R. Conte, M. Musette, Exact solutions to the partially integrable Eckhaus equation,
ТМФ, 1994, том 99, номер 2, 226–233

ТЕОРЕТИЧЕСКАЯ
И МАТЕМАТИЧЕСКАЯ
ФИЗИКА
Том 99, № 2
май, 1994

 R. Conte, M. Musette

EXACT SOLUTIONS TO THE PARTIALLY INTEGRABLE ECKHAUS EQUATION

A partially integrable extension of the Eckhaus equation is first converted to one real fourth order equation. The only integrable case is isolated by simply solving a diophantine equation, and its linearizing transformation, not obvious at first glance, is shown to be the singular part transformation of Painlevé analysis. In the partially integrable case, three exact solutions are found by the truncation procedure. The third one is a six-parameter solution, whose dependence on x is elliptic and dependence on t involves the equation of Chazy.

1. INTRODUCTION

The PDE

$$(1)\quad iU_t + \alpha U_{xx} + \left(\frac{\beta^2}{\alpha}|U|^4 + 2be^{i\gamma}(|U|^2)_x\right)U = 0,\ (\alpha,\beta,b,\gamma)\in\mathcal{R},\ \alpha\beta b\cos\gamma\neq 0,$$

was first derived [15], in the particular case $b^2 = \beta^2, \gamma = \pi/2$, as an explicit transform (Kundu's equations (3.10), (4.2), (4.3)) of the (linear) Schrödinger equation

$$(2)\qquad iV_t + \alpha V_{xx} = 0.$$

This linearizability persists [3, 4, 7] for $b^2 = \beta^2$. For $b^2 \neq \beta^2$, eq. (1) fails the Painlevé test [7] but still admits various reductions to a system of two coupled ODEs [12, 6, 13] in the reduced variables

$$(3)\qquad x-ct,\ \frac{xt-a_1}{t^2},\ \frac{x-ct}{\sqrt{t}},\ \frac{x-ct}{\sqrt{t^2+t_0^2}}\quad (c, a_1, t_0 - \text{const}).$$

and this allows to find particular solutions for $\beta^2/b^2 = 1 - 5\cos^2\gamma$ only [6].

In section 2., we represent the fields $|U|^2$ and grad arg U as algebraic transforms of a more convenient field u satisfying a single real PDE. This allows to reduce the Painlevé test to the resolution of one diophantine equation, without the need to undertake a Laurent expansion. In section 3., the single case isolated in previous section is linearized into (2) by the singular part transformation of Painlevé analysis, simply expressed as $u = (\alpha/(2\beta\cos\gamma))\,\mathrm{Log}\,\varphi$, thus providing *ipso facto* the inverse transformation. Sections 4. and 5. are devoted to the partially integrable case $b^2 \neq \beta^2$: we look for particular solutions described by one among the two families of movable singularities and we find three new solutions, two for arbitrary values of $(\beta^2/b^2, \gamma)$ and one for $\beta^2/b^2 = 1 - (35/3)\cos^2\gamma$.

2. THE EQUIVALENT SYSTEM, ITS PAINLEVÉ TEST

Variables $U, \bar{U}, |U|$ have movable algebraic branch points (their square behaves like a simple pole) and are thus uneasy to use in the Painlevé test [7]. Better variables are $|U|^2$ and grad arg U, which behave like simple poles and obey an equivalent system, still polynomial because of the parity in U. The imaginary part of equation (1) is a conservation law

$$\left[2\alpha(|U|^2\theta_x + b\sin\gamma)|U|^4\right]_x + (|U|^2)_t = 0,\ \theta = \arg U, \tag{4}$$

and provides the parametric representation

$$\arg U = \theta,\ |U|^2 = u_x,\ \theta_x = -\frac{u_t}{2\alpha u_x} - \frac{b\sin\gamma}{\alpha}u_x, \tag{5}$$

$$\theta_t = \alpha\left(\frac{u_{xxx}}{2u_x} - \frac{u_{xx}^2}{4u_x^2}\right) - \frac{u_t^2}{4\alpha u_x^2} + \frac{\beta^2 - (b\sin\gamma)^2}{\alpha}u_x^2 + 2b\cos\gamma u_{xx} - \frac{b\sin\gamma}{\alpha}u_t, \tag{6}$$

where u satisfies the fourth order real PDE [16]

$$\begin{aligned} E \equiv\ & \frac{\alpha}{2}(u_{xxxx}u_x^2 + u_{xx}^3 - 2u_xu_{xx}u_{xxx}) + 2\frac{\beta^2 - (b\sin\gamma)^2}{\alpha}u_x^4u_{xx} \\ & +2(b\cos\gamma)u_x^3u_{xxx} + \frac{1}{2\alpha}(u_{tt}u_x^2 + u_{xx}u_t^2 - 2u_tu_xu_{xt}) = 0. \end{aligned} \tag{7}$$

