)^{2+\alpha} > 0. \tag{3}$$

Then for every $\nu \in M(K)$ we have

$$\liminf_{n\to\infty} \frac{\|S_{N_n}(\nu)\|}{\log(N_n)} \geqslant \frac{4}{\pi^2}\frac{\alpha}{2+\alpha}\|\nu\|. \tag{4}$$

Above, $\mathrm{dist}(E, F)$ denotes the distance between the sets E and F. For the existence of sequences $\{N_n\}$ satisfying the assumptions of the theorem see Remark 2.

Proof. To simplify notation, in the proof we sometimes suppress the dependence of N on n and simply write N instead of N_n. Let $\varepsilon_N = \delta(n)/2$ and note that $1/\varepsilon_N = o(N)$ so that Lemma 2 is applicable. Take $\nu \in M(K)$ and let

$$\mu_n = \sum_{j=1}^{j_n} \nu(I_{n,j})\delta_{x_{n,j}}, \quad x_{n,j} \in I_{n,j}.$$

We have

$$\|S_{N_n}(\nu)\| \geqslant \|S_{N_n}(\mu_n)\| - \|S_{N_n}(\nu) - S_{N_n}(\mu_n)\|. \tag{5}$$

Using $N_n\lambda(n) \to 0$, we show that $\|S_N(\nu) - S_N(\mu_n)\| = o(\log N)$. Since

$$S_{N_n}(\nu)(x) - S_{N_n}(\mu_n)(x) = \int D_{N_n}(x-y)d\nu(y) - \sum_{j=1}^{j_n} \nu(I_{n,j})D_{N_n}(x - x_{n,j})$$

$$= \sum_{j=1}^{j_n} \int_{I_{n,j}} \big(D_{N_n}(x-y) - D_{N_n}(x - x_{n,j})\big)d\nu(y),$$

we see that

$$\|S_{N_n}(\nu) - S_{N_n}(\mu_n)\| \leqslant \sum_{j=1}^{j_n} \int_{I_{n,j}} \|D_{N_n}(\cdot - y) - D_{N_n}(\cdot - x_{n,j})\|\,|d\nu|(y).$$

Using Lemma 3, we get

$$\|S_{N_n}(\nu) - S_{N_n}(\mu_n)\| \leqslant C\lambda(n)\|\nu\|N_n\log(N_n).$$

Now, the first condition in (3) implies that

$$\frac{\|S_{N_n}(\nu) - S_{N_n}(\mu_n)\|}{\log(N_n)} \leqslant C\|\nu\| N_n \lambda(n) \to 0 \tag{6}$$

as $n \to \infty$.

Now, we consider $\|S_{N_n}(\mu_n)\|$. Let D_N^ε denote the truncated Dirichlet kernel defined by $D_N^\varepsilon(x) = D_N(x)$ for $|x| < \varepsilon$ and $D_N(x) = 0$ for $\varepsilon \leqslant |x| \leqslant \pi$. We have

$$\begin{aligned} \|S_N(\mu_n)\| = \|D_N * \mu_n\| &\geqslant \|D_N^{\varepsilon_N} * \mu_n\| - \|(D_N - D_N^{\varepsilon_N}) * \mu_n\| \\ &\geqslant \|D_N^{\varepsilon_N} * \mu_n\| - \|D_N - D_N^{\varepsilon_N}\|\|\nu\|. \end{aligned} \tag{7}$$

Next we compute the asymptotics for $\|D_N - D_N^{\varepsilon_N}\|$. We have

$$\|D_N - D_N^{\varepsilon_N}\| = \frac{1}{2\pi} \int\limits_{\varepsilon_N \leqslant |x| \leqslant \pi} |D_N(x)|dx = \frac{4}{\pi^2}\log(1/\varepsilon_N) + O(1), \tag{8}$$

where at the last step we used Lemma 2.

