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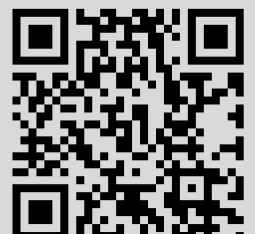
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INDUCTIVE SYSTEMS OF REPRESENTATIONS WITH SMALL HIGHEST WEIGHTS FOR NATURAL EMBEDDINGS OF SYMPLECTIC GROUPS

A. A. Osinovskaya, I. D. Suprunenko

Institute of Mathematics
of the National Academy of Sciences of Belarus
e-mail: anna@im.bas-net.by, suprunenko@im.bas-net.by
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For natural embeddings of symplectic groups, inductive systems of irreducible representations where the maximum of the highest weight value on the maximal root is equal to 2 are studied. For such embeddings of algebraic groups of type C_n in characteristic 3, the inductive system of representations generated by irreducible representations with highest weight $2\omega_n$ is determined. It is proved that any inductive system of representations of such groups consisting of representations with the value of the highest weight on the maximal root at most 2 and containing representations with such value equal to 2 contains the subsystem generated by the standard representations or the subsystem generated by the representations with highest weight ω_n . For algebraic groups of type C_n in characteristic 3, the restrictions of certain irreducible modules to subsystem subgroups of type C_{n-1} are described.

For natural embeddings of symplectic groups, inductive systems of irreducible representations where the maximum of the highest weight value on the maximal root is equal to 2 are studied. For such embeddings of algebraic groups of type C_n in characteristic 3, the inductive system of representations generated by irreducible representations with highest weight $2\omega_n$ is determined. It is proved that any inductive system of representations of such groups consisting of representations with the value of the highest weight on the maximal root at most 2 and containing representations with such value equal to 2 contains the subsystem generated by the standard representations or the subsystem generated by the representations with highest weight ω_n . The proof of the main result is based on an explicit description of the restrictions of certain irreducible modules for algebraic groups of type C_n in characteristic 3 to subsystem subgroups of type C_{n-1} .

In what follows K is an algebraically closed field of an odd characteristic p , \mathbb{Z} is the set of all integers, \mathbb{N} is the set of positive integers, \mathbb{C} is the field of complex numbers, $G_n = C_n(K)$, ω_i^n and α_i^n with $1 \leq i \leq n$ are the fundamental weights and the simple roots of G_n , $\langle \mu, \alpha \rangle$ is the value of a weight μ on a root α , $\varphi(\omega)$ is the irreducible representation with highest weight ω , $M(\omega)$ is the module affording $\varphi(\omega)$, $\varphi \downarrow H$ ($M \downarrow H$) is the restriction of a representation φ (a module M) to a subgroup H , M^μ is the weight component of weight μ in a module M , $\text{Irr } \varphi$ is the set of composition factors of a representation φ (multiplicities are not taken into account).

Let

$$\Gamma_1 \subset \Gamma_2 \subset \dots \subset \Gamma_n \subset \dots \quad (1)$$

be a series of fixed embeddings of algebraic groups Γ_n over K and Φ_n be a nonempty finite set of irreducible rational representations of Γ_n . A collection $\Phi = \{\Phi_n | n \in \mathbb{N}\}$ is called an *inductive system* of representations for the series (1) if

$$\bigcup_{\varphi \in \Phi_{n+1}} \text{Irr}(\varphi \downarrow \Gamma_n) = \Phi_n$$

for all $n \in \mathbb{N}$. The notion of an inductive system of representations was introduced by A. Zalesskii in [12]. Such systems can be regarded as an asymptotic version of the branching rules for the embeddings (1). Observe that in positive characteristic one cannot expect to find explicit analogues of the classical branching rules in characteristic 0 which have quite a lot of applications, so their asymptotic versions can be useful. Moreover, inductive systems can be applied to the study of ideals in group algebras of locally finite groups.

So far we know little about the structure of inductive systems. Minimal and minimal nontrivial inductive systems of modular representations for natural embeddings of special linear groups were classified by A. Baranov and I. Suprunenko [2]. In this case only the system that consists of the trivial representations is minimal, all inductive systems contain it; all minimal nontrivial inductive systems consist of nontrivial irreducible representations of the minimal dimension and trivial ones. For other classical groups the question on the minimal inductive systems is substantially more difficult and is far from solution. In [1] for the natural embeddings of the classical groups, inductive systems of representations with totally bounded weight multiplicities are classified and an analogue of the Steinberg tensor product theorem for arbitrary indecomposable inductive systems for such embeddings is proved.

Let $l \in \mathbb{N}$ or $l = 0$ and φ_n be an irreducible representation of Γ_n for $n > l$. We say that an inductive system of representations $\Phi = \{\Phi_n | n \in \mathbb{N}\}$ is generated by a collection $\{\varphi_n | n > l\}$ if $\varphi_n \in \Phi_n$ for $n > l$ and for any representation $\rho \in \Phi_n$ there exists $m > \max\{n, l\}$ such that $\rho \in \text{Irr}(\varphi_m \downarrow \Gamma_n)$. In this situation we shall use the notation $\Phi = \langle \varphi_n | n > l \rangle$. It follows from [1, Lemma 4.3] that the system Φ is correctly determined.

It is well known that for $n > k$ the root subgroups in G_n associated with the roots

$$\pm\alpha_{n-k+1}^n, \dots, \pm\alpha_n^n$$

generate a subgroup isomorphic to G_k . Identifying G_k with such subgroup, we get a series of natural embeddings

$$G_1 \subset G_2 \subset \dots \subset G_n \subset \dots \quad (2)$$

In what follows we consider only inductive systems for the series (2).