The field grad u has a simple pole-like movable singularity, so we directly analyze the PDE for u as indicated in [9], using the invariant formalism [8] of Painlevé analysis, equivalent to the WTC one [19]. In this formalism, the function ψ defining the movable singular manifold $\psi = 0$ obeys the linear system

$$\psi_{xx} + \frac{S}{2}\psi = 0, \tag{8}$$

$$\psi_t + C\psi_x - \frac{C_x}{2}\psi = 0, \tag{9}$$

$$X \equiv S_t + C_{xxx} + 2C_xS + CS_x = 0. \tag{10}$$

Looking for a dominant behaviour $u \sim u_0\psi^p, E \sim E_0\psi^q$, one finds $p = 0$ (i.e. $u \sim u_0 \operatorname{Log}\psi$) and $q = -6$, with two values for u_0

$$u \sim \frac{\alpha}{b\cos\gamma}A\operatorname{Log}\psi,\ 4\left(\frac{b^2 - \beta^2}{(b\cos\gamma)^2} - 1\right)A^2 + 8A - 3 = 0. \tag{11}$$

The set of four Fuchs indices for each family is $(-1, 0, 2, i_j = 3 - 4A_j), j = 1, 2$. A necessary condition for the Painlevé property (PP) is that, for each family, the Fuchs indices be distinct integers [10]. This generates the diophantine equation

$$(i_1 - 3/2)(i_2 - 3/2) = 9/4, \tag{12}$$

whose integer solutions are $(0, 0), (2, 6), (-3, 1)$. So, the only case where the PDE may have the PP is the last one, i.e. the Eckhaus case $b^2 = \beta^2$.

3. THE LINEARIZING TRANSFORMATION

In this Eckhaus case $b^2 = \beta^2$, the equations (5),(6),(7) have the following dependence on (β, γ, u)

$$\theta = -\frac{\beta \sin\gamma}{\alpha} u + F(u, \beta\cos\gamma),\ E(u, \beta\cos\gamma) = 0. \tag{13}$$

The two families are $A = 1/2$ and $A = 3/2$, and they have the indices $(-1, 0, 1, 2)$ and $(-3, -1, 0, 2)$. Let us apply the singular part transformation [17, 19] to the first family

$$u = \frac{A}{\beta\cos\gamma} \operatorname{Log}\varphi,\ A = \frac{1}{2}. \tag{14}$$

The representation (13) of the Eckhaus PDE becomes

$$\theta = -\frac{\tan\gamma}{2} \operatorname{Log}\varphi + F(\varphi, 0),\ E(\varphi, 0) = 0. \tag{15}$$

The variable $\theta + (\beta\sin\gamma)u/\alpha$ and the equation are obtained from (13) by the operation: change u to φ, set β to zero. Thus, the linearization of the Eckhaus equation into the Schrödinger equation (2), which constructively proves the PP, is best expressed in the natural singularity variables u and φ

$$u = \frac{\operatorname{Log}\varphi}{2\beta\cos\gamma} :\ \text{Eckhaus}(U^2 = u_x e^{2i\theta}) \Longleftrightarrow \text{Schrödinger}(V^2 = \varphi_x e^{2i\theta}). \tag{16}$$

Since U and V have the same argument, one retrieves the usually written form

$$U = \sqrt{\frac{\alpha}{2\beta\cos\gamma}} \frac{V}{\sqrt{\int^x |V|^2}}. \tag{17}$$