We consider $\|D_N^{\varepsilon_N} * \mu_n\|$. We note that since $2\varepsilon_N \leqslant \delta(n)$, the intervals $(x_{n,j} - \varepsilon_N, x_{n,j} + \varepsilon_N)$, $1 \leqslant j \leqslant j_n$, are disjoint. Since

$$D_N^{\varepsilon_N} * \mu_n(x) = \sum_{j=1}^{j_n} \nu(I_{n,j}) D_N^{\varepsilon_N}(x - x_{n,j}),$$

we obtain

$$\begin{aligned} \|D_N^{\varepsilon_N} * \mu_n\| &= \frac{1}{2\pi} \sum_{k=1}^{j_n} \int\limits_{x_{n,k}-\varepsilon_N}^{x_{n,k}+\varepsilon_N} |D_N^{\varepsilon_N} * \mu_n(x)|\, dx \\ &= \frac{1}{2\pi} \sum_{k=1}^{j_n} \int\limits_{x_{n,k}-\varepsilon_N}^{x_{n,k}+\varepsilon_N} \left| \sum_{j=1}^{j_n} \nu(I_{n,j}) D_N^{\varepsilon_N}(x - x_{n,j}) \right| dx \\ &= \frac{1}{2\pi} \sum_{k=1}^{j_n} \int\limits_{x_{n,k}-\varepsilon_N}^{x_{n,k}+\varepsilon_N} |\nu(I_{n,k}) D_N^{\varepsilon_N}(x - x_{n,k})|\, dx \\ &= \frac{1}{2\pi} \int\limits_{-\varepsilon_N}^{\varepsilon_N} |D_N(x)| dx \sum_{k=1}^{j_n} |\nu(I_{n,k})| \\ &= \frac{4}{\pi^2}\|\nu\| \log(\varepsilon_N N) + o(\log(\varepsilon_N N)), \end{aligned} \tag{9}$$

where at the last step we used Lemma 2 and Lemma 1.

From (7), (8), (9), and the relation $o(\log(\varepsilon_N N)) = o(\log N)$, we deduce that

$$\|S_{N_n}(\mu_n)\| \geqslant \frac{4}{\pi^2}\|\nu\| \log(\varepsilon_N^2 N) + o(\log N). \tag{10}$$

Now, (5), (6), and (10) imply

$$\|S_{N_n}(\nu)\| \geqslant \frac{4}{\pi^2}\|\nu\| \log(\varepsilon_N^2 N) + o(\log N).$$

By the second condition in (3) we have $\varepsilon_N^2 N \geqslant \text{const } N^{\alpha/(2+\alpha)}$, and (4) follows. •

Theorem 3. *Let $K \subset \mathbb{T}$ be a compact set such that there exist closed intervals $I_{n,j}$, $1 \leqslant j \leqslant j_n$, $n \geqslant 1$, with $K \subset \bigcup_{j=1}^{j_n} I_{n,j}$ and $I_{n,j} \cap I_{n,k} = \varnothing$ for $j \neq k$. Let $\lambda(n)$ and $\delta(n)$ be defined as in Theorem 2, and suppose that $\lambda(n) \to 0$. Assume that*

$$\lim_{n\to\infty} \frac{\lambda(n)}{\delta(n)^p} = 0 \quad \text{for every } p \geqslant 1. \tag{11}$$

Let N_n, $N_n \to \infty$, be a sequence of positive integers such that

$$\lim_{n\to\infty} N_n\lambda(n) = 0 \text{ and } \lim_{n\to\infty} \frac{1}{N_n\delta(n)^p} = 0 \text{ for all } p \geqslant 1. \tag{12}$$

Then for every $\nu \in M(K)$ we have

$$\lim_{n\to\infty} \frac{\|S_{N_n}(\nu)\|}{\log(N_n)} = \frac{4}{\pi^2}\|\nu\|. \tag{13}$$

For the existence of sequences $\{N_n\}$ satisfying the assumptions of the theorem see Remark 2.

Proof. It is clear that $\|S_N(\nu)\|/\log(N) \leqslant L_N\|\nu\|/\log(N) \to 4/\pi^2\|\nu\|$ as $N \to \infty$. Thus is suffices to show that $\liminf_{n\to\infty} \|S_N(\nu)\|/\log(N) \geqslant 4/\pi^2\|\nu\|$. But this is the limit case of Theorem 2 letting $\alpha \to \infty$. •

Remark 2. The first condition in (12) is that $N_n = o(1/\lambda(n))$. The second condition in (12) says that N_n increases faster than every power of $1/\delta(n)$. Fix K as in Theorem 3. One easily checks that $N_n = [\delta(n)/\lambda(n)^{1-\varepsilon}]$, $0 < \varepsilon < 1$, satisfies the assumptions of Theorem 3. Similar remarks are applicable to Theorem 2. Indeed, if K satisfies the assumptions of Theorem 2, then it is easily verified that $N_n = [\omega_n/\delta(n)^{2+\alpha}]$, where $\omega_n \to \infty$ slowly, satisfies (3).