Theorem 1. *Let $p = 3$. Then*

$$\langle \varphi(2\omega_n^n) | n > 0 \rangle_n = \{0, \varphi(\omega_1^n), \varphi(\omega_{n-2}^n + \omega_n^n), \varphi(2\omega_k^n), 1 \leq k \leq n, \varphi(\omega_j^n + \omega_{j+1}^n), 1 \leq j \leq n-1\}.$$

Corollary 1. *Let $p = 3$. Then $\langle \varphi(2\omega_{n-1}^n) | n > 1 \rangle = \langle \varphi(\omega_{n-1}^n + \omega_n^n) | n > 1 \rangle$.*

Let $\delta(\Phi)$ be the maximum of the values of the highest weights of representations from an inductive system Φ on the maximal roots of the corresponding groups. It follows from [1, Lemma 4.1] that the parameter $\delta(\Phi)$ is defined correctly. For $n \in \mathbb{N}$ set

$$\mathcal{R}_n^u = \{\omega_i^n | n+1-u \leq i \leq n\}.$$

By [3, Theorem 1.3], \mathcal{R}^{p-1} is a minimal inductive system.

The following lemma holds for all odd p .

Lemma 1. *Let $p > 2$ and Φ be an inductive system of representations for the series (2) with $\delta(\Phi) = 2$. Then Φ contains the subsystem generated by the standard representations, or the subsystem $\langle \varphi(2\omega_n^n) | n > 0 \rangle$, or \mathcal{R}^{p-1} .*

Corollary 2. *Let $p = 3$ and Φ be an inductive system of representations for the series (2) with $\delta(\Phi) = 2$. Then Φ contains the subsystem generated by the standard representations or \mathcal{R}^{p-1} .*

For $\omega \in \{2\omega_j^n, 1 \leq j \leq n, \omega_k^n + \omega_{k+1}^n, 1 \leq k \leq n-1, \omega_{n-2}^n + \omega_n^n\}$ the restrictions $\varphi(\omega) \downarrow G_{n-1}$ are described (Lemma 5).

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In what follows for the root $\pm\alpha_j^n$, $t \in K$ and $d \in \mathbb{N}$ denote by $X_{\pm j}$, $x_{\pm j}(t)$, $\mathcal{X}_{\pm j}$, and $X_{\pm j, d}$ the root element in the Lie algebra of G_n , the root element in G_n , the root subgroup in G_n , and the element in the hyperalgebra of G_n associated with $\pm\alpha_j^n$, t , and d , respectively; W is the Weyl group of G_n . It will always be clear from the context what group is considered. Denote by $G_n(\beta_1, \dots, \beta_k)$ the subgroup in G_n generated by all root subgroups for roots $\pm\beta_1, \dots, \pm\beta_k$. Set $G_n(i_1, \dots, i_k) = G_n(\alpha_{i_1}, \dots, \alpha_{i_k})$. Throughout the text if $H \subset G$ are semisimple algebraic groups with fixed maximal tori $T_1 \subset H$ and $T \subset G$ and $T_1 \subset T$, then $\lambda \downarrow H$ is the restriction of a weight λ from T to T_1 . For a G_n -module M denote by v^+ a highest weight vector, $\omega(v)$ is the weight of a weight vector v , $\omega_H(v) = \omega(v) \downarrow H$, $\Lambda(M)$ is the set of weights of M . Below $\omega(M)$ ($\omega(\varphi)$) is the highest weight of a module M (a representation φ). It is well known that

$$x_{\pm j}(t)(v) = v + \sum_{d=1}^{\infty} X_{\pm j, d} t^d(v)$$

and that $X_{\pm j, d} = X_{\pm j}^d / d!$ for $d < p$ (see, for instance, [4, Proposition 5.13]). We shall use this facts frequently without special comments.

We need some preliminary lemmas to prove the results stated above.

Lemma 2 [4, Lemma 5.14], [8, 1.5], and [10, 2.1].

(i) *For the operators $X_{i, d}$ the following equality holds:*

$$X_{-i} X_{i, d} = X_{i, d} X_{-i} - H_i X_{i, d-1} + (d-1) X_{i, d-1}. \quad (3)$$

Here $H_i = [X_i, X_{-i}]$.

(ii) *Let M be a G_n -module, μ be a weight of G_n , $v \in M^\mu \setminus \{0\}$, $X_{i, b} v = 0$ for $b > 0$, and $\langle \mu, \alpha_i \rangle = c \geq 0$. Then $X_i X_{-i, b} v = (c - b + 1) X_{-i, b-1} v$ and $X_{-i, c} v \neq 0$. In particular, if $0 < c < p$, one has $X_{-i, d} v \neq 0$ for $0 < d \leq c$.*

Theorem 2 (Jantzen [7], Smith [9]). *Let $H \subset G_n$ be the subgroup generated by $\mathcal{X}_{\pm i_1}, \dots, \mathcal{X}_{\pm i_j}$ and M be an irreducible G_n -module. Then $KHv^+ \subset M$ is an irreducible H -module with highest weight $\omega_H(v^+)$ and a direct summand of the H -module M .*

Lemma 3 [11, Lemma 2.46]. *Let G be a simple algebraic group over K of type $A_r, B_r,$ or C_r , ω_i be the fundamental weights of G , and α_i be its simple roots ($1 \leq i \leq r$). Suppose that V is an indecomposable G -module with highest weight $\sum_{k=1}^r m_k \omega_k$ and $v \in V$ is a nonzero highest weight vector. Let $1 \leq l, j \leq r$. Set $b_k = -\langle \alpha_{k+1}, \alpha_k \rangle$ and $c_k = -\langle \alpha_{k-1}, \alpha_k \rangle$. For an integer d with $0 < d \leq m_j$ define the vector $v(l, j, d)$ as follows. Put $d_j = d$. If $l < j$, set $d_k = m_k + d_{k+1} b_k$ for $l \leq k < j$. If $l > j$, put $d_k = m_k + d_{k-1} c_k$ for $l \geq k > j$. Now set*

$$v(l, j, d) = X_{-l, d_l} \dots X_{-k, d_k} \dots X_{-j, d} v.$$

For $l = j$ put $v(l, j, d) = X_{-l, d} v$. Assume that $0 < m_j < p$ or $d = m_j$. Then $v(l, j, d) \neq 0$ and $X_{t, b} v(l, j, d) = 0$ for positive $t \neq l$ and $b > 0$. Hence \mathcal{X}_t fixes $v(l, j, d)$.

Until the end of the paper $p = 3$, except Lemmas 1 and 6.

Lemma 4.

