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In honour of N. K. Nikolski on the occasion of his 80th birthday

STATIONARY PHASE METHOD, POWERS OF FUNCTIONS, AND APPLICATIONS TO FUNCTIONAL ANALYSIS

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The utility of the (weighted) van der Corput inequalities or of the stationary phase method is illustrated with various examples borrowed from: differentiability issues (Riemann's function and related); functional analysis on Banach spaces or algebras of analytic functions (composition operators); and local Banach space geometry (Schäffer's problem).

Contents

Ключевые слова: stationary phase method, powers of functions, Blaschke factors, Wiener space, Dirichlet series, composition operators, Schäffer's problem, Toeplitz operators.

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§1. Introduction

This paper is in part a survey of classical results, and in part a presentation of recent or quite recent results due to the authors.

In the sixties and seventies, a major concern of harmonic analysts was the determination of those functions $\varphi\colon \mathbb{T}\to\mathbb{C}$ and $\varphi\colon \mathbb{T}\to\mathbb{T}$, the unit circle, such that the superposition and composition operators $f\mapsto \varphi\circ f$ and $f\mapsto f\circ\varphi$ map the Wiener algebra W of absolutely convergent Fourier series

$$f(t)=\sum_{n\in\mathbb{Z}} c_n e^{int},\quad \sum_{n\in\mathbb{Z}}|c_n|<\infty,$$

to itself. Superposition (respectively, composition) operators were studied in detail respectively in [42, Chapter 6] and [42, Chapter 4]). It turns out that, in the latter example, since $f\circ\varphi=\sum_{n\in\mathbb{Z}} c_n e^{in\varphi}$, a necessary and sufficient condition for φ to induce a composition operator on W is that

$$\|e^{in\varphi}\|_W=O(1).$$

The van der Corput inequalities then intervene in a crucial way to show that φ must be affine ($\varphi=a\gamma$ with $|a|=1$ and $\gamma\in\widehat{G}$), as was established by P. Cohen [13] (see also [27]) in the more general context of compact Abelian groups G and their dual $\widehat{G}$. The utility of estimating (various norms of) powers of a given function and its Fourier coefficients (here $e^{i\varphi}$) is made clear here, and will be *the main theme of the present work* (even if another application to Riemann type functions

$$f(x)=\sum_{n\geqslant 1}\frac{\sin n^r\pi x}{n^r}$$

will be given). Before elaborating on this, we fix some notation.

1.1. Notation. First, $\mathbb{N}$ (respectively, $\mathbb{N}_0$) denotes the set of positive (respectively, nonnegative) integers. $\mathbb{T}$ (respectively, $\mathbb{D}$) will denote the unit circle equipped with its normalized Haar measure m (respectively, the open unit disk). An analytic self-map φ of $\mathbb{D}$ will be called a "symbol", the set of such symbols is denoted by S. The space of holomorphic functions on $\mathbb{D}$ is denoted by $\mathcal{H}(\mathbb{D})$. A symbol φ induces a so-called composition operator C_φ on $\mathcal{H}(\mathbb{D})$ defined by

$$C_\varphi(f)=f\circ\varphi.$$

We will occasionally consider self-maps of $\mathbb{T}$. We denote by $\mathbb{C}_\theta$ the vertical half-plane

$$\mathbb{C}_\theta=\{s\in\mathbb{C}:\mathfrak{Re}\, s>\theta\},\quad \theta\in\overline{\mathbb{R}}.$$

$A\ll B$ means that $|A|\leqslant C|B|$ where C is an absolute constant. $A\approx B$ means that $A\ll B$ and $B\ll A$. A Dirichlet series is a series of the form

$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then there exists an extended real number $\sigma_c(f)$, called the convergence abscissa of f, such that the series converges in $\mathbb{C}_\theta$ and diverges outside $\overline{\mathbb{C}_\theta}$. We define similarly the absolute convergence abscissa $\sigma_a(f)$ with

$$\sigma_c(f) \leqslant \sigma_a(f) \leqslant \sigma_c(f) + 1.$$

Let $\mathcal{D}$ denote the set of convergent Dirichlet series f, those for which $\sigma_c(f) < \infty$.

1.2. Content of the paper. This paper is divided in two main parts, with a common theme: the use of the stationary phase method (in particular the van der Corput inequalities) in various situations borrowed from complex and harmonic analysis, or local Banach space geometry. In particular, estimates from above or below on (weighted) oscillatory integrals, or sums, of the form

$$\int_a^b w(t) e^{iF(t)} dt \text{ or } \sum w_k e^{iF(k)} \tag{1.1}$$

will turn out to be of paramount importance.

We start with an application to differentiability properties of Riemann type functions

$$f(x) = \sum_{n \geqslant 1} \frac{\sin n^r \pi x}{n^r}.$$

Then, we study composition operators on the Wiener subalgebra W^+ of W formed by the absolutely convergent Taylor series

$$\sum_{n=0}^{\infty} a_n z^n.$$

A similar study is performed for a space of absolutely convergent Dirichlet series.

We finish with a study of composition operators on Hilbert spaces of analytic functions on the unit disk $\mathbb{D}$, equipped with the norm

$$\Big\| \sum_{n=0}^{\infty} a_n z^n \Big\|_H^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n$$

where (β_n) is a sequence of positive numbers (weights).

We switch to the second part of this paper. In the nineties, the first author was interested in the norm of the inverse of a matrix, in the continuation of the works of Schäffer, and later Gluskin, Meyer, Pajor. Schäffer's estimate was (for an arbitrary normed space E of dimension n and $T \colon E \to E$ linear and invertible):

$$|\det T| \times \|T^{-1}\| \leqslant C\sqrt{n} \|T\|^{n-1}. \tag{1.2}$$

In order to prove (1.2), Schäffer [43] used the Banach–Mazur compactum $\mathcal{B}_n$ of all n-dimensional Banach spaces, and asked for the optimality of (1.2). The Gluskin–Meyer–Pajor approach allows us to get rid of this compact metric space $\mathcal{B}_n$ by using the Wiener space W^+ *of absolutely convergent Taylor series*, notably finite Blaschke products, recovering (1.2) and showing that this is nearly optimal. The space W^+ will play an essential role throughout this work.

Using twisted Gaussian sums and probabilistic arguments, the first author showed that (1.2) is optimal, but his argument, using the cyclicity of the multiplicative group F_p^* (p a prime) was not really explicit. O. Szehr and the second author discovered a new and explicit approach, which involves weighted Blaschke products, the van der Corput inequalities, and standard methods for asymptotic analysis built on the stationary phase method. Moreover, an explicit $E \in \mathcal{B}_n$ and $T\colon E \to E$ (a truncated Toeplitz operator) are provided. This point will be discussed in detail in this second part. This is where our collaboration began.

In order to be more specific, we begin by recalling some estimates on oscillatory quantities.

§2. Oscillatory integrals or sums

The *van der Corput inequalities* give fairly sharp upper bounds for oscillating integrals as in (1.1), where F is real-valued and fairly regular (say C^k for some positive integer k). Here are two simple forms.

Proposition 2.1. *Consider $F\colon [a,b] \to \mathbb{R}$ and $w\colon [a,b] \to \mathbb{C}$ with F' monotonic and $|F'(t)| \geqslant \lambda_1 > 0$, and $|w(t)| \leqslant M$ for all $t \in [a,b]$, with total variation $V < \infty$. Then*

$$\left| \int_a^b w(t) e^{iF(t)} dt \right| \leqslant (3M + V)\lambda_1^{-1}.$$

In particular,

$$\left| \int_a^b e^{iF(t)} dt \right| \leqslant 3\lambda_1^{-1}.$$

When we write $e^{iF} = \frac{1}{iF'} iF' e^{iF}$, the proof consists of a simple integration by parts, see [32] (it is the form used by Newman to show the necessity of his condition in Theorem 3.1 to follow). But in many applications, F has a (single) critical point c (i.e., $F'(c) = 0$) inside $[a,b]$, and then the following substitute applies.

Proposition 2.2. *Let $F\colon [a,b] \to \mathbb{R}$ be a C^2 function, with F' monotonic and $|F''(t)| \geqslant \lambda_2$ throughout $[a,b]$. Then*

$$\Big|\int_a^b e^{iF(t)}dt\Big| \leqslant C\lambda_2^{-1/2}.$$

Here is a version of Proposition 2.2 for (weighted) sums.

Proposition 2.3. *Let $I =]u,v]$ be an interval of $\mathbb{R}$, with $u,v \in \mathbb{N}$, and let $F\colon I \to \mathbb{R}$ be a C^2-function satisfying, for some positive numbers λ_2, $\alpha \geqslant 1$, the two-sided estimate*

$$\lambda_2 \leqslant |F''(t)| \leqslant \alpha\lambda_2 \quad \textit{for all} \quad t \in I.$$

Then

$$\Big|\sum_{n\in I} e^{iF(n)}\Big| \ll (v-u)\alpha\lambda_2^{1/2} + \lambda_2^{-1/2}.$$

If $(w_n)_{u<n\leqslant v}$ is a monotone nonincreasing sequence of positive numbers, then

$$\Big|\sum_{n\in I} w_n e^{iF(n)}\Big| \ll \big[(v-u)\alpha\lambda_2^{1/2} + \lambda_2^{-1/2}\big]w_{u+1}.$$

We refer to the books [50] or [23], and to the nice survey [41] for a detailed proof of this inequality and more. The weighted case follows from the first one via an Abel summation by parts. A typical application of those inequalities is the Hardy–Littlewood estimate (where $\|\,.\,\|_\infty$ refers to the sup-norm on the circle)

$$\Big\|\sum_{n=1}^{N} e^{in\log n}z^n\Big\|_\infty \leqslant C\sqrt{N}.$$

Indeed, take $I =]2^p, 2^{p+1}]$ where p is a nonnegative integer, write $z = e^{i\theta}$ and let $F(t) = t\log t + \theta t$, so that $F''(t) = 1/t$. Proposition 2.3 with $\lambda_2 = 2^{-p-1}$ and $\alpha = 2$ gives

$$\Big|\sum_{2^p<n\leqslant 2^{p+1}} e^{in\log n}z^n\Big| \ll 2^p \times 2^{-p/2} + 2^{p/2} \ll 2^{p/2}.$$

Adding those estimates on consecutive dyadic blocks (the last one is incomplete) gives the result.

Sometimes, more precise results providing two-sided estimates are needed. Then we speak of the stationary phase method.

Theorem 2.4. *Let $F\colon [A,B] \to \mathbb{R}$, with $F \in C^3([a,b])$, be a function satisfying $F'' > 0$. Let c be a unique point in $]A,B[$ where $F'(c) = 0$. Assume that, with $\lambda_2,\ \Lambda_3,\ \eta > 0$, one has:*

1) $[c-\eta, c+\eta] \subseteq [A, B]$;
2) $F''(x) \geqslant \lambda_2$ *for all* $x \in [c-\eta, c+\eta]$;
3) $|F'''(x)| \leqslant \Lambda_3$ *for all* $x \in [A, B]$.