4. ONE-FAMILY PARTICULAR SOLUTIONS

In order to possibly extend the set of known particular solutions [12, 6, 13] in the partially integrable case $b^2 \neq \beta^2$, let us look for solutions described by one family of movable singularities, among the two existing ones, by performing the one-family truncation of WTC. This consists in representing u as

$$u = u_T = \frac{\alpha A}{b\cos\gamma}(\operatorname{Log}\psi + u_0), \tag{18}$$

where A is one of the two zeros of (11), ψ satisfies (8)–(9), u_0 is the arbitrary function arising at Fuchs index 0. After elimination of any derivative of ψ or order higher than or equal to $(2,0)$ or $(0,1)$ in (x,t), the LHS of (7) becomes a polynomial in ψ_x/ψ

$$E(u_T) \equiv \sum_{j=0}^{6} E_j(S, C, u_0)(\psi_x/\psi)^{j-6}, \tag{19}$$

whose identification the the null polynomial generates seven equations for (S, C, u_0), plus the ever present constraint (10).

One first solves these equations for (S, C, u_0) as functions of (x,t), then ψ is obtained as the general solution of the linear system (8)–(9). This provides a solution u defined by (18), and the physical solution U is obtained from the parametric representation (5)–(6).

5. RESULTS

Equation $E_0 = 0$ is identically zero by construction, equation $E_2 = 0$ is found identically zero, which means that no movable logarithm enters at index 2, and the first nonzero equation $E_1 = 0$ factorizes as $(A - 1/2)u_{0,x} = 0$. We discard the case $A = 1/2$ which represents the integrable case $b^2 = \beta^2$, and put $u_{0,x} = 0$ in the subsequent equations. The system to solve becomes

$$\begin{aligned}
&(20) && E_1 \equiv \alpha^2(2-4A)u_{0,x} = 0,\ A \neq 1/2,\\
&(21) && 2E_3 \equiv -2\alpha^2 A S_x - C_t - CC_x = 0,\\
&(22) && 8E_4 \equiv 3C_x^2 + 2C_{xt} + 2CC_{xx} - \alpha^2(S^2 + 2S_{xx}) + 4(C_x u_{0,t} + u_{0,tt} - u_{0,t}^2) = 0,\\
&(23) && 8E_5 \equiv -[(\alpha S)^2 + (C_x + 2u_{0,t})^2]_x = 0,\\
&(24) && 16E_6 \equiv -[(\alpha S)^2 + (C_x + 2u_{0,t})^2]S = 0,\\
&(25) && X \equiv S_t + C_{xxx} + 2C_x S + CS_x = 0.
\end{aligned}$$

The case $S = 0$ implies $C_t + CC_x = 0, C_{xx} = 0$, which integrates as $C = (x - x_0)/(t - t_0)$ (this includes $C = 0$ for $t_0 = \infty$) and leads to the solution

$$(26)\qquad \begin{aligned}
&A \text{ arbitrary},\ C = \frac{x}{t},\ S = 0,\ u_0 = \frac{\operatorname{Log} t}{2} - \operatorname{Log}(t - t_1),\\
&\frac{b\cos\gamma}{\alpha A}u = \operatorname{Log}\frac{x - ct}{t - t_1},\ \arg U = \frac{(x - ct_1)^2}{4\alpha(t - t_1)}.
\end{aligned}$$

In the case $S \neq 0$, the system is equivalent to

$$\begin{aligned}
&(27) && u_{0,x} = 0,\\
&(28) && u_{0,t} = -j(\alpha/2)S - C_x/2,\ j^2 = -1,\\
&(29) && C_t = -CC_x - 2\alpha^2 A S_x,\\
&(30) && S_t = -C_{xxx} - 2C_x S - CS_x,\\
&(31) && 0 = (C + j\alpha\partial_x)(C_{xx} + j\alpha C_x).
\end{aligned}$$

The cross-derivative condition $(u_{0,t})_x = (u_{0,x})_t$ provides

$$(32)\qquad C_{xx} = -j\alpha S_x,$$

and the condition $(C_{xx})_t = (C_t)_{xx}$ provides

$$(33)\qquad (2A + 1)S_{xxx} = -2SS_x,$$

which splits this case in two subcases $S_x = 0$ and $S_x \neq 0$. The subcase $S_x = 0$ leads to the solution

$$(34)\qquad \begin{aligned}
&A \text{ arbitrary},\ C = \frac{x}{t},\ S = -\frac{1}{2c_1^2 t^2},\ u_0 = -\frac{j\alpha}{4c_1^2 t} - \frac{\operatorname{Log} t}{2},\\
&\frac{b\cos\gamma}{\alpha A}u = \operatorname{Log}\cosh\frac{x - ct}{2c_1 t} - \frac{j\alpha}{4c_1^2 t},\\
&\arg U = \frac{(x - ct)^2}{4\alpha t} - \frac{\alpha(4A - 1)}{8c_1^2 t} - \frac{j}{2}\operatorname{Log}\sinh\frac{x - ct}{2c_1 t},
\end{aligned}$$

depending on two arbitrary parameters c, c_1 (plus the origins of x and t).