- 1) Let $\omega = 2\omega_2^2$ and $M = M(\omega)$. Then there exists a nonzero vector $v \in M$ with $\omega(v) = \omega - 2\alpha_1^2 - 2\alpha_2^2$ that is fixed by \mathcal{X}_2 .
- 2) Let $\omega = \omega_1^3 + \omega_3^3$ and $M = M(\omega)$. Then there exists a nonzero vector $w \in M$ with $\omega(v) = \omega - 2\alpha_1^3 - 2\alpha_2^3 - \alpha_3^3$ that is fixed by \mathcal{X}_i for $i = 2$ and 3 .

Proof. Set $\lambda = \omega - 2\alpha_1^2 - 2\alpha_2^2$ in Case 1 and $\lambda = \omega - 2\alpha_1^3 - 2\alpha_2^3 - \alpha_3^3$ in Case 2; put $\lambda' = \lambda + \alpha_2^n$. We shall show that in both cases $\dim M^\lambda > \dim M^{\lambda'}$. This implies that there exists a nonzero vector $v \in M^\lambda$ such that $X_2v = 0$. Hence \mathcal{X}_2 fixes v .

Case 1. Put

$$\begin{aligned} v_1 &= X_{-2}(X_{-1})^2X_{-2}v^+, \\ v_2 &= X_{-1}X_{-2}X_{-1}X_{-2}v^+. \end{aligned}$$

We claim that v_1 and v_2 are linearly independent. Suppose that $w = av_1 + bv_2 = 0$ with $a, b \in K$. By Formula (3), $X_2w = (2a + b)(X_{-1})^2X_{-2}v^+$ and $(X_1)^2w = (4a + 12b)(X_{-2})^2v^+$. Several applications of Lemma 2(ii) imply that $(X_{-1})^2X_{-2}v^+$ and $(X_{-2})^2v^+ \neq 0$. Hence $2a + b = 4a + 12b = 0$ which forces $a = b = 0$. Therefore v_1 and v_2 are linearly independent and $\dim M^\lambda \geq 2$. Observe that the weight λ' lies in the same W -orbit with the weight $\omega - \alpha_2^2$ and hence $\dim M^{\lambda'} = 1$. This completes the proof in Case 1.

Case 2. We claim that $\lambda + \alpha_3^3 \notin \Lambda(M)$. Indeed, $\lambda + \alpha_3^3$ lies in the same W -orbit with the weight $\mu = \omega - 2\alpha_1^3$. Since $\langle \mu, \alpha_1^3 \rangle = -3$, it is clear that μ and $\lambda + \alpha_3^3 \notin \Lambda(M)$. Hence \mathcal{X}_3 fixes all vectors of weight λ . Since λ' is in the same W -orbit with $\omega - \alpha_2^3 - \alpha_3^3$, it is clear that $\dim M^{\lambda'} = 1$. Set $\nu = \omega - \alpha_1^3 - \alpha_2^3 - \alpha_3^3$. Since λ and ν are in the same W -orbit, we have $\dim M^\lambda = \dim M^\nu$. Put

$$\begin{aligned} w_1 &= X_{-3}X_{-2}X_{-1}v^+, \\ w_2 &= X_{-1}X_{-2}X_{-3}v^+. \end{aligned}$$

Suppose that $w = aw_1 + bw_2 = 0$ with $a, b \in K$. Then $X_2w = X_3w = 0$. Several applications of Lemma 2(ii) yield that $X_{-2}X_{-1}v^+$ and $X_{-1}X_{-3}v^+ \neq 0$. Since $X_2w = (a + 2b)X_{-1}X_{-3}v^+$ and $X_3w = 2aX_{-2}X_{-1}v^+$ by Formula (3), we conclude that $a = b = 0$. Hence w_1 and w_2 are linearly independent and $\dim M^\lambda = \dim M^\nu > 2$. This completes the proof.

Put $\omega_0^n = \omega_{-k}^n = 0$ for $k \in \mathbb{N}$.

Lemma 5. Let $n > 1$. Then

- 1) $\varphi(2\omega_n^n) \downarrow G_{n-1} \cong 3\varphi(2\omega_{n-1}^{n-1}) \oplus 2\varphi(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus \varphi(2\omega_{n-2}^{n-1})$;
- 2) $\varphi(2\omega_{n-1}^n) \downarrow G_{n-1} \cong \varphi(2\omega_{n-1}^{n-1}) \oplus 2\varphi(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus 3\varphi(2\omega_{n-2}^{n-1}) \oplus 2\varphi(\omega_{n-3}^{n-1} + \omega_{n-2}^{n-1}) \oplus \varphi(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1}) \oplus \varphi(2\omega_{n-3}^{n-1})$ for $n > 2$;
 $\varphi(2\omega_1^2) \downarrow G_1 \cong 6\varphi(0) \oplus 2\varphi(\omega_1^1) \oplus \varphi(2\omega_1^1)$;
- 3) $\varphi(2\omega_k^n) \downarrow G_{n-1} \cong \varphi(2\omega_k^{n-1}) \oplus 2\varphi(\omega_{k-1}^{n-1} + \omega_k^{n-1}) \oplus 3\varphi(2\omega_{k-1}^{n-1}) \oplus 2\varphi(\omega_{k-2}^{n-1} + \omega_{k-1}^{n-1}) \oplus \varphi(2\omega_{k-2}^{n-1})$ for $1 \leq k \leq n-2$;
- 4) $\varphi(\omega_{n-1}^n + \omega_n^n) \downarrow G_{n-1} \cong 2\varphi(2\omega_{n-1}^{n-1}) \oplus 4\varphi(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus 2\varphi(2\omega_{n-2}^{n-1}) \oplus \varphi(\omega_{n-3}^{n-1} + \omega_{n-2}^{n-1}) \oplus 2\varphi(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})$ for $n > 2$;
 $\varphi(\omega_1^2 + \omega_2^2) \downarrow G_1 \cong 3\varphi(0) \oplus 4\varphi(\omega_1^1) \oplus 2\varphi(2\omega_1^1)$;

- 5) $\varphi(\omega_k^n + \omega_{k+1}^n) \downarrow G_{n-1} \cong \varphi(\omega_k^{n-1} + \omega_{k+1}^{n-1}) \oplus 2\varphi(2\omega_k^{n-1}) \oplus 3\varphi(\omega_{k-1}^{n-1} + \omega_k^{n-1}) \oplus 2\varphi(2\omega_{k-1}^{n-1}) \oplus \varphi(\omega_{k-2}^{n-1} + \omega_{k-1}^{n-1})$ for $1 \leq k \leq n-2$;
- 6) $\varphi(\omega_{n-2}^n + \omega_n^n) \downarrow G_{n-1} \cong \varphi(2\omega_{n-2}^{n-1}) \oplus 2\varphi(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus 3\varphi(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})$ for $n > 2$;
 $\varphi(\omega_2^2) \downarrow G_1 = 2\varphi(\omega_1^1) \oplus \varphi(0)$.