Then with an absolute constant in the O*-condition below*:

$$I := \int_A^B \exp[i\,F(x)]\,dx = \sqrt{2\pi}\,\frac{e^{i(F(c)+\pi/4)}}{|F''(c)|^{1/2}} + O\left(\frac{1}{\eta\lambda_2} + \eta^4\Lambda_3\right),$$

This theorem is a variant of a theorem in Titchmarsch's book [51, p. 72], see also [11, Lemma 2.2], or [41], with slightly modified assumptions, adapted to our purposes in [29].

§3. Typical applications

We first elaborate on an example concerning the Wiener algebra W^+ of absolutely convergent *Taylor series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ \|f\|_{W^+} := \sum_{n=0}^{\infty} |a_n| < \infty.$$

The situation is richer in this case than for the algebra W of all absolutely convergent Fourier series. This example is due to D. Newman [32], in a very nice (even if a little forgotten) paper on composition operators C_φ acting on W^+. Here, $\varphi \in S$, so that $C_\varphi\colon W^+ \to \mathcal{H}(\mathbb{D})$, and the question is to find necessary and sufficient conditions on φ to ensure that $C_\varphi\colon W^+ \to W^+$. Since $C_\varphi(f) = \sum_{n=0}^{\infty} a_n\varphi^n$ with $C_\varphi(z^n) = \varphi^n$, an obvious necessary and sufficient condition is once again the relation

$$\|\varphi^n\|_{W^+} = O(1) \tag{3.1}$$

(in particular, φ extends continuously to $\overline{\mathbb{D}}$, and we will even assume φ to be analytic on $\overline{\mathbb{D}}$). The problem becomes to find a tractable form, not too difficult to check, of this condition (3.1).

We first perform some reductions: one may assume that $\|\varphi\|_\infty = 1$, otherwise C_φ clearly maps W^+ to itself; one may also assume that $|\varphi|$ is not identical to one on $\mathbb{T}$; otherwise φ is a finite Blaschke product

$$\varphi(z) = u\prod_{j=1}^{d}\frac{z-a_j}{1-\overline{a_j}z}$$

with $|u| = 1$ and $a_j \in \mathbb{D}$. And then two cases occur.

• One of the a_j's is not zero. Then $\|\varphi^n\|_{W^+}$ is unbounded and more precisely

$$\|\varphi^n\|_{W^+} \geqslant \delta\sqrt{n} \text{ with } \delta > 0. \tag{3.2}$$

The qualitative part follows from Cohen's result [13], and the quantitative one from the van der Corput inequality of Proposition 2.3: see [32, Assertion (3)] or [27, p. 77]. But, $C_\varphi(z^n) = \varphi^n$ and hence this case is excluded in the sequel.

- φ is a monomial

$$\varphi(z) = az^d \text{ with } |a| = 1 \text{ and } d \in \mathbb{N}.$$

In this case, C_φ is an isometry; moreover, C_φ is surjective if and only if $d = 1$ as was observed by Harzallah [27].

If $|\varphi|$ is *not* identically 1, there are only finitely many points θ_0 where θ_0 is a maximum point for $|\varphi(e^{i\theta})|$. We call θ_0 an *ordinary maximum point* if, in the local expansion $\log \varphi(e^{i(\theta_0+t)}) = c_0 + c_1 t + \cdots + c_k t^k + \cdots$ the first nonvanishing c_k with $k > 1$ (the *index* of θ_0) is *not pure imaginary.* The necessary and sufficient condition found by Newman [32] reads as follows.

Theorem 3.1 (Newman)**.** *Assume that* $\|\varphi\|_\infty = 1$ *and* $\varphi(z) \neq az^d$*. Then a necessary and sufficient condition for* C_φ *to map* W^+ *to itself* (*i.e., for* (3.1)) *is that all its maximum points are ordinary.*

The subtlety of Newman's condition is illustrated by the following two quadratic examples, which at first glance look quite similar, see [32]:

$$\varphi_1(z) = \frac{1 + z - z^2}{\sqrt{5}}. \tag{3.3}$$

We have $\|\varphi_1\|_\infty = 1$ since $|\varphi_1(\pm i)| = 1$ and if $|z| = 1$, then

$$|1 + z - z^2| = |z(1 + \overline{z} - z)| = |1 + 2i\Im\mathrm{m}\, z| \leqslant \sqrt{5}.$$

The maximum points for $|\varphi_1(e^{i\theta})|$ are $\theta_0 = \pm\pi/2$ and a simple computation gives

$$\log \varphi_1(e^{i(\pm\pi/2+t)}) = \log(2 \pm i)5^{-1/2} + it - \frac{t^2}{2 \pm i} + \cdots$$

so that both maximum points are ordinary and C_{φ_1} maps W^+ to itself.

The second example is:

$$\varphi_2(z) = \frac{12 + 16z - 3z^2}{25}. \tag{3.4}$$

It is less obvious that this φ_2 has sup-norm equal to one. Newman indicates the identity (with $|z| = 1$)

$$|12 + 16z - 3z^2|^2 + 36|z - 1|^4 = 625$$

which gives the result. Indeed, if one looks for positive integers a, b, c, d such that

$$|a + bz - cz^2|^2 + d|z - 1|^4 = \text{ constant for } |z| = 1$$

writing $z = e^{it}$, one sees that

$|a + bz - cz^2| = |ae^{-it} + b - ce^{it}| = |(a-c)\cos t + b - i(a+c)\sin t|$, so that

$|a + bz - cz^2|^2 + d|z-1|^4 = \big((a-c)\cos t + b\big)^2 + (a+c)^2\sin^2 t + 4d(1-\cos t)^2$

and one finds $d = ac$, $\quad b = \frac{4ac}{a-c}$. Newman chose $(a,b,c,d) = (12,16,3,36)$.

In this case, a unique maximum point corresponds to $z = 1$, i.e., $\theta_0 = 0$, and we find

$$\varphi_2(e^{it}) = 1 + \frac{2it}{5} - \frac{2t^2}{5} + \frac{4it^3}{75} + \cdots$$

and

$$\log \varphi_2(e^{it}) = \frac{2it}{5} + ict^3 + \cdots \text{ with } c = \frac{8}{125}.$$

This is not an ordinary point, so that C_{φ_2} *does not* map W^+ to itself.

In [4], we sharpened Newman's result as follows.

Theorem 3.2. *Let $\varphi \in S$. The following are equivalent*:

(1) $C_\varphi \colon W^+ \to W^+$ *is compact*;
(2) $\lim_{n\to\infty} \|\varphi^n\|_{W^+} = 0$;
(3) $\|\varphi\|_\infty < 1$.

Proof. The spectrum of the Banach algebra W^+ is $\overline{\mathbb{D}}$, hence we have the spectral radius formula

$$\|\varphi\|_\infty = \lim_{n\to\infty} (\|\varphi^n\|_{W^+})^{1/n} = \inf_{n\geqslant 1} (\|\varphi^n\|_{W^+})^{1/n}, \tag{3.5}$$

which shows that (2) and (3) are equivalent. If (2) is fulfilled, let $\varepsilon_N = \sup_{n>N} \|\varphi^n\|_{W^+}$. Denote by T_N the finite rank operator defined by

$$T_N(f) = \sum_{n=0}^{N} \widehat{f}(n)\varphi^n, \quad \text{with } f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n.$$

One clearly has $\|C_\varphi - T_N\| \leqslant \varepsilon_N$, giving the compactness of C_φ. Finally, assume that C_φ is compact. We use a general and simple criterion of J. Shapiro [44] saying here that C_φ is compact if and only if, for any sequence $(f_n) \in W^+$ with $\|f_n\|_{W^+} = O(1)$ and f_n converging to zero uniformly on compact sets of $\mathbb{D}$, $C_\varphi(f_n) \to 0$. Testing that criterion with $f_n(z) = z^n$, we get $\|\varphi^n\|_{W^+} \to 0$, ending the proof. □

The study of $\|\varphi^n\|$ *in some Banach space of analytic functions* for a given $\varphi \in S$ turns out to be essential, and will reappear later in a Hilbertian context.

It is our purpose in this work to illustrate the use of the stationary phase method on various examples, which reflect various paths followed by both authors. To end this section, we concentrate on the following three topics.

• Differentiability of sums of weakly lacunary Fourier series.

• Composition operators on the space of absolutely convergent Dirichlet series.

• Composition operators on weighted Hardy–Hilbert spaces of analytic functions.

Here are now our three applications.

3.1. Weakly lacunary Fourier series. Long ago, in the seventies, one of us became acquainted with a simple version of van der Corput's inequalities while studying the differentiability of sums of weakly lacunary Fourier series like

$$f_r(x) = \sum_{n=1}^{\infty} \frac{\sin n^r \pi x}{n^r}$$

where $r > 1$ is an integer. The story of $R = f_2$ is curious: Riemann had reportedly claimed to Weierstrass that R is an example of nowhere differentiable function. Hardy proved that it is not differentiable at irrational points, and at some rational points. Today, we could summarize things as follows.

• R satisfies the functional equations [25]

$$R(x+2) = R(x), \quad R(x) \equiv R(-1/x)$$

where the sign $\equiv$ means that, for $0 < x < 2$, one has

$$R(x) = \varphi(x) + \psi(x)R(-1/x)$$

where φ, ψ are C^∞ on $(0,2)$ with ψ nonvanishing. Hence, since R is not differentiable at 0 (quite easy to prove, indeed $R(x) \geqslant \delta\sqrt{x}$ for small $x > 0$), it is not differentiable at x_0, an element of the orbit of 0 under Γ_0, the θ_0-modular group; i.e., the subgroup of the modular group generated by $\sigma,\ \tau$ where

$$\sigma(z) = -1/z,\ \tau(z) = z + 2.$$

This orbit is the set of reduced fractions p/q with p *or* q even. Similarly, the orbit of 1 under Γ_0 is the set of reduced fractions p/q with p *and* q odd. So, to validate Riemann's claim, it only remains to check that R is not differentiable at the point 1. But it is!! This is the breakthrough by J. Gerver see [17, 18]), who showed that $R'(1)$ exists and equals $-\pi/2$. A little later one of us, using the van der Corput techniques, gave a simpler proof of Gerver's result, extended this result, and among other things proved that f_3 is differentiable at certain points (see [36, 37]). Let us elaborate on this.

