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A STABLE FOLIATION TO INFINITY IN THE PHASE SPACE OF THE HÉNON MAP

ABSTRACT. The phase space of quadratic area-preserving Hénon map of the plane is considered. The stable and unstable foliations to infinity are constructed and their differentiability in the real case is proved. Main conjectures on the foliation behavior are discussed for the complex case. The presentation of a dynamical system in the form of a continued fraction is used.

§1. INTRODUCTION: BASIC DEFINITIONS AND THE MAIN THEOREM

One of instruments to explore dynamical systems is continued fractions. V. F. Lazutkin suggests to use continued fractions for construction a stable foliation to infinity in the complexified phase space of some symplectic maps of the plane. This idea was realized in a joint work of V. F. Lazutkin and M. A. Pankratov [1] for the case of the Standard map.

We will consider iterations of the *Hénon map*, HM, as a self-map of the symplectic space $\mathbb{R}^2$ defined by the formulae $HM: (x, y) \longmapsto (x_1, y_1)$ where

$$\begin{aligned} x_1 &= x + y_1, \\ y_1 &= y + \varepsilon x(1 - x). \end{aligned} \tag{1.1}$$

Here ε is a small positive parameter. We take a standard symplectic form

$$\Omega = dx \wedge dy = dx_1 dy_2 - dx_2 dy_1. \tag{1.2}$$

The HM preserves an area and an orientation:

$$dx_1 \wedge dy_1 = dx \wedge dy \tag{1.3}$$

and possesses a *reversor*:

$$R: (x, y) \longmapsto (x - y, -y), \tag{1.4}$$

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which satisfies the equations

$$\begin{aligned} R^2 &= id, \\ R \circ HM &= HM^{-1} \circ R. \end{aligned} \tag{1.5}$$

Also HM possesses a *symmetry*

$$S\colon (x, y) \mapsto (x, -y) \tag{1.4'}$$

which satisfies

$$\begin{aligned} S^2 &= id, \\ S \circ HM &= HM \circ S. \end{aligned} \tag{1.5'}$$

The origin (0,0) is a fixed point of HM, the matrix of the linear part being

$$\begin{pmatrix} 1+\varepsilon & 1 \\ \varepsilon & 1 \end{pmatrix}, \tag{1.6}$$

The eigenvalues of (1.6) are $\lambda,\ \lambda^{-1}$, where

$$\lambda = 1 + \frac{\varepsilon}{2} + \sqrt{\varepsilon + \frac{\varepsilon^2}{4}} > 1. \tag{1.7}$$

It is well known that the stable and unstable manifolds

$$\begin{aligned} W^s &= \left\{(X, Y)\colon \lim_{n\to+\infty} HM^n(X, Y) = (0, 0)\right\}, \\ W^u &= \left\{(X, Y)\colon \lim_{n\to+\infty} HM^{-n}(X, Y) = (0, 0)\right\} \end{aligned}$$

are analytic curves passing through (0,0), the tangent directions coinciding with the eigenvectors of the linear part at (0,0). Also we have $R(W_1^u) = W_1^s$.

The main purpose of this work is to prove the following theorem.

Theorem 1 (Main). *There exists a map*

$$\Phi^u : \mathbb{C} \times \mathbb{R} \to \mathbb{C}^2$$

with the following properties:

(1) Φ^u *is injective;*
(2) Φ^u *conjugates the model map* $M\colon (\xi, \eta) \mapsto (\lambda\xi, \lambda^{-1}\eta)$ *with the restriction of HM onto* $\Phi^u(\mathbb{C} \times \mathbb{R})$*;*

(3) *for every* $\eta \in \mathbb{R}$ *the restriction of* Φ^u *onto the plane* $\mathbb{C} \times \{\eta\}$ *is analytic;*

(4) $\Phi^u\Big|_{\mathbb{C}\times\{0\}}$ *coincides with the map* φ^u *defined in Proposition 1 with a certain choice of* (a_0, b_0)*;*

(5) $\Phi^u\Big|_{\{0\}\times\mathbb{R}}$ *coincides with the restriction of* φ^s*, defined in Proposition 1, onto the real plane, also with a suitable choice of* (a_0, b_0)*;*

(6) Φ^u *is continuous.*

What are φ^u, φ^s, and (a_0, b_0)?

To understand this we have to prove the following proposition.