Here and below the expression $\varphi \downarrow G_{n-1} \cong a_1\varphi_1 \oplus \dots \oplus a_k\varphi_k$ means that such restriction is a direct sum of a_1 copies of φ_1 , \dots , and a_k copies of φ_k .

Proof. Set $\Gamma_n = SL(2n, K)$. Denote by $\theta(\mu)$ the irreducible representation of Γ_n with highest weight μ and by λ_i^n , $1 \leq i \leq 2n-1$, the fundamental weights of Γ_n . We assume that $\lambda_0^n = \lambda_{-k}^n = 0$ for $k \in \mathbb{N}$ and $\lambda_2^1 = 0$. The group $G_n = Sp(2n, K)$ can be embedded into Γ_n in a natural way. There exist maximal tori T_1 and T in G_n and Γ_n , respectively, such that $T_1 = T \cap G_n$. One can assume that $\lambda_i^n \downarrow T_1 = \omega_i^n$ for $1 \leq i \leq n$ and $\lambda_i^n \downarrow T_1 = \omega_{2n-i}$ for $n < i \leq 2n-1$. By [14, Proposition 2.2], for $n > 1$

$$\theta(2\lambda_n^n) \downarrow G_n \cong \varphi(2\omega_n^n) \oplus \varphi(\omega_{n-2}^n + \omega_n^n), \quad (4)$$

$$\theta((2-j)\lambda_k^n + j\lambda_{k+1}^n) \downarrow G_n \cong \varphi((2-j)\omega_k^n + j\omega_{k+1}^n) \quad (5)$$

for $j \in \{0, 1\}$, $k \in \{0, \dots, n-1\}$, and

$$\theta((2-j)\lambda_k^n + j\lambda_{k+1}^n) \downarrow G_n \cong \varphi((2-j)\omega_{2n-k}^n + j\omega_{2n-k-1}^n) \quad (6)$$

for $j \in \{0, 1\}$, $k \in \{n, \dots, 2n-1\}$, $(2-j)\lambda_k^n + j\lambda_{k+1}^n \neq 2\lambda_n^n$. It is clear that

$$\theta(2\lambda_1^1) \downarrow G_1 \cong \varphi(2\omega_1^1), \quad \theta(\lambda_1^1) \downarrow G_1 \cong \varphi(\omega_1^1)$$

since $G_1 = \Gamma_1$.

It follows from [14, Corollary 1.5] that

$$\theta(2\lambda_n^n) \downarrow \Gamma_{n-1} \cong \theta(2\lambda_{n-2}^{n-1}) \oplus 2\theta(\lambda_{n-2}^{n-1} + \lambda_{n-1}^{n-1}) \oplus 3\theta(2\lambda_{n-1}^{n-1}) \oplus 2\theta(\lambda_{n-1}^{n-1} + \lambda_n^{n-1}) \oplus \theta(2\lambda_n^{n-1}).$$

Combining the formulas above, one can deduce that

$$(\varphi(2\omega_n^n) \oplus \varphi(\omega_{n-2}^n + \omega_n^n)) \downarrow G_{n-1} \cong 2\varphi(2\omega_{n-2}^{n-1}) \oplus 4\varphi(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus 3\varphi(2\omega_{n-1}^{n-1}) \oplus 3\varphi(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1}) \quad (7)$$

for $n > 2$ and

$$(\varphi(2\omega_2^2) \oplus \varphi(\omega_2^2)) \downarrow G_1 \cong 2\varphi(0) \oplus 4\varphi(\omega_1^1) \oplus 3\varphi(2\omega_1^1). \quad (8)$$

It follows from results of [13] (Formula (1) and the proof of the main theorem) that

$$\varphi(\omega_2^2) \downarrow G_1 \cong 2\varphi(\omega_1^1) \oplus \varphi(0).$$

Therefore (8) yields that

$$\varphi(2\omega_2^2) \downarrow G_1 \cong \varphi(0) \oplus 2\varphi(\omega_1^1) \oplus 3\varphi(2\omega_1^1).$$

Set $U_1 = M(2\omega_n^n) \downarrow G_{n-1}$ and $U_2 = M(\omega_{n-2}^n + \omega_n^n) \downarrow G_{n-1}$. To prove Items 1 and 6 of the lemma for $n > 2$, we shall construct explicitly certain composition factors of U_1 and U_2 . By (7), U_1 and U_2 are completely reducible. Below in this proof $M = M(2\omega_n^n)$ or $M(\omega_{n-2}^n + \omega_n^n)$, ω is the highest weight of M . For a fixed nonnegative integer j denote by Λ_j the subset in $\Lambda(M)$ consisting of the weights of the form $\omega - j\alpha_1^n - \sum_{i=2}^n x_i\alpha_i^n$ and by M_j the subspace in M generated by all M^μ with $\mu \in \Lambda_j$. It is clear that M_j is a direct summand of the G_{n-1} -module M (naturally,

$M_j = 0$ for large enough j). Hence if $L \in \text{Irr } M_j$ for at least s different j , then the multiplicity of L as a composition factor of $M \downarrow G_{n-1}$ is at least s . Set $H = G_n(1, \dots, n-2)$.