Theorem 3.3. *Let* $r\in\mathbb{N}$, $r\geqslant 2$. *Then the function* f_r *is differentiable at* 1 *and*

$$f_r'(1)=-\pi/2.$$

Proof of Theorem 3.3. We drop the index r, temporarily fix $0<x<1$ (with later $x\to 0^+$), and set

$$S(t)=\frac{\sin\pi t}{t},\ T(t)=S(t^r x)=:S\circ g. \tag{3.6}$$

Here, we need a technical lemma.

Lemma 3.4. *For* $t>0$, *the following holds*:

$$|T'(t)|\ll 1/t,\ |T''(t)|\ll 1/t^2+xt^{r-2}. \tag{3.7}$$

Proof of Lemma 3.4. The Leibnitz formula easily gives:

$$|S'(t)|\ll 1/t,\ |S''(t)|\ll 1/t+1/t^2.$$

Since $T'=g'(S'\circ g),\ T''=g''(S'\circ g)+g'^2(S''\circ g)$, this implies $|T'|\ll\frac{g'}{g}$ and

$$|T''|\ll\frac{g''}{g}+\frac{g'^2}{g^2}+\frac{g'^2}{g}.$$

The lemma follows. □

Since $f(1)=0$ and $f(x)/x=\sum_{k=1}^{\infty}T(k)$, we now see that

$$g(x):=-\frac{f(x+1)-f(1)}{x}=\sum_{k=1}^{\infty}\big[T(2k-1)-T(2k)\big]=:\sum_{k=1}^{\infty}\Delta(k). \tag{3.8}$$

We will show that $\lim_{x\to 0^+}g(x)=\pi/2$, which gives the result because $f(x+1)$ is odd. We accordingly fix large integers $0<a<b$ depending on x as follows (we omit the integer value issues and use the fact that $3/(3r-2)<1/(r-1)$):

$$a=x^{-\alpha},\ b=x^{-\beta},\ \text{with}\ \ 0<\alpha<1/r<3/(3r-2)<\beta<1/(r-1). \tag{3.9}$$

We split the previous sum into three parts, namely we write

$$\begin{gathered}\sum_{k=1}^{\infty}\Delta(k)=:I_1+I_2+I_3\ \text{with}\\ I_1=\sum_{1\leqslant k<a}\Delta(k),\quad I_2=\sum_{a\leqslant k\leqslant b}\Delta(k),\quad I_3=\sum_{k>b}\Delta(k).\end{gathered} \tag{3.10}$$

We estimate separately I_1,I_2,I_3 and first show: I_1 is *smooth*, and

$$I_1=O(a^r x)=o(1). \tag{3.11}$$

Indeed, by (3.7) and the mean value theorem, we have

$$|\Delta(k)|\ll k^{r-1}x,\ |I_1|\ll a^r x=x^{1-\alpha r}=o(1).$$

We next show that

$$I_2 = \pi/2 + O\Big(b^{r-1}x + \frac{1}{a}\Big) = \pi/2 + o(1). \tag{3.12}$$

The idea is to approximate the difference $T(2k-1) - T(2k)$ by the "*telescoping term*" $\frac{1}{2}\big[T(2k-1) - T(2k+1)\big]$. Indeed, write

$$T(2k-1) - T(2k) = \frac{1}{2}\big[T(2k-1) - T(2k+1)\big] + \varepsilon_k \quad \text{with}$$

$$\varepsilon_k = \frac{1}{2}\big[T(2k-1) - 2T(2k) + T(2k+1)\big] = \frac{1}{2}\int\int_{0\leqslant u,v\leqslant 1} T''(2k-1+u+v)dudv.$$

The error term ε_k is a *second difference*, which is estimated with the help of (3.7):

$$|\varepsilon_k| \ll \sup_{2k-1\leqslant t\leqslant 2k+1} |T''(t)| \ll k^{-2} + k^{r-2}x.$$

Summing up gives

$$I_2 = \frac{1}{2}\big[T(2a-1) - T(2b+1)\big] + O\Big(b^{r-1}x + \frac{1}{a}\Big) = \frac{\pi}{2} + o(1).$$

We have also used that, by (3.9), $b^r x \sim x^{1-\beta r} \to \infty$, whence $T(2b+1) \to 0$, and that $(2a-1)^r x \sim x^{1-\alpha r} \to 0$, whence $T(2a-1) \to \pi$. Similarly, we have $b^{r-1}x \sim x^{1-\beta(r-1)} \to 0$. We finally show that I_3 is *small* and, more precisely,

$$I_3 = O(x^\delta) = o(1) \text{ with some constant } \delta > 0. \tag{3.13}$$

It is here that the *weighted van der Corput inequality* (2.3) intervenes. No more cancellation is to be awaited, and we dominate separately $\sum_{k>b} T(2k)$ and $\sum_{k>b} T(2k-1)$, considering only the first sum (the second is similar). We first write

$$\sum_{k>b} T(2k) = \sum_{l=1}^{\infty} U_l \quad \text{with} \quad U_l = \sum_{lb<k\leqslant(l+1)b} T(2k).$$

Proposition 2.3 with the choice

$$u = lb,\ v = (l+1)b,\ F(t) = \pi t^r x,\ w_k = F(k)^{-1},\ \lambda_2 = (lb)^{r-2}x,\ \alpha = 2^{r-2}$$

gives

$$|U_l| \ll \frac{1}{(lb)^r x}\Big[b(lb)^{(r-2)/2}x^{1/2} + (lb)^{(2-r)/2}x^{-1/2}\Big].$$

Setting $\gamma = 1 + r/2$ and using $1 - 3r/2 \leqslant -r/2 - 1$, we deduce the estimate

$$|U_l| \ll l^{-\gamma}\big[b^{-r/2}x^{-1/2} + b^{1-3r/2}x^{-3/2}\big].$$

Summing up over l and recalling that $b = x^{-\beta}$ gives

$$\Big|\sum_{k>b} T(2k)\Big| \ll x^{(\beta r-1)/2} + x^{[\beta(3r-2)-3]/2}. \tag{3.14}$$

By the choice of β in (3.9), both exponents of x on the right-hand side are positive, giving (3.13). Putting things together, it ensues that $\lim_{x\to 0^+} g(x) = \pi/2$, which ends the proof of Theorem 3.3. □

Remark. For more on f_2, see Jaffard [26]. Fairly recent work on the generalized Riemann functions

$$f_{k,r}(x) = \sum_{n=1}^{\infty} \frac{e^{i\pi n^k x}}{n^r}$$

with two parameters was done by Gerver himself (see [19]) and next by Chamizo and Ubis ([11]), with the nice use of the Poisson summation formula. In particular, the almost everywhere nondifferentiability of some functions $f_{k,r}$ was proved. But we thought it worthwhile, as an illustration of the utility of van der Corput's inequalities, to give a simple proof of the differentiability of the function $f_r = \mathfrak{Im}\, f_{r,r}$ at the point 1 (see [36, 37] for other points than 1).

3.2. Absolutely convergent Dirichlet series. We refer to the introduction for the notation of this section. Much later than for Theorem 3.3, in 2008, one of us turned back to the van der Corput inequalities in the study of composition operators on the Wiener algebra $\mathcal{A}^+$ of absolutely convergent Dirichlet series [4]. This algebra is the set of Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ such that

$$\|f\|_{\mathcal{A}^+} = \sum_{n=1}^{\infty} |a_n| < \infty.$$

The elements of $\mathcal{A}^+$ "live" in the right half-plane $\mathbb{C}_0$ and we will set

$$\|f\|_\infty = \sup_{s\in\mathbb{C}_0} |f(s)|.$$

The spectrum of $\mathcal{A}^+$, that is the set of characters of $\mathcal{A}^+$, is the set M of completely multiplicative functions $\chi\colon \mathbb{N} \to \overline{\mathbb{D}}$ such that $\chi(1) = 1$ (also called characters). The action of $\chi \in M$ on $f \in \mathcal{A}^+$ is given by

$$\chi(f) = \sum_{n=1}^{\infty} a_n \chi(n).$$

A character χ is determined by its values on the sequence $(p_j)_{j\geqslant 1}$ of primes, namely by the vector

$$h(\chi) = \big(\chi(p_j)\big)_{j\geqslant 1} \in \overline{\mathbb{D}}^{\infty}. \tag{3.15}$$

We need the following important fact.

Lemma 3.5. *The spectral radius $r(f)$ of any $f \in \mathcal{A}^+$ is given by*

$$r(f) = \|f\|_\infty.$$

Proof. We will use the so-called *Bohr lift* Δf of $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{A}^+$ defined as follows: if $n \in \mathbb{N}$ has the prime decomposition

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r},$$

we set $\alpha(n) = (\alpha_1, \dots, \alpha_r, 0, 0, \dots)$. And if $z = (z_j)_{j \geqslant 1} \in \overline{\mathbb{D}}^\infty$, we set $z^{\alpha(n)} = z_1^{\alpha_1} \cdots z_r^{\alpha_r}$, as well as

$$\Delta f(z) = \sum_{n=1}^\infty a_n z^{\alpha(n)},\ z = (z_j) \in \overline{\mathbb{D}}^\infty. \tag{3.16}$$

The distinguished maximum principle shows that

$$\|\Delta f\|_\infty := \sup_{z \in \overline{\mathbb{D}}^\infty} |\Delta f(z)| = \sup_{z \in \mathbb{T}^\infty} |\Delta f(z)|. \tag{3.17}$$

We have the (obvious) important identity

$$\chi(f) = \Delta f\big(h(\chi)\big). \tag{3.18}$$

Using this identity and (3.17), we see that

$$r(f) = \sup_{\chi \in M} |\chi(f)| = \sup_{\chi \in M} |\Delta f\big(h(\chi)\big)| = \sup_{z \in \mathbb{T}^\infty} |\Delta f(z)| = \sup_{\chi \in M_u} |\chi(f)|,$$

where M_u denotes the set of unimodular characters of M. Now, the special characters $\chi_t(n) = n^{-it}$, $t \in \mathbb{R}$, are dense in M_u, by the Kronecker simultaneous approximation theorem, see [40, p. 50]. So that, again by the maximum principle (in $\mathbb{C}_0$), the latter supremum is none other than

$$\sup_{t \in \mathbb{R}} |f(it)| = \sup_{s \in \mathbb{C}_0} |f(s)| = \|f\|_\infty. \tag{3.19}$$

This ends the proof. □

We refer to ([40], Chapter 6) for more details and applications.

Our next example can be seen as a multivariate extension of Newman's work already discussed. We are interested in the analytic self-maps φ of $\mathbb{C}_0$ such that the composition operator C_φ maps $\mathcal{A}^+$ to itself. This strongly restricts the form of φ as indicated by the following lemma [22] (see also [4] and [40]).