Proposition 1. *Let* (a_0,b_0) *be an eigenvector of the matrix* (1.6) *corresponding to the eigenvalue* λ*. There exists a unique entire function* $\big(x^u(\xi), y^u(\xi)\big)$ *satisfying*

(1) $\big(x^u(0), y^u(0)\big) = (0, 0)$,

(2) $\big(x^{u\prime}(0), y^{u\prime}(0)\big) = (a_0, b_0)$,

(3) *the map* $\varphi^u : \xi \mapsto \big(x^u(\xi), y^u(\xi)\big)$ *is an injective immersion of* $\mathbb{C}$ *into* $\mathbb{C}^2$,

(4) $\varphi^u(\mathbb{C}) = W^u$,

(5) φ^u *conjugates the linear dilatation* $\xi \mapsto \lambda\xi$ *with the restriction of HM onto* W^u*,* λ *being given by the formulae* (1.1),

(6) *the map* $\varphi^s = R \circ \varphi^u$ *parameterizes the stable manifold in the similar sense, i.e., its image coincides with* W^s *and it conjugates the map* $\eta \mapsto \lambda^{-1}\eta$ *with the restriction of HM onto* W^s*.*

We shall find a representation of $x^u(\xi)$, $y^u(\xi)$ in the form of series

$$x^u(\xi) = \sum_{i=0}^{\infty} a_i \xi^{i+1}, \quad y^u(\xi) = \sum_{i=0}^{\infty} b_i \xi^{i+1}, \tag{1.8}$$

and prove that for some positive constant A the coefficients can be estimated as follows:

$$|a_i| \leqslant A^{i+1}, \quad |b_i| \leqslant A^{i+1}. \tag{1.9}$$

Remark. The only uncertainty in the construction of series (1.8) is the choice of the eigenvector (a_0, b_0). Taking another eigenvector $(a_0', b_0') = c(a_0, b_0)$, $c \neq 0$, is equivalent to rescaling of the variable

$$\xi \mapsto \xi' = c^{-1}\xi. \tag{1.10}$$

The substitution (1.10) gives all the diversity of the parameterizations of W^u satisfying condition (5). The same remark is true for the stable manifold.

§2. The stable foliation of HM in the real plane

In this section we investigate the phase portrait of HM in the real plane $\mathbb{R}^2$. We start with a simple proposition concerning the behaviour of the stable and the unstable manifolds in $\mathbb{R}^2$. For definiteness we suppose that a_0 in (1.8) is positive. We get the following.

Proposition 2. *All derivatives of the functions* $x^u(\xi)$, $y^u(\xi)$, $\xi \in \mathbb{R}^-$ *are negative.*

Proof. This proposition follows immediately from equations (1.1) and from the symmetries $(1.4^{(\prime)})$–$(1.5^{(\prime)})$. □

Corollary. *The curves* $W^u \cap \mathbb{R}^- \times \mathbb{R}$ *and* $W^u \cap \mathbb{R}^- \times \mathbb{R}$ *are the graph of a strictly decreasing function.*

Now let us deal with geometrical aspects of the phase plane of HM. The complement $\mathbb{R}^2 \backslash (W^u \cup W^s)$ consists of four components. Denote them by Ω_I, Ω_{II}, Ω_{III}, and Ω_{IV} as is shown on Fig. 1.

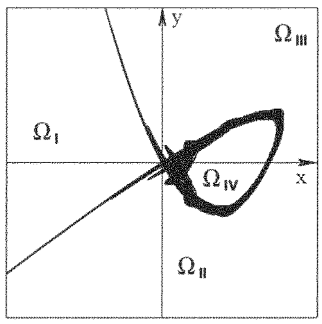

Fig. 1.

The origin (0,0) divides each curve $W^u \cap \mathbb{R}^2$ and $W^s \cap \mathbb{R}^2$ into two components. We denote the latter by $W^{u,s}_{+,-}$ where the sign $+,-$ reflects the half plane $y > 0$ or $y < 0$ in which the component starts.We obtain the following decomposition of $\mathbb{R}^2$ into eight invariant subsets

$$\mathbb{R}^2 = \{(0,0)\} \cup W^u_+ \cup W^u_- \cup W^s_+ \cup W^s_- \cup \Omega_I \cup (\Omega_{II} \cup \Omega_{III}) \cup \Omega_{IV}. \quad (2.1)$$

Reversor R given by (1.4) and the symmetry S defined by (1.4$'$) permute the decomposition (2.1).