First let $\omega = 2\omega_n^n$. By Theorem 2, $M(2\omega_{n-1}^{n-1})$ is a direct summand of U_1 . Put

$$\begin{aligned} v_1 &= X_{-1}X_{-2} \dots X_{-n}v^+, \\ v_2 &= X_{-1,2}X_{-2,2} \dots X_{-(n-1),2}X_{-n}v^+, \\ v_3 &= X_{-1,3}X_{-2,3} \dots X_{-(n-1),3}X_{-n,2}v^+, \\ v_4 &= X_{-1,4}X_{-2,4} \dots X_{-(n-1),4}X_{-n,2}v^+. \end{aligned}$$

We claim that the vectors $v_i \neq 0$ and are fixed by \mathcal{X}_l for $2 \leq l \leq n$. For v_2 and v_4 this follows from Lemma 3. Set $v'_1 = X_{-(n-1)}X_{-n}v^+$ and $v'_3 = X_{-(n-1),3}X_{-n,2}v^+$. By Lemma 2(ii), $v'_1 \neq 0$. Using Formula (3) and Lemma 2(ii), we conclude that $X_nX_{n-1}v'_3 = 2X_{-(n-1),2}X_{-n}v^+ \neq 0$. Hence $v'_3 \neq 0$. Consider v_1 and v_3 as vectors in the H -modules generated by v'_1 and v'_3 , respectively, and conclude that $v_1 = v'_1(1, n-2, 1)$ and $v_3 = v'_3(1, n-2, 3)$. Then Lemma 3 implies that v_1 and $v_3 \neq 0$ and are fixed by \mathcal{X}_j for $2 \leq j \leq n-2$. We claim that \mathcal{X}_{n-1} and \mathcal{X}_n fix v_1 and v_3 as well. It suffices to show that $X_{n-1}v_1 = X_nv_1 = X_{n-1,s}v_3 = X_nv_3 = 0$ with $1 \leq s \leq 3$. It is clear that $\omega(v_1) + \alpha_n^n \notin \Lambda(M)$ and hence $X_nv_1 = 0$. Set

$$\begin{aligned} u_1 &= X_{-(n-2)}X_{-(n-1)}X_{-n}v^+, \\ u_2 &= X_{-(n-1),3}X_{-n,2}v^+, \\ u_3 &= X_{-(n-2),3}u_2. \end{aligned}$$

Taking into account the commutation relations for the operators $X_{\pm i, k}$, we reduce the question to proving the following equalities: $X_{n-1}u_1 = X_nu_2 = X_{n-1,s}u_3 = 0$. Set $\mu = \omega(u_1) + \alpha_{n-1}^n$, $\lambda = \omega(u_2) + \alpha_n^n$, and $\nu(s) = \omega(u_3) + s\alpha_{n-1}^n$. Observe that $\langle \mu, \alpha_{n-2}^n \rangle = -2$, $\langle \lambda, \alpha_{n-1}^n \rangle = -4$, and $\langle \nu(s), \alpha_{n-2}^n \rangle < -3$. Since $\Lambda(M)$ is invariant under the action of W , this implies that μ , λ , and $\nu(s) \notin \Lambda(M)$ and yields the desired equalities. Hence \mathcal{X}_j fixes v_i for $1 \leq i \leq 4$ and $j > 1$.

Therefore v_i generates an indecomposable G_{n-1} -module V_i with highest weight $\delta_i = \omega_{G_{n-1}}(v_i)$. We have $\delta_1 = \delta_3 = \omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}$ and $\delta_2 = \delta_4 = 2\omega_{n-1}^{n-1}$.

By Lemma 4(1), there exists a nonzero vector $v \in M$ of weight $\omega - 2\alpha_{n-1}^n - 2\alpha_n^n$ that is fixed by \mathcal{X}_n . Put $u = X_{-1,2}X_{-2,2} \dots X_{-(n-2),2}v$. Applying Lemma 3 to the H -module generated by v , we conclude that $u \neq 0$ and is fixed by \mathcal{X}_j for $2 \leq j \leq n-2$. Observe that the weight $\omega(u) + \alpha_{n-1}^n$ lies in the same W -orbit with the weight $\tau = \omega(v) - 2\alpha_{n-2}^n + \alpha_{n-1}^n \notin \Lambda(M)$ (the latter is clear since $\langle \tau, \alpha_{n-2}^n \rangle = -3$). So \mathcal{X}_{n-1} fixes u as well. The group \mathcal{X}_n commutes with $X_{-j,d}$ for $j < n$ and so it fixes u as it fixes v . Therefore u generates an indecomposable G_{n-1} -module with highest weight $\omega_{G_{n-1}}(u) = 2\omega_{n-2}^{n-1}$. It is clear that $V_i \subset M_i$ for $1 \leq i \leq 4$. Now we can conclude that U_1 has at least 3 composition factors isomorphic to $M(2\omega_{n-1}^{n-1})$, at least 2 factors isomorphic to $M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1})$, and at least 1 factor isomorphic to $M(2\omega_{n-2}^{n-1})$.

Now let $\omega = \omega_{n-2}^n + \omega_n^n$. The arguments are close to those for $\omega = 2\omega_n^n$. By Theorem 2, $M(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})$ is a direct summand of U_2 (for $n = 3$ it has the form $M(\omega_2^2)$). Set

$$\begin{aligned} w_1 &= X_{-1}X_{-2} \dots X_{-(n-2)}v^+, \\ w_2 &= X_{-1,2}X_{-2,2} \dots X_{-(n-2),2}X_{-(n-1)}X_{-n}v^+, \\ w_3 &= X_{-1,3}X_{-2,3} \dots X_{-(n-2),3}X_{-(n-1),2}X_{-n}v^+, \\ w_4 &= X_{-1,4}X_{-2,4} \dots X_{-(n-3),4}X_{-(n-2),3}X_{-(n-1),3}X_{-n,2}X_{-(n-1)}X_{-(n-2)}v^+ \end{aligned}$$

for $n > 3$, and

$$w_4 = X_{-(n-2),3}X_{-(n-1),3}X_{-n,2}X_{-(n-1)}X_{-(n-2)}v^+$$

for $n = 3$.

We claim that $w_i \neq 0$ and are fixed by the groups \mathcal{X}_l for $2 \leq l \leq n$. For $i = 1$ and 3 this follows from Lemma 3. The proof for w_2 is similar to that for v_1 above. We conclude that $X_{-(n-1)}X_{-n}v^+ \neq 0$, then apply Lemma 3 to the relevant H -module and show that $w_2 \neq 0$ and is fixed by \mathcal{X}_l for $2 \leq l \leq n-2$, and finally show that $\omega(w_2) + \alpha_{n-1}^n$ and $\omega(w_2) + \alpha_n^n \notin \Lambda(M)$ and hence \mathcal{X}_{n-1} and \mathcal{X}_n fix w_2 as well.