Lemma 3.6. *Let $\varphi\colon \mathbb{C}_0 \to \mathbb{C}$ be such that $k^{-\varphi} \in \mathcal{D}$ (the set of convergent Dirichlet series) for $k = 2, \dots$. Then φ must have the form*

$$\varphi(s) = c_0 s + \psi(s) \text{ with } c_0 \in \mathbb{N}_0 \text{ and } \psi \in \mathcal{D},\ \psi(s) = \sum_{n=1}^\infty c_n n^{-s}. \tag{3.20}$$

If τ is a real number and $\varphi\colon \mathbb{C}_0 \to \mathbb{C}_\tau$, then $\psi\colon \mathbb{C}_0 \to \mathbb{C}_\tau$ as well.

We will therefore only consider, in the sequel, symbols φ of the form (3.20). We will moreover assume φ nonconstant, to avoid trivialities. The integer c_0 is called the *characteristic* of the symbol φ. When $c_0 = 0$, we are back to $\mathcal{D}$. The case where $c_0 \geqslant 1$ should be interpreted as

$$\varphi(\infty) = \infty.$$

It is good to have in mind, throughout this subsection, the following obvious fact: **the map $f \mapsto n^{-c_0 s} f,\ c_0 \in \mathbb{N}_0$, is an isometry of $\mathcal{A}^+$.**

It is also useful to note that, if $v \in \mathcal{A}^+$ and $r \geqslant 1$ is a *real number*, then

$$r^{-v} \in \mathcal{A}^+ \text{ and } \|r^{-v}\|_{\mathcal{A}^+} \leqslant r^{\|v\|_{\mathcal{A}^+}}. \tag{3.21}$$

Indeed

$$r^{-v} = \exp(-v \log r) = \sum_{k=0}^{\infty} \frac{(-\log r)^k}{k!} v^k \in \mathcal{A}^+$$

because v belongs to the Banach algebra $\mathcal{A}^+$. Moreover,

$$\|r^{-v}\|_{\mathcal{A}^+} \leqslant \sum_{k=0}^{\infty} \frac{(\log r)^k}{k!} (\|v\|_{\mathcal{A}^+})^k = r^{\|v\|_{\mathcal{A}^+}}.$$

The following theorem is a simple analog of Newman's Theorem 3.1, and the role of φ^n is now played by $n^{-\varphi}$. But if $n = 2^j$, we are back to powers, and then we will interpolate between two powers of 2. This will be made clear in the sequel.

Theorem 3.7. *Let $\varphi\colon \mathbb{C}_0 \to \mathbb{C}$ be of the form* (3.20). *The following holds.*

1. *If $\varphi\colon \mathbb{C}_0 \to \mathbb{C}_0$ and $C_\varphi\colon \mathcal{A}^+ \to \mathcal{A}^+$, then $n^{-\varphi} \in \mathcal{A}^+$ and $\|n^{-\varphi}\|_{\mathcal{A}^+} \leqslant C$, $n = 1, 2, \dots,$ where C does not depend on n.*
2. *Conversely, if $(n^{-\varphi})_{n\geqslant 1}$ is a bounded sequence in $\mathcal{A}^+$, then $\varphi\colon \mathbb{C}_0 \to \mathbb{C}_0$ and C_φ is a bounded operator on $\mathcal{A}^+$.*
3. *Assume that $\varphi(s) = c_0 s + \psi(s)\colon \mathbb{C}_0 \to \mathbb{C}_0$ with $\psi \in \mathcal{A}^+$. Then $C_\varphi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is compact if and only if $\varphi(\mathbb{C}_0) \subset \mathbb{C}_\delta$ for some $\delta > 0$.*

Proof. 1. This is clear, because $n^{-\varphi} = C_\varphi(e_n)$ with $e_n(s) = n^{-s}$.

2. For $s \in \mathbb{C}_0$ and $n \in \mathbb{N}$, we write

$$n^{-\Re\mathrm{e}\,\varphi(s)} = |n^{-\varphi(s)}| \leqslant \|n^{-\varphi}\|_\infty \leqslant \|n^{-\varphi}\|_{\mathcal{A}^+} \leqslant C.$$

Letting n go to infinity, we get $\Re\mathrm{e}\,\varphi(s) \geqslant 0$, so that $\Re\mathrm{e}\,\varphi(s) > 0$ because φ is not constant.

3. Suppose first that $C_\varphi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is compact. Let $f \in \mathcal{A}^+$ be a cluster point of $(n^{-\varphi})_{n\geqslant 1}$, and let (n_k) be a sequence of integers such that

$\|n_k^{-\varphi} - f\|_{\mathcal{A}^+} \to 0$. But $n_k^{-\varphi(s)} \to 0$ for each $s \in \mathbb{C}_0$ because $\varphi\colon \mathbb{C}_0 \to \mathbb{C}_0$. Hence, $f = 0$ and $\|n^{-\varphi}\|_{\mathcal{A}^+} \to 0$. Now, let $\varepsilon = \inf_{s\in\mathbb{C}_0} \mathfrak{Re}\,\varphi(s) \geqslant 0$. Since

$$n^{-\varepsilon} = \|n^{-\varphi}\|_\infty \leqslant \|n^{-\varphi}\|_{\mathcal{A}^+} \to 0,$$

we see that $\varepsilon > 0$.

Conversely, assume that $\varepsilon := \inf_{s\in\mathbb{C}_0} \mathfrak{Re}\,\varphi(s) > 0$. Let $\omega = 2^{-\psi}$ and $0 < \varepsilon' < \varepsilon$. Lemma 3.5 gives us

$$r(\omega) = \|\omega\|_\infty = \sup_{s\in\mathbb{C}_0} 2^{-\mathfrak{Re}\,\psi(s)} \leqslant 2^{-\varepsilon} < 2^{-\varepsilon'}.$$

By Gelfand's spectral radius formula, this implies for some constant C:

$$\|(2^j)^{-\psi}\|_{\mathcal{A}^+} \leqslant C2^{-j\varepsilon'} \text{ for all } j \in \mathbb{N}_0.$$

We now interpolate to an arbitrary integer n: let $j \in \mathbb{N}$ be such that $2^j \leqslant n < 2^{j+1}$, and set $q = n2^{-j}$, with $1 \leqslant q < 2$. Consequently, since $\psi \in \mathcal{A}^+$:

$$\|n^{-\varphi}\|_{\mathcal{A}^+} = \|n^{-\psi}\|_{\mathcal{A}^+} = \|(2^j)^{-\psi} \times q^{-\psi}\|_{\mathcal{A}^+} \leqslant \|(2^j)^{-\psi}\|_{\mathcal{A}^+} \times \|q^{-\psi}\|_{\mathcal{A}^+}.$$

We thus get, using (3.21):

$$\|n^{-\varphi}\|_{\mathcal{A}^+} \leqslant C2^{-j\varepsilon'}2^{\|\psi\|_{\mathcal{A}^+}} \leqslant C'n^{-\varepsilon'} \text{ for all } n \in \mathbb{N}.$$

Hence, C_φ is compact. This ends the proof. □

An important corollary is the following.

Corollary 3.8. *Let $\varphi(s) = c_0 s + c_1 + \sum_{n=2}^\infty c_n n^{-s} =: c_0 s + c_1 + \varphi_0(s)$ with $\varphi_0 \in \mathcal{A}^+$. Then C_φ is bounded if $\mathfrak{Re}\,c_1 \geqslant \sum_{n=2}^\infty |c_n| = \|\varphi_0\|_{\mathcal{A}^+}$ and is compact if $\mathfrak{Re}\,c_1 > \sum_{n=2}^\infty |c_n|$.*

Proof. For each positive integer n, we have via (3.21):

$$\|n^{-\varphi}\|_{\mathcal{A}^+} = n^{-\mathfrak{Re}\,c_1}\|n^{-\varphi_0}\|_{\mathcal{A}^+} \leqslant n^{-\mathfrak{Re}\,c_1} n^{\|\varphi_0\|_{\mathcal{A}^+}} = n^{-\mathfrak{Re}\,c_1 + \|\varphi_0\|_{\mathcal{A}^+}}.$$

The result follows via Theorem 3.7. □

We now discuss an analog of Newman's quadratic examples.

Theorem 3.9. *Let $\varphi(s) = c_0 s + c_1 + c_r r^{-s} + c_{r^2} r^{-2s}$ with $r \in \mathbb{N}$, $r \geqslant 2$, and $c_r,\ c_{r^2} > 0$.*

1. *If*

$$\mathfrak{Re}\,c_1 > c_{r^2} + \frac{c_r^2}{8c_{r^2}}, \tag{3.22}$$

then $\varphi\colon \mathbb{C}_0 \to \mathbb{C}_0$ and $C_\varphi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is compact.

2. *Conversely, if $\varphi\colon \mathbb{C}_0 \to \mathbb{C}_0$ and $C_\varphi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is bounded and moreover $c_r \leqslant 4c_{r^2}$, we must have*

$$\mathfrak{Re}\,c_1 \geqslant c_{r^2} + \frac{c_r^2}{8c_{r^2}}, \tag{3.23}$$

with strict inequality whenever C_φ *is compact.*

3. *Assume that*

$$\Re\mathrm{e}\, c_1 = c_{r^2} + \frac{c_r^2}{8c_{r^2}}. \tag{3.24}$$

Then $C_\varphi\colon \mathcal{A}^+ \to \mathcal{A}^+$ *is bounded if and only if* $c_r \neq 4c_{r^2}$.

Proof. 1. Without loss of generality, we may and shall assume that the integer r equals 2. Let $\delta = \inf_{s\in\mathbb{C}_0} \Re\mathrm{e}\,\varphi(s)$. We observe that

$$\delta = \Re\mathrm{e}\, c_1 - c_2 + c_4 > \Re\mathrm{e}\, c_1 - c_4 - \frac{c_2^2}{8c_4} \quad \text{if } c_2 > 4c_4,$$

by the arithmetic-geometric mean inequality, while clearly

$$\delta = \inf_{\theta\in\mathbb{R}} \big(\Re\mathrm{e}\, c_1 + c_2\cos\theta + c_4\cos 2\theta\big) = \Re\mathrm{e}\, c_1 - c_4 - \frac{c_2^2}{8c_4} \quad \text{if } c_2 \leqslant 4c_4. \tag{3.25}$$

So, the result is a straightforward consequence of Theorem 3.7.