The following lemma has been communicated to the author by M. E. Andersson and, in principle, is well known (cf. [4, Vol. II, Chapter XII, Theorem 10.9]). For the sake of completeness we include a sketch of the proof.

Lemma 4. *$M_l = \{\mu \in M(\mathbb{T}) : \|S_N(\mu)\| = o(\log N)\}$ is an L-ideal, i.e., M_l is a closed ideal in $M(\mathbb{T})$ such that if $\mu \in M_l$, $\lambda \in M(\mathbb{T})$, and $\lambda \ll \mu$ (λ is absolutely continuous with respect to μ), then $\lambda \in M_l$.*

Proof. It is easily verified that M_l is a closed ideal in $M(\mathbb{T})$. Fix $\mu \in M_l$. By the Radon–Nikodym Theorem, we must prove that if $d\lambda = fd\mu$, $f \in L^1(|\mu|)$, then $\lambda \in M_l$. It is easily checked that $e^{ikx}d\mu \in M_l$ for every $k \in \mathbb{Z}$. Hence, $pd\mu \in M_l$ for every trigonometric polynomial p, and, by an approximation argument $fd\mu \in M_l$ whenever $f \in C(\mathbb{T})$. By regularity of μ, $\chi_E d\mu \in M_l$ for every Borel set $E \subset \mathbb{T}$. Approximating by simple functions we get $\lambda \in M_l$. •

Theorem 4. *Let $\mu \in M_l$, i.e., $\mu \in M(\mathbb{T})$ and $\|S_N(\mu)\| = o(\log N)$, and let K satisfy the assumptions of Theorem 2. Then $|\mu|(K) = 0$.*

Proof. Let $\mu \in M_l$. By Lemma 4 we have $\mu|_K \in M_l$. By Theorem 2, $|\mu|(K) = \|\mu|_K\| = 0$. •

Remark 3. By an obvious modification of the proof of Theorem 4 we obtain the following formally stronger result:

Let K and N_n satisfy the assumptions of Theorem 2. If $\mu \in M(\mathbb{T})$ and

$$\liminf_{n\to\infty} \|S_{N_n}(\mu)\|/\log(N_n) = 0,$$

then $|\mu|(K) = 0$.

The prototype of sets satisfying the requirements of Theorem 2, 3, and 4 are sets of Cantor type. We proceed to construct sets satisfying the assumptions of Theorem 3. Let $\mathrm{s} = \{s_n\}_{n=1}^{\infty}$ be a sequence such that $0 < s_n < 1$ for all n. The Cantor set $C(\mathrm{s})$ is constructed as follows. Let

$$C(s_1) = [0,1] \setminus \big((1-s_1)/2, (1+s_1)/2\big).$$

For $n \geqslant 1$, we write

$$C(s_1, \ldots, s_n) = \bigcup_{j=1}^{2^n} S_{n,j},$$

where the $S_{n,j}$, $1 \leqslant j \leqslant 2^n$, are disjoint nonempty closed intervals of equal length. The set $C(s_1, \ldots, s_{n+1})$ is obtained from $C(s_1, \ldots, s_n)$ by removing an open subinterval of length $s_{n+1}|S_{n,j}|$ from the exact middle of each $S_{n,j}$. Now, the *Cantor set* $C(\mathrm{s})$ is defined by

$$C(\mathrm{s}) = \bigcap_{n=1}^{\infty} C(s_1, \ldots, s_n).$$

Let $\lambda(n)$ and $\delta(n)$ be as in Theorem 2 for $K = C(\mathrm{s})$ and $I_{n,j} = S_{n,j}$. It is straightforward to see that

$$\lambda(n) = \prod_{k=1}^{n} \frac{1 - s_k}{2}, \quad \delta(n) = s_n \prod_{k=1}^{n-1} \frac{1 - s_k}{2},$$

where for the validity of the second formula s must be such that $s_{n+1} \leqslant 2s_n/(1-s_n)$ for all n; for instance, if all $s_n \geqslant 1/3$, this is true. We also observe that $\lambda(n) \leqslant 1/2^n \to 0$ as $n \to \infty$. By the above formulas for $\lambda(n)$ and $\delta(n)$, it is immediate that (11) is fulfilled if $s_n \to 1$ sufficiently fast.

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Division of Mathematics
and Computer Science
Vrije Universiteit
De Boelelaan 1081A
1081 HV Amsterdam
The Netherlands
E-mail: anderso@matematik.su.se

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