Given $(x,y)\in\mathbb{R}^2$, $n\in\mathbb{Z}$, we denote

$$(x_n, y_n) = HM^n(x,y). \tag{2.2}$$

Proposition 3. *If* $(x,y)\in W_-^u\cup\Omega_I\cup\Omega_{II}\cup\Omega_{III}$ *then* $\lim_{n\to\infty}(x_n,y_n) = (-\infty,-\infty)$.

Proof. If $(x,y)\in W_-^u$ then the assertion is true due to Proposition 2 and item (5) of Theorem 1.

The rest of the assertion follows from the equations

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1},\\ y_{n+1} &= y_n + \varepsilon x_n(1-x_n). \end{aligned} \tag{2.3}$$

We exclude not large enough values of x and y from domain Ω_{III}. Proposition becomes obvious. □

Differentiating (1.1) yields

$$\frac{dx_1}{dy_1} = 1 + \cfrac{1}{\varepsilon(1-2x) + \cfrac{1}{\frac{dx}{dy}}}. \tag{2.4}$$

Define a function T on $\mathbb{R}^2$ as the following infinite continued fraction

$$\begin{aligned} T(x,y) &= 1 + \cfrac{1}{\varepsilon(1-2x_{-1}) + \cfrac{1}{1 + \cfrac{1}{\varepsilon(1-2x_{-2}) + \dots}}} = \\ &= [1,\ \varepsilon(1-2x_{-1}),\ 1,\ \varepsilon(1-2x_{-2}),\ 1,\ \dots]. \end{aligned} \tag{2.5}$$

Note that if $x<1/2-\delta$, $0<\delta<1/2$, all the entries of continued fraction (2.5) are positive and bounded from below by a constant δ. Let $T_n^+(x,y)$, $T_n^-(x,y)$ denote the truncated continuous fractions:

$$T_n^+(x,y) = [1,\ \varepsilon(1-2x_{-1}),\ 1,\ \varepsilon(1-2x_{-2}),\ 1,\ \dots,\ 1,\ \varepsilon(1-2x_{-n})], \tag{2.6}$$

$$T_n^-(x,y) = [1,\ \varepsilon(1-2x_{-1}),\ 1,\ \varepsilon(1-2x_{-2}),\ 1,\ \dots,\ \varepsilon(1-2x_{-n}),\ 1].$$

From the theory of continued fractions it follows [2] that there exists the limit

$$\lim_{n\to\infty} T_n^+(x,y) = \lim_{n\to\infty} T_n^-(x,y) = T(x,y) \tag{2.7}$$

and

$$T_n^- < T(x,y) < T_n^+, \qquad \text{for all } n. \tag{2.8}$$

Simple manipulations with finite fractions give us the inequality

$$1 < T(x,y) < 1 + \frac{1}{\varepsilon}. \tag{2.9}$$

Proposition 4. *$T(x,y)$ is C^1 at $(x,y) \in \mathbb{R}^2 \backslash (W^u \cup \Omega_{III})$.*

Remark. It have to be proved later that $T(x,y)$ is not differentiable at points of W^u.

Proof of Proposition 4. From (2.5) and (2.6) it follows that

$$T_n^-(x,y) = 1 + \cfrac{1}{\varepsilon(1-2x_{-1}) + \cfrac{1}{1 + \cfrac{1}{T_{n-1}^-(x_{-1},y_{-1})}}}, \qquad n = 1,\ 2, \dots, \qquad T_0^- = 1. \tag{2.10}$$

Denote

$$D_1 T_n^-(x,y) \equiv \frac{\partial T_n^-}{\partial x}(x,y), \quad D_2 T_n^-(x,y) \equiv \frac{\partial T_n^-}{\partial y}(x,y), \quad \vec{v}_n = \begin{pmatrix} D_1 T_n^- \\ D_2 T_n^- \end{pmatrix}.$$