We present an alternative proof, which seems instructive to us. By the proof of Theorem 3.7, it suffices to show that $\lim_{n\to\infty} \|n^{-\varphi}\|_{\mathcal{A}^+} = 0$. Instead of Gelfand's formula, we use the Hermite polynomials $(H_k)_{k\geqslant 0}$, defined by

$$H_k(\lambda) = (-1)^k e^{\lambda^2} \frac{d^k}{d\lambda^k}(e^{-\lambda^2}) = (2\lambda)^k + \text{ terms of lower degree}, \tag{3.26}$$

whose exponential generating function is

$$\sum_{k=0}^{\infty} \frac{H_k(\lambda)}{k!} x^k = \exp(2\lambda x - x^2). \tag{3.27}$$

We have the following sharp estimate (see [24]) in which $\lambda > 0$:

$$|H_k(\lambda)| \leqslant (2^k\, k!)^{1/2} \exp(\lambda^2/2). \tag{3.28}$$

It follows that, for $x > 0$ and $a > 1$:

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant C_a \exp(ax^2 + \lambda^2/2). \tag{3.29}$$

Indeed, the Cauchy–Schwarz inequality and (3.29) imply

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k &= \sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{(k!)^{1/2}(2a)^{k/2}}\, \frac{(2a)^{k/2} x^k}{(k!)^{1/2}} \\ &\leqslant \Big(\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|^2}{k!(2a)^k}\Big)^{1/2} \Big(\sum_{k=0}^{\infty} \frac{(2a)^k x^{2k}}{k!}\Big)^{1/2}\end{aligned}$$

$$\leqslant e^{\lambda^2/2}\Big(\sum_{k=0}^{\infty} a^{-k}\Big)^{1/2}\exp(ax^2) =: C_a \exp\big(ax^2+\lambda^2/2\big).$$

Now, we observe that

$$n^{-\varphi(s)} = (n^{c_0})^{-s} n^{-c_1}\exp(-c_2 2^{-s}\log n - c_4 4^{-s}\log n),$$

which allows us to write

$$n^{-\varphi(s)} = (n^{c_0})^{-s} n^{-c_1}\exp(2\lambda_n x - x^2) \quad \text{with}$$

$$x = 2^{-s}x_n, \quad x_n = \sqrt{c_4 \log n}, \quad \lambda_n = \frac{-c_2}{2\sqrt{c_4}}\sqrt{\log n}.$$

Observe next that

$$x_n^2 + \frac{\lambda_n^2}{2} = \log n\Big(c_4 + \frac{c_2^2}{8c_4}\Big). \tag{3.30}$$

Thus

$$n^{-\varphi(s)} = (n^{c_0})^{-s} n^{-c_1}\sum_{k=0}^{\infty}\frac{H_k(\lambda_n)}{k!}x_n^k(2^k)^{-s}$$

and

$$\|n^{-\varphi}\|_{\mathcal{A}^+} = n^{-\Re\mathrm{e}\, c_1}\sum_{k=0}^{\infty}\frac{|H_k(\lambda_n)|}{k!}x_n^k. \tag{3.31}$$

Using the assumption (3.22), let $0<\delta'<\delta$ (see (3.25)) and then choose $a>1$ with

$$\delta' = \Re\mathrm{e}\, c_1 - a\big(c_4 + \frac{c_2^2}{8c_4}\big).$$

Now, (3.29), (3.30), and (3.31) give, for any such δ':

$$\|n^{-\varphi}\|_{\mathcal{A}^+} \leqslant C_{\delta'} n^{-\Re\mathrm{e}\, c_1 + a(c_4+\frac{c_2^2}{8c_4})} = C_{\delta'} n^{-\delta'} =: \varepsilon_n \to 0. \tag{3.32}$$

2. Let $\theta\in\mathbb{R}$. Identity (3.31) implies that

$$n^{\Re\mathrm{e}\, c_1}\|n^{-\varphi}\|_{\mathcal{A}^+} \geqslant \Big|\sum_{k=0}^{\infty}\frac{H_k(\lambda_n)}{k!}x_n^k e^{ik\theta}\Big| = \big|\exp(2\lambda_n x_n e^{i\theta} - x_n^2 e^{2i\theta})\big|$$
$$= \exp\big(2\lambda_n x_n\cos\theta - x_n^2\cos 2\theta\big).$$

We now maximize the right-hand side by taking $\cos\theta = \frac{\lambda_n}{2x_n} = \frac{-c_2}{4c_4}$, which is allowed because $c_2 \leqslant 4c_4$. For this value of θ, we get first

$$2\lambda_n x_n\cos\theta - x_n^2\cos 2\theta = \frac{\lambda_n^2}{2} + x_n^2 = \log n\Big(c_4 + \frac{c_2^2}{8c_4}\Big)$$

and then

$$\|n^{-\varphi}\|_{\mathcal{A}^+} \geqslant n^{-\Re\mathrm{e}\, c_1 + \frac{c_2^2}{8c_4} + c_4}, \; n = 1,2,\dots, \; c_2 \leqslant 4c_4.$$

Since $\|n^{-\varphi}\|_{\mathcal{A}^+} = O(1)$, we get (3.23) (not a surprise), with strict inequality if C_φ is compact.

3. This part is more delicate, and we separate several cases.

(i) First, if $c_2 > 4c_4$, we saw in item 1 that $\mathfrak{Re}\,\varphi(s) \geqslant \delta$, for some constant $\delta > 0$, and the result follows. We may therefore assume $c_2 \leqslant 4c_4$. Then:

$$\begin{aligned}\|(2^j)^{-\varphi}\|_{\mathcal{A}^+} &= \|(2^{-j\varphi}\|_{\mathcal{A}^+} = \|(2^{-c_1-c_2 2^{-s}-c_4 4^{-s}})^j\|_{\mathcal{A}^+}\\ &= \big\|\big[\exp\big(-(c_1+c_2 2^{-s}+c_4 4^{-s})\log 2\big)\big]^j\big\|_{\mathcal{A}^+} = \|\chi^j\|_{W^+},\end{aligned}$$

where $\chi(z) = \exp\big(-(c_1 + c_2 z + c_4 z^2)\log 2\big)$. We now apply Newman's Theorem 3.1 on ordinary points to see if $\|\chi^j\|_{W^+}$ is bounded or not. Let $\theta_0 \in [0, 2\pi]$ with $|\chi(e^{i\theta_0})| = 1$, and find the coefficient of t^2 in the Taylor expansion of

$$\log\chi(e^{i(\theta_0+t)}) = -(c_1 + c_2 e^{i\theta_0}e^{it} + c_4 e^{2i\theta_0}e^{2it})\log 2.$$

This coefficient is $\big(\frac{c_2}{2}e^{i\theta_0} + 2c_4 e^{2i\theta_0}\big)\log 2$ and its real part is

$$\Big(\frac{c_2}{2}\cos\theta_0 + 2c_4(2\cos^2\theta_0 - 1)\Big)\log 2. \tag{3.33}$$

Now, observe that the condition $|\chi(e^{i\theta_0})| = 1$ means that

$$\mathfrak{Re}\,c_1 = -c_2\cos\theta_0 - c_4(2\cos^2\theta_0 - 1),$$

which gives, in view of (3.24), $(4c_4\cos\theta_0 + c_2)^2 = 0$, that is $\cos\theta_0 = -\frac{c_2}{4c_4}$. Hence, (3.33) is equal to 0 if and only if $c_2 = 4c_4$. In this case, $\theta_0 = \pi$; the Taylor expansion becomes

$$\log\chi(e^{i(\theta_0+t)}) = d_0 + d_1 t + 0.t^2 + i\log 2\frac{2c_4}{c_2}t^3 + \cdots,$$

the point θ_0 is of index 3 and is not an ordinary point. This implies as above that the sequence $(n^{-\varphi})_{n\geqslant 1}$ is unbounded and that C_φ is not bounded on $\mathcal{A}^+$.

In the case where $c_2 < 4c_4$, the point θ_0 is ordinary of index 2, and the sequence $(\|(2^j)^{-\varphi}\|_{W^+})$ is bounded. We interpolate to an arbitrary integer n as in Theorem 3.7 to get

$$\|n^{-\varphi}\|_{\mathcal{A}^+} = O(1).$$

Hence, C_φ is bounded. This ends the proof of Theorem 3.9. □

3.3. Composition operators on weighted analytic Hilbert spaces.

• Quite recently, O. Szehr and the second author, see [46], obtained an *explicit proof,* free of arithmetic considerations, of the optimality of (1.2). Their work, given an automorphism

$$\varphi = \varphi_a,\ \varphi_a(z) = \frac{a-z}{1-\overline{a}z},\ a \in \mathbb{D},$$

heavily relies on estimates of φ^n for various norms, and also, more precisely, on estimates of the Fourier coefficients $\widehat{\varphi^n}(m)$ for all integers m. They made use of standard methods for the asymptotic analysis of Laplace type integrals, see [53, 6], which are also used in the theory of orthogonal polynomials [45]. See also the forthcoming paper [8].

• Even more recently, in collaboration with P. Lefèvre, D. Li, L. Rodríguez-Piazza, the first author studied the boundedness of C_φ acting on general weighted Hilbert spaces of analytic functions on $\mathbb{D}$, namely

$$H^2(\beta) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \ : \ \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n < \infty\} \subset \mathcal{H}(\mathbb{D}).$$

Here, $\beta = (\beta_n)$, with $\liminf_{n\to\infty} \beta_n^{1/n} \geqslant 1$, is a sequence of weights. A necessary and sufficient condition when β_n is monotone nonincreasing [29] then emerged, namely:

(β_n) satisfies a Δ_2-*Orlicz type property,* meaning that

$$\beta_{2n} \geqslant \delta\beta_n \text{ for some positive constant } \delta > 0. \tag{3.34}$$

The proof of sufficiency will not be discussed here (see [29]). Then, to show that this condition was also necessary, we happened to need again the behavior of $\widehat{\varphi_a^n}(m)$; and we used a stationary phase method ($k = 3$), recovering some results of Szehr–Zarouf in a less precise form, but in a simple way. As we already said, this convergence of our concerns on the implied problems led to the present joint article. We give a sketchy proof (see [29] for more details). We will use Theorem 2.4 with a special choice of the parameters therein. We consider the Blaschke factor

$$T_a(z) = \frac{z+a}{1+az}, \text{ with } T_a^n(z) =: \sum_{m=0}^{\infty} a_{m,n} z^m, \ 0 < a < 1.$$

We will prove, denoting $\varphi = T_a = -\varphi_{-a}$ and $a_{m,n} = \widehat{\varphi^n}(m)$, the following key lemma.