In the second subcase $S_x \neq 0$, the condition $(S_t)_{xxx} = (S_{xxx})_t$ provides

$$(8A+3)S_x S_{xx} = 0, \tag{35}$$

i.e. $A = -3/8$. The dependence on x is easy: the functions S and C are Weierstrass functions

$$S = -\frac{3}{2}\wp(x-g_0, g_2, g_3),\ C = -\frac{3j\alpha}{2}\zeta(x-g_0, g_2, g_3) + (x-g_0)g_4 + g_5, \tag{36}$$

and the equation (8) for ψ is a Lamé equation with a noninteger index 1/2, whose general solution (Halphen 1880, quoted in [14,p.93]), involves elliptic functions in half the argument

$$\psi = (\wp'((x-g_0)/2, g_2, g_3))^{-1/2}(g_6\wp((x-g_0)/2, g_2, g_3) + g_7). \tag{37}$$

Then, the seven functions g_i of t are determined by the two equations (29) and (8), involving the partial derivatives of $\wp$ and ζ with respect to g_2 and g_3. Equation (29) (notation $\Delta = g_2^3 - 27g_3^2$)

$$\begin{aligned} &\frac{3}{4}\alpha^2 S_x - C_t - CC_x \\ &= \frac{3}{2}j\alpha\left(g_4 + \frac{g_2' g_2^2 - 18 g_3' g_3}{4\Delta}\right)(\zeta - (x-g_0)\wp') \\ &-\frac{3}{2}j\alpha\left(\frac{6g_3' g_2 - 9g_2' g_3}{4\Delta} - \frac{3}{4}j\alpha\right)(\wp' + 2\zeta\wp) \\ &+(g_5 - g_0')(g_4 - \frac{3}{2}j\alpha\wp) - g_5' \\ &+\left(j\alpha g_2\frac{6g_3' g_2 - 9g_2' g_3}{16\Delta} - g_4' - g_4^2\right)(x-g_0) = 0, \end{aligned} \tag{38}$$

generates five coupled ODFs for g_0, g_2, g_3, g_4, g_5, and, taking them into account, equation (8) generates five linear ODEs for g_6, g_7, equivalent to

$$g_6' = g_4 g_6 + \frac{3}{4}j\alpha g_7, \tag{39}$$

$$g_7' = \frac{5}{16}j\alpha g_2 g_6 - g_4 g_7. \tag{40}$$

Function g_4 satisfies the ODE

$$g_4''' + 12 g_4 g_4'' - 18 g_4'^2 = 0, \tag{41}$$

belonging to a class studied by Chazy (class III) [5]. Its general solution, only defined inside or outside a circle whose center and radius depend on the three integration parameters, is single valued and its only singularity is the movable natural boundary defined by the circle. Its explicit expression was given by Bureau [1, 2]

$$-6g_4 = \left[\mathrm{Log}(y'^3 y^{-2}(y-1)^{-2})\right]', \tag{42}$$

in which $y(t)$ is the general solution of the Hermite modular equation, linearizable into the hypergeometric equation.

The third solution is finally defined by

$$
\begin{aligned}
(43)\qquad & A=-\frac{3}{8},\ \beta^2=(1-(35/3)\cos^2\gamma)b^2,\\
& g_4 \text{ solution of } g_4'''+12g_4g_4''-18g_4'^2=0,\\
& g_2=-\frac{16}{3\alpha^2}(g_4'+g_4^2),\ g_3=\frac{16j}{27\alpha^3}(g_4''+6g_4g_4'+4g_4^3),\\
& g_6 \text{ solution of } g_6''-\frac{9}{4}(g_4'+g_4^2)g_6=0,\\
& g_7=-\frac{4j\alpha}{3}(g_6'-g_4g_6),\\
& S=-\frac{3}{2}\wp(x-ct,g_2,g_3),\ C=c-\frac{3j\alpha}{2}\zeta(x-ct,g_2,g_3)+(x-ct)g_4,\\
& u_0=-\frac{1}{2}\int^t g_4\mathrm{d}t,\\
& \frac{b\cos\gamma}{\alpha A}u=-\frac{1}{2}\int^t g_4\mathrm{d}t-\frac{1}{2}\operatorname{Log}\wp'(\frac{x-ct}{2},g_2,g_3)\\
& \qquad +\operatorname{Log}(g_6\wp(\frac{x-ct}{2},g_2,g_3)+g_7).
\end{aligned}
$$