Rewriting (1.1) as

$$x_{-1} = x - y,$$
$$y_{-1} = y - \varepsilon x(1-x)$$

we can explicitly differentiate (2.10) and thus obtain

$$\frac{\partial T_n^-}{\partial x} = \frac{D_1 T_{n-1}^- - \varepsilon(1-2x_{-1})\, D_2 T_{n-1}^- + 2\varepsilon (T_{n-1}^-)^2}{(T_{n-1}^- \varepsilon(1-2x_{-1}) + 1)^2},$$
$$\frac{\partial T_n^-}{\partial y} = \frac{-D_1 T_{n-1}^- + \big(1 + \varepsilon(1-2x_{-1})\big)\, D_2 T_{n-1}^- - 2\varepsilon (T_{n-1}^-)^2}{(T_{n-1}^- \varepsilon(1-2x_{-1}) + 1)^2}.$$

To simplify these formulae we denote

$$M(x_{-1},y_{-1}) = \frac{1}{(T_{n-1}^- \varepsilon(1-2x_{-1}) + 1)^2} \begin{pmatrix} 1 & -\varepsilon(1-2x_{-1}) \\ -1 & 1 + \varepsilon(1-2x_{-1}) \end{pmatrix},$$
$$\vec{f}(x_{-1},y_{-1}) = \frac{2\varepsilon (T_{n-1}^-)^2}{(T_{n-1}^- \varepsilon(1-2x_{-1}) + 1)^2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

One can easily establish the following estimates:

$$\|M(x_{-1}, y_{-1})\| \leqslant \frac{1}{1+\varepsilon}, \quad |\vec{f}(x_{-1}, y_{-1})| \leqslant \frac{2}{\varepsilon}. \tag{2.11}$$

Now $\vec{v}_n$ can be calculated by the recurrence

$$\vec{v}_n(x,y) = M(x_{-1}, y_{-1})\, \vec{v}_{n-1}(x_{-1}, y_{-1}) + \vec{f}(x_{-1}, y_{-1})$$

where $\vec{v}_0 = 0$ since

$$T_0^- = 1, \ \vec{v}_1 = \vec{f}(x_{-1}, y_{-1}),$$

$$\vec{v}_2 = M(x_{-1}, y_{-1})\, \vec{f}(x_{-2}, y_{-2}) + \vec{f}(x_{-1}, y_{-1}),$$

etc.

Iterating we obtain

$$\vec{v}_n(x,y) = \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} M(x_{-j}, y_{-j}) \right) \vec{f}(x_{-k}, y_{-k}).$$

Estimates (2.11) give

$$\left| \sum_{k=n}^{m} \left(\prod_{j=1}^{k-1} M(x_{-j}, y_{-j}) \right) \vec{f}(x_{-k}, y_{-k}) \right| \leqslant \left(\frac{1}{1+\varepsilon} \right)^{n-1} \frac{2}{\varepsilon^2}$$

which implies uniform convergence of the series to

$$\vec{v}(x,y) = \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} M(x_{-j}, y_{-j}) \right) \vec{f}(x_{-k}, y_{-k}).$$

Hence C^1 is proved. □

Consider the vector field

$$\vec{v}^u(x,y) = \begin{pmatrix} T(x,y) \\ 1 \end{pmatrix}. \tag{2.12}$$

It follows

$$(HN_*^{-1}\, \vec{v}^u)(x,y) = \tilde{\lambda}(x,y)\, \vec{v}^u(x,y) \tag{2.13}$$

from (2.10) where

$$\tilde{\lambda}(x,y) = 1 + \varepsilon(1-2x)\,T(x,y) \gg 1,$$

provided that x_{-1} is large enough. Since T satisfies condition (2.9), this means that HM^{-1} strongly contracts the lines in the direction of $\vec{v}^u$ with very large contracting coefficient (2.13). In other words the direction of $\vec{v}^u$ is unstable direction.

Further we can make an analytic continuation of $\vec{v}^u$. Since $\vec{v}^u$ is C^1 on the plane, may be with a cut through the origin, except at the points of $W^u \cap \mathbb{R}^2$, this field can be integrated and thus we obtain the real unstable foliation $\mathcal{W}^u$ on $\mathbb{R}^2$, may be with a cut which is of class C^1. Applying the reversor to this foliation we produce the stable foliation.

It may be shown the leaves of either foliation, $\mathcal{W}^u$ and $\mathcal{W}^s$, are analytic curves suffering a singularity when approaching the corresponding manifold, W^u or W^s. More precisely the foliation $\mathcal{W}^u$ is C^1 at $\mathbb{R}\backslash W^u$ and it is continuous at W^u, but its derivative is discontinuous at W^u. The same is true for $\mathcal{W}^s$ with respect to W^s.

References

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