Lemma 3.10. *Let $a \in (0,1)$. We set:*

$$\tau = \frac{1+a}{1-a} > 1 \tag{3.35}$$

and write:

$$\tau^{-1} = 1 - 3\mu\,, \tag{3.36}$$

with $\mu = \mu_a \in (0, 1/3)$. For n a fixed positive integer, let:

$$J_n = [(1-2\mu)\,n, (1-\mu)\,n]\,. \tag{3.37}$$

Then there exists a number $\delta = \delta_a > 0$*, and for each* n *a set of indices* $E_n \subseteq J_n$*, with cardinality* $|E_n| \geqslant \delta n$*, such that*:

$$m \in E_n \quad \Longrightarrow \quad |a_{m,n}| \geqslant \delta n^{-1/2}\,. \tag{3.38}$$

Proof. We sketch an argument and refer once more to [29] for details. We have, for $-\pi \leqslant x \leqslant \pi$:

$$T_a(e^{ix}) = \exp[ih_a(x)] \text{ with } h_a(x) = \int_0^x P_{-a}(t)dt,$$

where P_{-a} is the Poisson kernel (for the unit disk) at $-a$. Hence

$$\pi a_{m,n} = \mathfrak{Re}\, I_{m,n},$$

where

$$I_{m,n} = \int_0^\pi \exp[i(nh_a(x) - mx)]dx =: \int_0^\pi \exp[i\,F_m(x)]dx.$$

Let us fix $m \in J_n$. We check that F_m has a single critical point

$$c_{m,n} = c_m \in [0, \pi]$$

satisfying, for some $\alpha = \alpha_a > 0$:

$$K_m := [c_m - \alpha, c_m + \alpha] \subset [0, \pi], \quad \sin x \geqslant \alpha \text{ for } x \in K_m.$$

We deduce that $F''_m \approx n$ throughout K_m and $|F'''_m| \ll n$ throughout $[0, \pi]$. Therefore we can apply Theorem 2.4 with the parameters

$$A = 0,\ B = \pi,\ F_m(x) = nh_a(x) - mx,\ \lambda_2 \approx n,\ \Lambda_3 \approx n.$$

We adjust $\eta > 0$ to have $\frac{1}{\eta\lambda_2} \sim \eta^4\Lambda_3$, say $\eta = n^{-2/5} \leqslant \alpha$. We get

$$I_{m,n} = \sqrt{2\pi}\frac{e^{i\theta_m}}{\sqrt{|F''_m(c_m)|}} + O(n^{-3/5}),\ \theta_m := F_m(c_m) + \pi/4. \tag{3.39}$$

Finally, we prove that $\cos\theta_m \geqslant \delta = \delta_a > 0$ for a positive proportion E_n of the indices $m \in J_n$, say $|E_n| \geqslant \delta n$. Now, for $m \in E_n$, (3.39) gives us (with δ varying from one formula to another):

$$\pi a_{m,n} = \mathfrak{Re}\, I_{m,n} = \sqrt{2\pi}\frac{\cos\theta_m}{\sqrt{|F''_m(c_m)|}} + O(n^{-3/5}) \geqslant \delta n^{-1/2} + O(n^{-3/5}).$$

This ends the proof of Lemma 3.10. □

Here is now the main theorem of this section, with the notations of Lemma 3.10.

Theorem 3.11. *Assume that, for some $a \in (0,1)$, the composition operator $C_\varphi\colon H^2(\beta) \to H^2(\beta)$, where $\varphi = T_a$, is bounded. Then there exist two numbers $\mu = \mu_a \in (0, 1/3)$, $\delta = \delta_a > 0$, and for each n a finite set E_n of integers such that*

- $E_n \subset [(1-2\mu)n,\ (1-\mu)\,n]$;
- $|E_n| \geqslant \delta n$;
- $\beta_n \geqslant \delta^3 \frac{1}{|E_n|} \sum_{m\in E_n} \beta_m$.

In particular, when β is monotone nonincreasing, β must satisfy the Δ_2-property.

Proof. We abbreviate $H^2(\beta)$ to H, and take for E_n the set given by Lemma 3.10. Our boundedness assumption implies, since $\varphi^n(z) = \sum_{m=0}^{\infty} a_{m,n} z^m$:

$$\sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m = \|\varphi^n\|_H^2 = \|C_\varphi(z^n)\|_H^2 \leqslant C^2 \|z^n\|_H^2 = C^2 \beta_n.$$

This gives the general necessary condition

$$C^2\beta_n \geqslant \sum_{m=0}^{\infty} |a_{m,n}|^2 \beta_m \geqslant \sum_{m\in E_n} |a_{m,n}|^2 \beta_m \geqslant \sum_{m\in E_n} \delta^2 n^{-1} \beta_m \geqslant \delta^3 \biggl(\frac{1}{|E_n|} \sum_{m\in E_n} \beta_m\biggr).$$

When β is monotone nonincreasing, we get $C^2\beta_n \geqslant \delta^3 \beta_{(1-\mu)n}$, and (β_n) satisfies Δ_2. □

§4. Schäffer's question on norms of inverses

In this section we deal with a problem which goes back to the early 70's and to studies of B. L. van der Waerden, W. A. Coppel and J. J. Schäffer (see [43] and the references therein). Let $K = K_n$ be the smallest C (indeed, $K_n < \infty$) such that

$$|\det T| \|T^{-1}\| \leqslant C \|T\|^{n-1}$$

for any invertible operator T acting on *any* n-dimensional *Banach* space. By homogeneity, replacing T by $T/\|T\|$, we see that K_n is the best possible constant with

$$\|T^{-1}\| \leqslant K_n \biggl(\prod_i |\lambda_i(T)|\biggr)^{-1},$$

for any invertible *contraction* T acting on *any* n-dimensional Banach space, where the $\lambda_i(T)$ are the eigenvalues of T counted with multiplicities.

4.1. Schäffer's upper bound. Making use of Cayley–Hamilton's theorem, Coppel proved that

$$K_n \leqslant 2^n - 1,$$

see [38, Theorem 2.1] for a detailed proof. In 1970 J. J. Schäffer [43] proved that

$$K_n \leqslant \sqrt{en} \tag{4.1}$$

making use of a famous theorem by F. John [38, Theorem 2.3]. The latter bound was rediscovered by E. Gluskin, M. Meyer and A. Pajor [21, Proposition 3] and later by N. K. Nikolski [33, Theorem 3.20]. It was also recently generalized in [48, Theorem 8], providing an answer to the (more general) question of bounding the norm of the resolvent of T instead of the norm T^{-1}. The proofs in [33] and [48] are based on an analytic expression for K_n provided in [21] in terms of a "max-min-type" optimization problem,

$$K_n = \sup_{(\lambda_1,\ldots,\lambda_n)\in\mathbb{D}^n} F\left(\lambda_1,\ldots,\lambda_n\right), \tag{4.2}$$

where F is given by

$$\begin{aligned} &F(\lambda_1,\ldots,\lambda_n) \\ &:= \inf\left\{\sum_{k=1}^{\infty}|a_k|; f(z)=\prod_{i=1}^{n}\lambda_i+\sum_{k=1}^{\infty}a_k z^k,\ f(\lambda_i)=0,\ i=1\ldots n\right\}. \end{aligned} \tag{4.3}$$

We first provide a short and elegant proof of the upper bound obtained by Schäffer, due to Nikolski [33, Theorem 3.20]. The proof combines Nikolski's ideas together with an argument of F. L. Nazarov. As it is stated, [33, Theorem 3.20] actually gives the upper bound $\sqrt{n+1}$ instead of $\sqrt{n}$ in (4.1), and below we adapt Nikolski's proof to recover Schäffer's original inequality.

Proof of Schäffer's inequality (4.1)**.** Let T be a contraction acting on an n-dimensional Banach space ($n \geqslant 2$) and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues counted with multiplicities. Let $r=\sqrt{1-\frac{1}{n}}$, so that $r^{n-1}\geqslant e^{-1/2}$, and let $\widetilde{\varphi}$ be defined by

$$\widetilde{\varphi}(z)=\prod_{i=1}^{n}\frac{z-r\lambda_i}{1-r\overline{\lambda_i}z},\quad \widetilde{\varphi}_r=\widetilde{\varphi}(rz).$$

We consider the rational function

$$R(z)=\frac{\widetilde{\varphi}_r(0)-\widetilde{\varphi}_r(z)}{z\widetilde{\varphi}_r(0)},$$

which is analytic in $\mathbb{D}$. Observing that $\widetilde{\varphi}_r(T) = 0$, we get $T^{-1} = R(T)$ and therefore

$$\begin{aligned}\|T^{-1}\| &= \|\sum_{k\geqslant 0} \widehat{R}(k)T^k\| \\ &\leqslant \sum_{k\geqslant 0} |\widehat{R}(k)| \\ &= \frac{1}{r^n\prod_{i=1}^n |\lambda_i|}\left\|\frac{\widetilde{\varphi}_r(0)-\widetilde{\varphi}_r(z)}{z}\right\|_{W^+} \\ &= \frac{1}{r^n\prod_{i=1}^n |\lambda_i|}\sum_{k\geqslant 1} r^k|\widehat{\widetilde{\varphi}}(k)|,\end{aligned}$$

where we have observed that for the kth-Fourier coefficients of $\widetilde{\varphi}_r$ we have

$$\widehat{\widetilde{\varphi}_r}(k) = r^k\widehat{\widetilde{\varphi}}(k).$$

Applying the Cauchy–Schwarz inequality, we get

$$|\det T|\|T^{-1}\| \leqslant r^{-n}\sum_{k\geqslant 1} r^k|\widehat{\widetilde{\varphi}}(k)| \leqslant r^{-(n-1)}\left(1-r^2\right)^{-1/2}\|\widetilde{\varphi}\|_{H^2},$$

and it remains to notice that $\|\widetilde{\varphi}\|_{H^2} \leqslant \|\widetilde{\varphi}\|_{H^\infty} \leqslant 1$ to conclude. (Here H^∞ stands for the classical Hardy space (algebra) of the disk, endowed with the sup norm on $\mathbb{D}$.) □

4.2. Schäffer's conjecture. Obtaining $K_n = 2$ for $\mathbb{R}^n$ endowed with the ℓ^1 norm, Schäffer conjectured that the sequence $(K_n)_{n\geqslant 1}$ is uniformly bounded. This was refuted in the early 90's by E. Gluskin, M. Meyer, and A. Pajor, see [21, Theorem 1], who employed a probabilistic approach to prove that

$$K_n \geqslant \frac{c}{\log\log n}\sqrt{\frac{n}{\log n}}$$

where $c > 0$ is an absolute constant. Subsequent contributions of J. Bourgain [21, Appendix] and of the first author [39, Theorem 1] provided increasing lower estimates on K_n, see below for details. Those estimates make use of the Gluskin–Meyer–Pajor identity (4.2) and are built on a lemma by J. Bourgain, which provides a lower estimate on F.