This solution depends on six arbitrary parameters: c, the origin of x, the three constants of integration of (41), one constant of integration of the ODE for g_6 (the second one does not contribute).

For the two-parameter particular solution of Chazy's equation

$$
(44)\qquad g_4=\frac{1}{t-t_1}+\frac{t_0}{(t-t_1)^2},\ (t_0,t_1) \text{ arbitrary constants,}
$$

the discriminant $\Delta=g_2^3-27g_3^2$ of the elliptic function vanishes, and the six-parameter elliptic solution (43) degenerates to a five-parameter trigonometric solution

$$
\begin{aligned}
(45)\qquad & g_2=-\frac{16t_0^2}{3\alpha^2(t-t_1)^4},\ g_3=\frac{64jt_0^3}{27\alpha^3(t-t_1)^6},\\
& G=\frac{g_6}{t} \text{ solution of } G_{TT}+\frac{9t_0^2}{4}TG=0 \text{ (Airy)},\ t=T^{-1},\\
& g_7=-\frac{4j\alpha}{3}(g_6'-g_4g_6),\\
& \wp(x-ct,g_2,g_3)=\frac{2jt_0}{3\alpha(t-t_1)^2}\left(1+3\sinh^{-2}\sqrt{\frac{2jt_0}{\alpha}}\frac{x-ct}{t-t_1}\right),\\
& \zeta(x-ct,g_2,g_3)=-\frac{2ja}{3\alpha(t-t_1)^2}(x-ct)+\sqrt{\frac{2jt_0}{\alpha}}\frac{1}{t-t_1}\coth\sqrt{\frac{2jt_0}{\alpha}}\frac{x-ct}{t-t_1}.
\end{aligned}
$$

This solution degenerates to a four-parameter rational solution for $t_0=0$

$$
(46)\qquad g_2=g_3=0,\ C=-\frac{3j\alpha}{2(x-ct)}+\frac{x-ct_1}{t-t_1},\ S=-\frac{3}{2(x-ct)^2},
$$

$$g_6 = -\frac{3jt_2}{4} + (t-t_1),\ g_7 = \frac{t_2}{(t-t_1)},\ t_2 \text{ arbitrary constant,}$$
$$u_0 = -\frac{1}{2}\operatorname{Log}(t-t_1),$$
$$\frac{b\cos\gamma}{\alpha A}u = -\frac{3}{2}\operatorname{Log}(t-t_1) - \frac{1}{2}\operatorname{Log}(x-ct)$$
$$+ \operatorname{Log}[(x-ct)^2 - 3jt_2(t-t_1) + 4(t-t_1)^2].$$

Remarks.

• None of the solutions (26), (34), (43) belongs to the lists previously established [12, 6, 13], and to our knowledge they are new.

• We have not found *reductions* of the fourth order PDE (7) to an ODE, only *solutions* to the PDE. The solutions for u defined by (26), (34) and the degeneracies (45), (46) of (43) can be considered as depending on the reduced variables

$$\frac{x-ct}{t-t_1},\ \frac{x}{t},\ \frac{x-ct}{\sqrt{t-t_1}} \quad (c, t_1 - \text{const}), \tag{47}$$

but the six-parameter solution (43) cannot. So, the truncation procedure and the Lie symmetries are complementary methods to find exact solutions.

• We have denoted j the square root of -1 arising from the resolution of the truncation equations to insist on the absence of relationship with the i in the definition of Eckhaus equation. The values for the modulus and the argument of U as given by (5)–(6) are complex for some solutions, since they depend on j. Such a situation, usual [11] for complex PDEs, simply means that the real modulus and the real argument result from an additional computation.

• The solution (43) is the richest one we have ever seen arising from the Weiss truncation procedure.

6. CONCLUSION

The solutions for U^2 are best expressed with a complex modulus and the gradient of a complex argument, both singlevalued.