Now put $S = G_n(1, \dots, n-1)$ and $m = X_{-n,2}X_{-(n-1)}X_{-(n-2)}v^+$. By Lemma 3, $m \neq 0$ and is fixed by \mathcal{X}_l for $l \neq n$. Next, we regard w_4 as a vector in the S -module generated by m and observe that $w_4 = m(1, n-1, 3)$. Then Lemma 3 implies that $w_4 \neq 0$ and is fixed by \mathcal{X}_l for $2 \leq l \leq n-1$. Observe that the weight $\omega(w_4) + \alpha_n^n$ lies in the same W -orbit with the weight $\gamma = \omega - \alpha_{n-2}^n - 4\alpha_{n-1}^n - \alpha_n^n$. Since $\langle \gamma, \alpha_{n-1}^n \rangle = -5 < -4$, we deduce that $\gamma \notin \Lambda(M)$. Hence $X_n w_4 = 0$ and so \mathcal{X}_n fixes w_4 .

Hence the vectors w_i generate indecomposable G_{n-1} -modules with highest weights $\mu_i = \omega_{G_{n-1}}(w_i)$. We have $\mu_1 = \mu_3 = \omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}$, $\mu_2 = 2\omega_{n-2}^{n-1}$, $\mu_4 = \omega_{n-3}^{n-1} + \omega_{n-1}^{n-1}$ for $n > 3$, and $\mu_4 = \omega_{n-1}^{n-1}$ for $n = 3$.

By Lemma 4(2), there exists a nonzero vector w of weight $\omega - 2\alpha_{n-2}^n - 2\alpha_{n-1}^n - \alpha_n^n$ that is fixed by the groups \mathcal{X}_{n-1} and \mathcal{X}_n . Put $t = X_{-1,2}X_{-2,2} \dots X_{-(n-3),2}w$ for $n > 3$ and $t = w$ otherwise. We claim that $t \neq 0$ and is fixed by \mathcal{X}_l for $2 \leq l \leq n$. For $n = 3$ this is trivial. Let $n > 3$. Here the arguments are similar to those for the vector u considered earlier in the case where $\omega = 2\omega_n^n$. First we apply Lemma 3 to the $G_n(1, \dots, n-3)$ -module generated by w and show that $t \neq 0$ and is fixed by the groups \mathcal{X}_l for $2 \leq l \leq n-3$. The weight $\omega(t) + \alpha_{n-2}^n$ lies in the same W -orbit with $\chi = \omega - 2\alpha_{n-3}^n - \alpha_{n-2}^n - 2\alpha_{n-1}^n - \alpha_n^n$. As $\langle \chi, \alpha_{n-3}^n \rangle = -3 < -2$, the weight $\chi \notin \Lambda(M)$. Hence $X_{n-2}t = 0$ and \mathcal{X}_{n-2} fixes t . Arguing as for the vector u and for the group \mathcal{X}_n earlier, we conclude that \mathcal{X}_{n-1} and \mathcal{X}_n fix t as well. Therefore t generates an indecomposable G_{n-1} -module with highest weight $\omega_{G_{n-1}}(t) = \omega_{n-3}^{n-1} + \omega_{n-1}^{n-1}$.

Since the vectors $w_i \in M_i$ for $1 \leq i \leq 4$ and $t \in M_2$, we can conclude that U_2 has at least 3 composition factors isomorphic to $M(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})$, at least 2 composition factors isomorphic to $M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1})$, and at least 1 composition factor isomorphic to $M(\omega_{n-2}^{n-1})$.

Now Formula (7) implies that

$$M(2\omega_n^n) \downarrow G_{n-1} \cong 3M(2\omega_{n-1}^{n-1}) \oplus 2M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus M(2\omega_{n-2}^{n-1});$$

and

$$M(\omega_{n-2}^n + \omega_n^n) \downarrow G_{n-1} \cong M(2\omega_{n-2}^{n-1}) \oplus 2M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1}) \oplus 3M(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1}).$$

This proves Items (1) and (6) of the lemma.

For $k < n$ it follows from [14, Corollary 1.5] that

$$\theta(2\lambda_k^n) \downarrow \Gamma_{n-1} \cong \theta(2\lambda_{k-2}^{n-1}) \oplus 2\theta(\lambda_{k-2}^{n-1} + \lambda_{k-1}^{n-1}) \oplus 3\theta(2\lambda_{k-1}^{n-1}) \oplus 2\theta(\lambda_{k-1}^{n-1} + \lambda_k^{n-1}) \oplus \theta(2\lambda_k^{n-1})$$

and

$$\begin{aligned} & \theta(\lambda_k^n + \lambda_{k+1}^n) \downarrow \Gamma_{n-1} \cong \\ & \cong \theta(\lambda_{k-2}^{n-1} + \lambda_{k-1}^{n-1}) \oplus 2\theta(2\lambda_{k-1}^{n-1}) \oplus 3\theta(\lambda_{k-1}^{n-1} + \lambda_k^{n-1}) \oplus 2\theta(2\lambda_k^{n-1}) \oplus \theta(\lambda_k^{n-1} + \lambda_{k+1}^{n-1}). \end{aligned}$$

Using Formulas (4)–(6), we prove Items (2)–(5) of the lemma.

We have proved that the restrictions of the modules mentioned in Lemma 5 to G_{n-1} are completely reducible. For $\varphi(\omega_1^n)$, $\varphi(2\omega_k^n)$ with $1 \leq k \leq n$, and $\varphi(\omega_j^n + \omega_{j+1}^n)$ with $1 \leq j \leq n-1$

this was proved earlier in [14]. The complete reducibility of the restriction $\varphi(\omega_{n-2}^n + \omega_n^n) \downarrow G_{n-1}$ is a new result.