Lemma 4.1 (Bourgain). *For any sequence $(\lambda_1,\ldots,\lambda_n) \in \mathbb{D}^n$ we have*

$$F(\lambda_1,\ldots,\lambda_n) \geqslant \frac{n\prod_{i=1}^n |\lambda_i|}{\max_{k\geqslant 1}\left|\sum_{i=1}^n \lambda_i^k\right|} - \prod_{i=1}^n |\lambda_i|.$$

Using Bourgain's lemma, the key point to improve Gluskin–Meyer–Pajor's estimate on K_n is to find a sequence $(\lambda_i)_{i=1}^n$ such that:

(1) $|\lambda_i| = 1 - 1/n$, for all $i = 1, \ldots, n$ (so that the sequence $(\prod_{i=1}^{n} |\lambda_i|)_{n\geqslant 1}$ is bounded from below),
(2) there exists a sequence $(u_n)_n$ satisfying

$$\max_{k\geqslant 1} |\sum_{i=1}^{n} \lambda_i^k| \ll u_n \tag{4.4}$$

with

$$u_n \ll \sqrt{n \log n} \log \log n$$

and ideally (since we aim at proving the sharpness of Schäffer's inequality) such that

$$u_n \approx \sqrt{n}.$$

Bourgain [21, Appendix] combined Lemma 4.1 with a probabilistic approach to prove that

$$K_n \gg \sqrt{\frac{n}{\log n}}. \tag{4.5}$$

More precisely, Bourgain considered n independent complex valued random variables $(Z_i)_{i=1}^n$ uniformly distributed on the unit circle $\mathbb{T}$ and observed that for any integer $k \geqslant 1$ the sequence of random variables $(Z_i^k)_{i=1}^n$ follows the same distribution as the sequence $Z_1, \ldots, Z_n$. Applying the classical Bernstein inequality to these sequences, he found that

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} Z_i^k\right| \leqslant 4\sqrt{n \log(1+k)}, \quad \forall k \geqslant 1\right) > 0$$

and therefore that there exists $(z_1, \ldots, z_n) \in \mathbb{T}^n$ such that

$$\max_{k\geqslant 1} \left|\sum_{i=1}^{n} z_i^k\right| \leqslant 4\sqrt{n \log(1+k)}. \tag{4.6}$$

To conclude, Bourgain put $\lambda_i = (1 - 1/n)z_i$ for all $i = 1, \ldots, n$ and applied Lemma 4.1 to the sequence $(\lambda_1, \ldots, \lambda_n)$ to prove (4.5).

Later the first author [39] combined Lemma 4.1 with a number theoretic approach to improve Bourgain's inequality (4.5) showing that $\sqrt{n}$ in Schäffer's original result is actually asymptotically sharp as $n \to \infty$. More precisely he proved, see [39, Theorem 1], that

$$K_n \geqslant \sqrt{\frac{n}{2e}}, \tag{4.7}$$

and that

$$K_n \geqslant \sqrt{n}(1 - \mathcal{O}(1/n)), \tag{4.8}$$

which is the currently best known lower estimate on K_n. To prove (4.7) the first author first assumes that $n = p-1$ where p is a prime number and considers a character $\chi \mod p$ of order $p-1$. Considering such a character he puts

$$z_j = \chi(j)e^{ij/p}, \quad j = 1, \dots, p-1$$

and uses the theory of Gauss sums to prove that

$$\max_{1 \leqslant k < p(p-1)} \left| \sum_{i=1}^{n} z_i^k \right| \leqslant \sqrt{p}.$$

Then he puts $\lambda_i = (1-1/p)z_i$ for all $i = 1, \dots, p-1$ and thereby satisfies (4.4) with $u_p \approx \sqrt{p}$, which proves the sharpness of Schäffer's inequality when $n = p-1$. The rest of the proof of (4.7) and the one of (4.8) is based on an application of Bertrand–Chebyshev theorem.

The first author mentions however [38, Remark 4.7] that the choice of $z_1, \dots, z_{p-1}$ above is not so explicit because nobody knows an explicit generator of $\mathbb{F}_p^*$ (the cyclic multiplicative group of nonzero elements of $\mathbb{Z}/p\mathbb{Z}$), or equivalently of its dual $\widehat{\mathbb{F}_p^*}$ whose generator is denoted above by χ. It was also later observed [2] that in essence satisfying (4.4) with $u_n \approx \sqrt{n}$ is the so-called Turan's tenth problem which to date has no constructive solution [2, 52]. Observe that the construction of explicit solutions to such problems appears to be a well-studied but open problem in number theory [52, 40, 15, 2, 3]. Moreover Gluskin–Meyer–Pajor also mention [21, p. 2] that they do not know a concrete example of $(\lambda_1, \dots, \lambda_n)$ for which $F(\lambda_1, \dots, \lambda_n)$ is growing. These remarks and observations motivated the second author and O. Szehr [46, 47] first to find an explicit example of sequence $(\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$ such that

$$\sup_n F(\lambda_1, \dots, \lambda_n) = \infty \tag{4.9}$$

and second to find an explicit example of a sequence $(\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$ that asymptotically achieves Schäffer's upper bound, i.e., such that

$$F(\lambda_1, \dots, \lambda_n) \gg \sqrt{n}.$$

This will be done in two steps on the basis of results proved in [47] and [46] respectively.

(1) In Subsection 4.3 below, we will show using fairly elementary methods that for any $\lambda \in \mathbb{D} \setminus \{0\}$ we have

$$F(\lambda, \dots, \lambda) \gg n^{1/3}, \tag{4.10}$$

where λ is repeated according to its multiplicity n. To this end we will use a simple duality method to prove an analog of Bourgain's lemma (see Lemma 4.2 below), which circumvents the power sum theory. Our version of Bourgain's lemma relates Schäffer's question to the study of asymptotics for Fourier coefficients of powers of functions. More precisely, it will allow us to bound F from below by $\|(\widehat{\varphi^n}(k))_{k\geqslant 0}\|_{l^\infty}$ with $\varphi = \varphi_\lambda = \frac{z-\lambda}{1-\bar{\lambda}z}$. Therefore we will see that (4.9) is a direct consequence of Lemma 4.2 and of the fact that

$$|a_{n,k}| \ll n^{-1/3} \tag{4.11}$$

uniformly for $k \geqslant 0$, where the sequence $(a_{n,k})_{k\geqslant 0}$ was defined above in Subsection 3.3, under the assumption that $a = \lambda \in (0,1)$. The proof of (4.11) (see Proposition 4.4 below) is based on repeated application of the classical van der Corput inequality stated in Proposition 2.1 above.

(2) In Subsection 4.5 we will use more sophisticated methods to see that in fact a stronger estimate holds:

$$F(\lambda, \dots, \lambda) \gg \sqrt{n} \tag{4.12}$$

for any $\lambda \in \mathbb{D} \setminus \{0\}$. In particular, the proof of (4.12) requires the use of a modified version of our analog of Bourgain's lemma, together with a careful investigation of the asymptotic behavior of the sequence $(a_{n,k})_{k\geqslant 0}$ and of a certain suitable linear combination of the $a_{n,k}$'s. The latter can be done by using standard methods for asymptotic analysis: the stationary phase method (see Theorem 2.4), the method of the steepest descent [53, 14, 10], and uniform versions of those methods [6, Chapter 9], [9]. We refer to the recent work [8] for a detailed application of those methods to obtain asymptotic formulas for $a_{n,k}$ as $n \to \infty$ and $k \geqslant 0$.

4.3. $F(\lambda, \dots, \lambda)$ grows at least as $n^{1/3}$.

4.3.1. *The first analog of Bourgain's lemma.* Building on the duality approach developped in [46], we prove the first analog of Bourgain's lemma. It relates the problem of bounding $F(\lambda_1, \dots, \lambda_n)$ from below to the question of finding an upper bound (as sharp as possible asymptotically as $n \to \infty$) on the Fourier coefficients of the finite Blaschke product associated with the sequence $(\lambda_1, \dots, \lambda_n)$. To formulate this analog we need to introduce the space l^∞_A of analytic functions f on $\mathbb{D}$ with bounded Taylor coefficients,

$$l^\infty_A = \left\{ f = \sum_{k\geqslant 0} \widehat{f}(k) z^k \in \mathcal{H}(\mathbb{D});\ \|f\|_{l^\infty_A} = \sup_{k\geqslant 0} |\widehat{f}(k)| < \infty \right\}.$$

As a direct application of our lemma, we deduce that, given a sequence $(\lambda_1, \dots, \lambda_n)$ in the disk, if the corresponding finite Blaschke product

$$\varphi(z) = \prod_{i=1}^{n} \frac{z - \lambda_i}{1 - \bar{\lambda}_i z}$$

satisfies

$$\lim_n \|\varphi\|_{l^\infty_A} = 0,$$

then the sequence $(\lambda_1, \dots, \lambda_n)$ realizes (4.9). We will see that this is already the case of the sequence $(\lambda, \dots, \lambda)$ where λ is arbitrary in $\mathbb{D} \setminus \{0\}$.

Lemma 4.2. *For any sequence* $(\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$ *we have*

$$F(\lambda_1, \dots, \lambda_n) \geqslant \frac{1}{\|\varphi\|_{l^\infty_A}} - \prod_{i=1}^{n} |\lambda_i|,$$

where $\varphi(z) = \prod_{i=1}^{n} \frac{z-\lambda_i}{1-\bar{\lambda}_i z}$, *and* F *is defined by* (4.3).

Remark 4.3. Comparing this lemma with Lemma 4.1, we observe that the numerator $\prod_{i=1}^{n} |\lambda_i|$ disappeared. Therefore, it is no longer necessary to assume that $|\lambda_i| = 1 - 1/n$ for all $i = 1, \dots, n$.

Proof. Let $L^2(\mathbb{T})$ be the usual L^2 space on the unit circle equipped with the standard scalar product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(e^{it}) \overline{g(e^{it})} \frac{dt}{2\pi}.$$

For $f = \sum_k \widehat{f}(k) z^k$, $g = \sum_k \widehat{g}(k) z^k \in \mathcal{H}(\mathbb{D})$, the $L^2(\mathbb{T})$ scalar product can be written as

$$\langle f, g \rangle = \sum_{k \geqslant 0} \widehat{f}(k) \overline{\widehat{g}(k)}.$$

To bound F from below, we will apply Hölder's inequality in the form

$$|\langle f, g \rangle| \leqslant \|f\|_{l^\infty_A} \|g\|_{W^+},$$

where, as before, W^+ stands for the Wiener algebra of absolutely convergent Taylor series in $\mathbb{D}$ (see §3 above for its definition). Note that φ maps the unit disk onto itself and satisfies $\overline{\varphi(z)} = \frac{1}{\varphi(z)}$ for $z \in \mathbb{T}$, [16]. It is easily verified that if f is in W^+ and $f(\lambda) = 0$ with $\lambda \in \mathbb{D}$, then $\left\| \frac{f}{z-\lambda} \right\|_{W^+} \leqslant \frac{\|f\|_{W^+}}{1-|\lambda|}$. Hence, $\frac{f}{z-\lambda}$ is also in W^+, (this is sometimes called the division property of W^+, see for instance [1, p. 22] for more general algebras satisfying this property). Therefore for any $h \in W^+$ with $h(\lambda_i) = 0$ we have $\frac{h}{\varphi} \in W^+$. Now we rewrite the Gluskin–Meyer–Pajor expression for F using the norm of h in W^+:

$$\begin{aligned} &F(\lambda_1, \dots, \lambda_n) \\ &\quad = \inf\Big\{ \|h\|_{W^+} - |h(0)| \;\Big|\; h \in W^+,\ h(0) = \prod_{i=1}^{n} \lambda_i,\ h(\lambda_i) = 0,\ i = 1 \dots n \Big\}. \end{aligned}$$