Other particular solutions could be obtained by looking for two-family truncations [18, 11].

References

[1] F. J. Bureau, Integration of some nonlinear systems of ordinary differential equations, Annali di Matematica pura ed applicata **XCIV** (1972) 345-360.

[2] F. J. Bureau, Sur des systèmes différentiels du troisième ordre et les équations différentielles associées, Bulletin de la Classe des Sciences **LXXIII** (1987) 335-353.

[3] F. Calogero and W. Eckhaus, Nonlinear evolution equations, rescalings, model PDEs and their integrability, I, Inverse problems **3** (1987) 229-262.

[4] F. Calogero and S. de Lillo, The Eckhaus PDE $i\psi_t + \psi_{xx} + 2(|\psi|^2)_x\psi + |\psi|^4\psi = 0$, Inverse problems **3** (1987) 633-681. Errata **4** (1988) 571.

[5] J. Chazy, Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, Thèse, Paris (1910); Acta Math. **34** (1911) 1-69.

[6] P. A. Clarkson, Dimensional reductions and exact solutions of a generalized nonlinear Schrödinger equation, Nonlinearity **5** (1992) 453-472.
[7] P. A. Clarkson and C. M. Cosgrove, The Painlevé property and a generalised derivative nonlinear Schrödinger equation, J. Phys. A **20** (1987) 2003-2024.
[8] R. Conte, Invariant Painlevé analysis of partial differential equations, Phys. Lett. A **140** (1989) 383-390.
[9] R. Conte, Unification of PDE and ODE versions of Painlevé analysis into a single invariant version, *Painlevé transcendents, their asymptotics and physical applications*, 125–144, eds. D. Levi and P. Winternitz (Plenum, New York, 1992).
[10] R. Conte, A. P. Fordy and A. Pickering, A perturbative Painlevé approach to nonlinear differential equations, Physica D **69** (1993) 33–58.
[11] R. Conte and M. Musette, Linearity inside nonlinearity: exact solutions to the complex Ginzburg-Landau equation, Physica D **69** (1993) 1–17.
[12] M. Florjańczyk and L. Gagnon, Exact solutions for a higher-order nonlinear Schrödinger equation, Phys. Rev. A **41** (1990) 4478–4485.
[13] M. Florjańczyk and L. Gagnon, Dispersive-type solutions for the Eckhaus equation, Phys. Rev. A **45** (1992) 6881-6883.
[14] G.-H. Halphen, Sur la réduction d'équations différentielles linéaires aux formes intégrables, *Œuvres*, Gauthier-Villars, Paris, tome 3, 1–260 (1921).
[15] A. Kundu, Landau-Lifshitz and higher order nonlinear systems gauge generated from nonlinear Schrödinger type equations, J. Math. Phys. **25** (1984) 3433–3438.
[16] M. Musette, Nonlinear partial differential equations, *An introduction to methods of complex analysis and geometry for classical mechanics and nonlinear waves*, 145–195, eds. D. Benest and C. Frœschlé (Éditions Frontières, Gif-sur-Yvette, 1994).
[17] P. Painlevé, Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, Acta Math. **25** (1902) 1–85.
[18] A. Pickering, A new truncation in Painlevé analysis, J. Phys. A **26** (1993) 4395–4405.
[19] J. Weiss, M. Tabor and G. Carnevale, The Painlevé property for partial differential equations, J. Math. Phys. **24** (1983) 522-526.

Service de physique de l'état condensé, Centre d'études de Saclay,
F-91190 Gif-sur-Yvette Cedex, France

Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel,
B-1050 Brussel, Belgium

Р. Конте, М. Мюзет

ТОЧНЫЕ РЕШЕНИЯ ЧАСТИЧНО ИНТЕГРИРУЕМОГО УРАВНЕНИЯ ЕКОСА

Частично интегрируемое расширение уравнения Екоса преобразовано в уравнение четвертого порядка. Единственный интегрируемый случай выделяется посредством решения диофантова уравнения. Показано, что линеаризующее его преобразование совпадает с сингулярным преобразованием разделения переменных в анализе Пенлеве. В случае частичной интегрируемости с использованием процедуры транкирования найдены три точных решения. Третье решение является шестипараметрическим, с эллиптической зависимостью от x и зависимостью от t, определяемой уравнением Чази.