Proof of Theorem 1. By Theorem 2, $\varphi(\omega_{n-i}^n + \omega_{n-j}^n) \in \text{Irr}(\varphi(\omega_{n+1-i}^{n+1} + \omega_{n+1-j}^{n+1}) \downarrow G_n)$ for $j \leq i \leq n$. Hence the inductive system $\langle \varphi(\omega_{n-i}^n + \omega_{n-j}^n) | n \geq i \rangle$ is correctly defined for fixed i and j with $j \leq i$.

Put

$$\mathcal{F}_n = \{0, \varphi(\omega_1^n), \varphi(\omega_{n-2}^n + \omega_n^n), \varphi(2\omega_k^n), 1 \leq k \leq n, \varphi(\omega_j^n + \omega_{j+1}^n), 1 \leq j \leq n-1\}.$$

First observe that $\mathcal{F} = \{\mathcal{F}_n | n \in \mathbb{N}\}$ is an inductive system by Lemma 5 and Theorem 2.

For $m > n$ set $\text{Irr}_n^m = \text{Irr}(\varphi(2\omega_m^m) \downarrow G_n)$. Let $n = k + l > k$. We claim that $\varphi(2\omega_k^n) \in \text{Irr}_n^{n+l}$. Use induction on l . For $l = 1$ the claim follows directly from Lemma 5(1). Now assume that $l > 1$ and the claim holds for $l - 1$. Then $\varphi(2\omega_{k+1}^{n+1}) \in \text{Irr}_{n+1}^{n+l}$. Items (2) and (3) of Lemma 5 yield that $\varphi(2\omega_k^n) \in \text{Irr}(\varphi(2\omega_{k+1}^{n+1}) \downarrow G_n)$ and complete the proof of the claim.

Similar arguments imply that $\varphi(\omega_k^n + \omega_{k+1}^n) \in \text{Irr}_n^{n+l}$ for $n = k + l$ and $l > 0$. Here we apply Lemma 5(1) for $l = 1$, then use induction on l and apply Items (4) and (5) of that lemma.

We have shown above that $\varphi(2\omega_1^{n+1}) \in \text{Irr}_{n+1}^{2n}$. Then Items (2) and (3) of Lemma 5 imply that $\varphi(\omega_1^n) \in \text{Irr}_n^{2n}$.

Recall that $\varphi(\omega_{n-2}^n + \omega_n^n) \in \text{Irr}_n^{n+1}$ for $n > 2$ by Lemma 5(6). It is well known that

$$\varphi(0) \in \text{Irr}(\varphi(\omega_1^{n+1}) \downarrow G_n).$$

This completes that proof since $\langle \varphi(2\omega_n^n) | n > 0 \rangle \subset \mathcal{F}$.

Proof of Corollary 1. This follows from Theorem 1 and the fact that $\varphi(2\omega_{n-1}^{n-1})$ is a composition factor of the restrictions $\varphi(2\omega_{n-1}^n) \downarrow G_{n-1}$ and $\varphi(\omega_{n-1}^n + \omega_n^n) \downarrow G_{n-1}$ by Lemma 5, Items (2) and (3).

The symbols ε_j^n , $1 \leq j \leq n$, denote the weights of the standard G_n -modules defined in [5, Ch. VIII, § 13].

Proof of Lemma 1. It is well known that for $l \geq j \geq n$ the subgroup

$$\Gamma = G_l(\alpha_{j-n+1}^l, \dots, \alpha_{j-1}^l, 2\varepsilon_j^l)$$

is conjugate to G_n . Hence $\text{Irr}(\varphi \downarrow \Gamma) = \text{Irr}(\varphi \downarrow G_n)$ for every representation φ of G_l .

First suppose that there exists an integer k such that for any $m > k$ the set Φ_m contains a representation $\varphi_m = \varphi(\omega_{m-j}^m + \omega_{m-i}^m)$ with $i \leq j \leq k$. Fix such k . For $n > 0$ and $l \geq n + k$ put

$$H_n = G_l(\alpha_{i-k-n+1}^l, \dots, \alpha_{i-k-1}^l, 2\varepsilon_{i-k}^l).$$

Observe that $\omega(\varphi_l) \downarrow H_n = 2\omega_n^n$. Hence $\varphi(2\omega_n^n) \in \text{Irr}(\varphi_l \downarrow H_n) \subset \Phi_n$ for any $n > 0$.

Now assume that there is no such k .

Suppose that for every n there exists m such that Φ_m contains a representation

$$\psi_m = \varphi(\omega_i^m + \omega_j^m)$$

with $i \leq j$ and $m - j \geq n$. In this case $\varphi(\omega_1^{m-j+1})$ or $\varphi(2\omega_1^{m-j+1}) \in \text{Irr}(\psi_m \downarrow G_{m-j+1})$ by Theorem 2. Lemma 5, Items (2) and (3), yields that $\varphi(\omega_1^{m-j}) \in \Phi_{m-j}$ and hence $\varphi(\omega_1^n) \in \Phi_n$.

Finally suppose that for some integer l the difference $m - j < l$ for all m and all

$$\varphi(\omega_i^m + \omega_j^m) \in \Phi_m.$$

Our previous assumptions imply that for every n there exist t and a representation

$$\chi = \varphi(\omega_i^t + \omega_j^t) \in \Phi_t$$

with $j - i \geq n$. Put

$$H_n = G_t(\alpha_{j-n+1}^t, \dots, \alpha_{j-1}^t, 2\varepsilon_j^t).$$

Then $\omega(\chi) \downarrow H_n = \omega_n^n$ and hence $\varphi(\omega_n^n) \in \text{Irr}(\chi \downarrow H_n) \subset \Phi_n$. The lemma is proved.

Proof of Corollary 2. The assertion follows from Theorem 1 and Lemma 1.

It is clear that the inductive system generated by the standard representations contains the trivial subsystem.

The following lemma implies that the inductive system $\langle \varphi(2\omega_n^n) | n > 0 \rangle$ in characteristic 3 strongly differs from the similar system in characteristic 0.

Lemma 6. *Let $n > 4$, $G_n^{\mathbb{C}} = C_n(\mathbb{C})$, and $M(\omega)^{\mathbb{C}}$ be the irreducible $G_n^{\mathbb{C}}$ -module with highest weight ω . Then*

- 1) $M(2\omega_n^n)^{\mathbb{C}} \downarrow G_{n-1}^{\mathbb{C}} \cong 3M(2\omega_{n-1}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1})^{\mathbb{C}} \oplus M(2\omega_{n-2}^{n-1})^{\mathbb{C}};$
- 2) $M(2\omega_{n-1}^n)^{\mathbb{C}} \downarrow G_{n-1}^{\mathbb{C}} \cong M(2\omega_{n-1}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1})^{\mathbb{C}} \oplus 3M(2\omega_{n-2}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{n-3}^{n-1} + \omega_{n-2}^{n-1})^{\mathbb{C}} \oplus M(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})^{\mathbb{C}} \oplus M(2\omega_{n-3}^{n-1})^{\mathbb{C}};$
- 3) $M(2\omega_k^n)^{\mathbb{C}} \downarrow G_{n-1}^{\mathbb{C}} \cong M(2\omega_k^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{k-1}^{n-1} + \omega_k^{n-1})^{\mathbb{C}} \oplus 3M(2\omega_{k-1}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{k-2}^{n-1} + \omega_{k-1}^{n-1})^{\mathbb{C}} \oplus M(2\omega_{k-2}^{n-1})^{\mathbb{C}}$ for $1 \leq k \leq n-2$;
- 4) $M(\omega_{n-1}^n + \omega_n^n)^{\mathbb{C}} \downarrow G_{n-1}^{\mathbb{C}} \cong 2M(2\omega_{n-1}^{n-1})^{\mathbb{C}} \oplus 4M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1})^{\mathbb{C}} \oplus 2M(2\omega_{n-2}^{n-1})^{\mathbb{C}} \oplus M(\omega_{n-3}^{n-1} + \omega_{n-2}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})^{\mathbb{C}};$
- 5) $M(\omega_k^n + \omega_{k+1}^n)^{\mathbb{C}} \downarrow G_{n-1}^{\mathbb{C}} \cong M(\omega_k^{n-1} + \omega_{k+1}^{n-1})^{\mathbb{C}} \oplus 2M(2\omega_k^{n-1})^{\mathbb{C}} \oplus 4M(\omega_{k-1}^{n-1} + \omega_k^{n-1})^{\mathbb{C}} \oplus \oplus 2M(2\omega_{k-1}^{n-1})^{\mathbb{C}} \oplus M(\omega_{k-2}^{n-1} + \omega_{k-1}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{k-1}^{n-1} + \omega_{k+1}^{n-1})^{\mathbb{C}} \oplus M(\omega_{k-2}^{n-1} + \omega_{k+1}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{k-2}^{n-1} + \omega_k^{n-1})^{\mathbb{C}}$ for $1 \leq k \leq n-2$;
- 6) $M(\omega_{n-2}^n + \omega_n^n)^{\mathbb{C}} \downarrow G_{n-1}^{\mathbb{C}} \cong M(2\omega_{n-2}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{n-2}^{n-1} + \omega_{n-1}^{n-1})^{\mathbb{C}} \oplus 4M(\omega_{n-3}^{n-1} + \omega_{n-1}^{n-1})^{\mathbb{C}} \oplus \oplus 2M(\omega_{n-4}^{n-1} + \omega_{n-1}^{n-1})^{\mathbb{C}} \oplus 2M(\omega_{n-3}^{n-1} + \omega_{n-2}^{n-1})^{\mathbb{C}} \oplus M(\omega_{n-4}^{n-1} + \omega_{n-2}^{n-1})^{\mathbb{C}}.$

Proof. Let ω be the same as in the left hand side of the formulas in Items 1-6 of the lemma. Apply the classical branching rules [6, 8.1.5]. To state them, we need some notation. When we write weights of $G_n(\mathbb{C})$ in terms of ε_i^n , (b_1, \dots, b_n) means $\sum_{i=1}^n b_i \varepsilon_i^n$. The set of all weights of $G_n^{\mathbb{C}}$ is equal to

$$\{(b_1, \dots, b_n) | b_1 \geq b_2 \geq \dots \geq b_n \geq 0, b_i \in \mathbb{Z}, 1 \leq i \leq n\}$$

and $(b_1, \dots, b_n) > (c_1, \dots, c_{n-1})$ means that all $b_j \in \mathbb{Z}$ and $b_1 \geq c_1 \geq b_2 \geq b_{n-1} \geq c_{n-1} \geq b_n$. It is well known that $\omega_i^n = \varepsilon_1^n + \dots + \varepsilon_i^n$ for $i = 1, \dots, n$. Hence for $1 \leq i \leq j \leq n$ the weight $\omega = \omega_i^n + \omega_j^n = (2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$ where the sequence in the right hand side contains i twos and $j - i$ ones. Denote this sequence by $b = b(\omega)$.

The branching rules imply that

$$M(\omega)^{\mathbb{C}} \downarrow G_{n-1}^{\mathbb{C}} \cong \oplus_{d < c} \oplus_{c < (b, 0)} M(d)^{\mathbb{C}}.$$

Writing down this formula for all relevant ω , we obtain the assertion of the lemma.

Note that by [13, Theorem, Item C] for the natural embeddings of groups of type C_n in characteristic 5

$$\langle \varphi(2\omega_n^n) | n > 0 \rangle_n = \{\varphi(2\omega_n^n), \varphi(\omega_{n-1}^n + \omega_n^n)\}.$$

Therefore, such inductive system contains no trivial or standard representations for groups of rank greater than 1 and does not contain \mathcal{R}^{p-1} .

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A. A. Osinovskaya, I. D. Suprunenko
Inductive systems of representations with small highest weights
for natural embeddings of symplectic groups

Summary

For natural embeddings of symplectic groups, inductive systems of irreducible representations where the maximum of the highest weight value on the maximal root is equal to 2 are studied. For such embeddings of algebraic groups of type C_n in characteristic 3, the inductive system of representations generated by irreducible representations with highest weight $2\omega_n$ is determined. It is proved that any inductive system of representations of such groups consisting of representations with the value of the highest weight on the maximal root at most 2 and containing representations with such value equal to 2 contains the subsystem generated by the standard representations or the subsystem generated by the representations with highest weight ω_n . For algebraic groups of type C_n in characteristic 3, the restrictions of certain irreducible modules to subsystem subgroups of type C_{n-1} are described.