We are now ready to bound F from below. Let $h \in W^+$ with $h(\lambda_i) = 0$ and $h(0) = \prod_{i=1}^{n} \lambda_i$, and let $g = \frac{h}{\varphi} \in W^+$. We have

$$\begin{aligned} \langle h, \varphi\rangle &= \langle g, 1\rangle \\ &= g(0) = (-1)^n. \end{aligned}$$

Applying Hölder's inequality, we conclude that

$$1 \leqslant \|h\|_{W^+} \|\varphi\|_{l^\infty_A} .$$

It follows that any function h in the definition of F satisfies

$$\|h\|_{W^+} \geqslant \frac{1}{\|\varphi\|_{l^\infty_A}}$$

and the proof of Lemma 4.2 follows. □

If we consider the simplest choice $\lambda_1 = \cdots = \lambda_n = \lambda \in \mathbb{D} \setminus \{0\}$, Lemma 4.2 leads to the question of finding an asymptotically sharp upper bound on $\|\varphi_\lambda^n\|_{l^\infty_A}$ as $n \to \infty$, where $\varphi_\lambda(z) = \frac{z-\lambda}{1-\overline{\lambda} z}$. Due to rotation invariance of the l^∞_A-norm, we may assume that $\lambda = a \in (0,1)$ without loss of generality, and we have

$$\|\varphi_\lambda^n\|_{l^\infty_A} = \sup_{k \geqslant 0} |a_{n,k}|$$

where the sequence $(a_{n,k})_{k\geqslant 0}$ was defined in Subsection 3.3.

4.3.2. *Upper bounds on* $\left|\widehat{\varphi_\lambda^n}(k)\right|$. Our aim in this subsection is to prove the proposition below, which was originally stated in [48]. The van der Corput inequality stated in Proposition 2.1 will be the key ingredient for obtaining the upper estimates (2), (3), and (4) below.

Proposition 4.4. *Suppose that $\varphi = \varphi_\lambda$ with $\lambda \in (0,1)$, $n \geqslant 1$, and $k \geqslant 0$. Set $\alpha_0 := \frac{1-\lambda}{1+\lambda}$ and choose a fixed $\alpha \in (0, \alpha_0)$. The following assertions hold true depending on the region to which k belongs.*

(1) *If $k/n \leqslant \alpha$, then $|\widehat{\varphi^n}(k)|$ decays exponentially and uniformly over k as n tends to ∞. Similarly, if $k/n \geqslant \alpha^{-1}$, then $|\widehat{\varphi^n}(k)|$ decays exponentially and uniformly over k as n tends to ∞.*

(2) *If* $k/n \in (\alpha, \alpha_0 - n^{-2/3}) \cup (\alpha_0^{-1} + n^{-2/3}, \alpha^{-1})$, *then*

$$|\widehat{\varphi^n}(k)| \ll \max\left\{\frac{1}{|\alpha_0 n - k|}, \frac{1}{|\alpha_0^{-1} n - k|}\right\}.$$

(3) *If* $k/n \in \left[\alpha_0 - n^{-2/3}, \alpha_0 + n^{-2/3}\right) \cup \left(\alpha_0^{-1} - n^{-2/3}, \alpha_0^{-1} + n^{-2/3}\right]$, *then*

$$|\widehat{\varphi^n}(k)| \ll \frac{1}{n^{1/3}}.$$

(4) *If* $k/n \in (\alpha_0 + n^{-2/3}, \alpha_0^{-1} - n^{-2/3})$, *then*

$$|\widehat{\varphi^n}(k)| \ll \max\left\{\frac{1}{n^{1/2}|\alpha_0 - \frac{k}{n}|^{1/4}}, \frac{1}{n^{1/2}|\alpha_0^{-1} - \frac{k}{n}|^{1/4}}\right\}.$$

In particular, for the l^∞_A*-norm of* φ^n *we have*

$$\|\varphi^n\|_{l^\infty_A} \ll \frac{1}{n^{1/3}}.$$

To apply the van der Corput inequality given in Proposition 2.1, we rewrite $\widehat{\varphi^n}(k)$ in a convenient way. First $\varphi(e^{it}) \in \mathbb{T}$ for any $t \in (-\pi, \pi]$ and there exists a real-valued and continuously differentiable function f_λ so that

$$\varphi(e^{it}) = e^{if_\lambda(t)}, \quad t \in (-\pi, \pi].$$

Differentiating the above identity with respect to t, we find

$$ie^{it}\frac{1-\lambda^2}{(1-\lambda e^{it})^2} = if'_\lambda(t)\varphi(e^{it})$$

which shows that

$$f'_\lambda(t) = \frac{1-\lambda^2}{|1-\lambda e^{it}|^2} = \frac{1-\lambda^2}{1+\lambda^2-2\lambda\cos t}, \quad t \in (-\pi, \pi].$$

Taking into account the fact that λ is real, we can write

$$\widehat{\varphi^n}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi^n(e^{it})e^{-ikt}\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ig(t)}\mathrm{d}t = \frac{1}{\pi}\Re\mathrm{e}\left\{\int_0^{\pi} e^{ig(t)}\mathrm{d}t\right\}, \quad (4.13)$$

where

$$g(t) = nf_\lambda(t) - kt \quad t \in [0, \pi].$$

Computing derivatives we find that $g'(0) = n\alpha_0^{-1} - k$, $g'(\pi) = n\alpha_0 - k$, and

$$g''(t) = -\frac{2\lambda n(1-\lambda^2)\sin t}{(1+\lambda^2-2\lambda\cos t)^2}.$$

This implies that g' is strictly monotone decreasing on $(0,\pi)$ with

$$g'(0) = n\alpha_0^{-1} - k > g'(t) > n\alpha_0 - k = g'(\pi).$$

Proof of Proposition 4.4. The proof of Proposition 4.4 when k is closer to $n\alpha_0$ than to $n\alpha_0^{-1}$ was provided in [48]. Below we adapt the proof of [48, Proposition 3] to ensure (1)–(4) for k closer to $n\alpha_0^{-1}$ than to $n\alpha_0$.

(1) This is a direct application of [45, Proposition 6, point (1)]. It is well known (see [16]) that for $z, w \in \mathbb{D}$ we have the following upper and lower bounds on φ_w:

$$\frac{|z| - |w|}{1 - |z||w|} \leqslant \left| \frac{z - w}{1 - \bar{w}z} \right| \leqslant \frac{|z| + |w|}{1 + |z||w|}.$$

The Fourier coefficients $\widehat{\varphi^n}(k)$ can be expressed via the usual contour integral

$$\widehat{\varphi^n}(k) = \frac{1}{2i\pi} \oint_{|z|=s} z^{-k-1} \left(\frac{z - \lambda}{1 - \lambda z} \right)^n \mathrm{d}z,$$

where $s \in (1, 1/\lambda)$. For the magnitude of the integral (bounding it roughly) we find that if $s \in (1, 1/\lambda)$, then

$$|\widehat{\varphi^n}(k)| \leqslant \max_{|z|=s} \frac{|\varphi^n(z)|}{|z|^k} = \frac{\varphi^n(s)}{s^k}.$$

If $k/n \geqslant \alpha^{-1}$, then there exists $s^* \in (1, 1/\lambda)$ such that

$$\frac{\varphi(s^*)}{s^{*k/n}} \leqslant \frac{\varphi(s^*)}{s^{*\alpha^{-1}}} < 1,$$

see [45, Proposition 6, item (1)] for more details and for the computation of s^*.

To prove items (2)–(4), we will apply van der Corput's inequality directly to the right-hand side in (4.13).

(2) If $k/n \in (\alpha_0^{-1} + n^{-2/3}, \alpha^{-1})$, then $g'(0) = n\alpha_0^{-1} - k < 0$. In particular, for any $t \in [0, \pi]$ we have

$$g'(t) \leqslant n\alpha_0^{-1} - k < 0,$$

and both g and g' are strictly monotone decreasing on this interval. Applying Proposition 2.1, we get

$$|\widehat{\varphi^n}(k)| \ll \frac{1}{k - n\alpha_0^{-1}}.$$

(3) If $k/n \in [\alpha_0^{-1} - n^{-2/3}, \alpha_0^{-1} + n^{-2/3}]$, then $g'(0) = n\alpha_0^{-1} - k$ may be positive or negative depending on the choice of k. We fix a constant $c > 0$ (independent of n) whose exact value is to be specified later. We split the integral

$$\int_0^{\pi} e^{ig(t)} \mathrm{d}t = \int_0^{cn^{-1/3}} e^{ig(t)} \mathrm{d}t + \int_{cn^{-1/3}}^{\pi} e^{ig(t)} \mathrm{d}t$$

and first observe that

$$\left| \int_0^{cn^{-1/3}} e^{ig(t)} \mathrm{d}t \right| \leqslant cn^{-1/3}.$$

To estimate the second integral, we will apply Proposition 2.1 once again, which requires a lower estimate on $|g'(t)|$ for $t \in [cn^{-1/3}, \pi]$. To achieve this, we first expand the function f'_λ in a neighborhood of 0:

$$f'_\lambda(t) = \alpha_0^{-1} - \frac{\lambda(1+\lambda)}{(1-\lambda)^3} t^2 + \mathcal{O}(t^4)$$

as t tends to 0, and compute the asymptotic value of $g'(cn^{-1/3})$ as $n \to \infty$. We have

$$\begin{aligned} g'(cn^{-1/3}) &= nf'_\lambda(cn^{-1/3}) - k \\ &= (\alpha_0^{-1} n - k) - c^2 \frac{\lambda(1+\lambda)}{(1-\lambda)^3} n^{1/3} + \mathcal{O}(n^{-1/3}) \\ &\leqslant \left(1 - c^2 \frac{\lambda(1+\lambda)}{(1-\lambda)^3} + \mathcal{O}(n^{-2/3}) \right) n^{1/3}, \end{aligned}$$

where we have made use of the assumption $\alpha_0^{-1} n - k \leqslant n^{1/3}$. In particular, there exists $c = c(\lambda) > 0$ such that

$$1 - c^2 \frac{\lambda(1+\lambda)}{(1-\lambda)^3} + \mathcal{O}(n^{-2/3}) \leqslant -\widetilde{c} < 0,$$

where $\widetilde{c} = \widetilde{c}(\lambda) > 0$. The function g' being monotone decreasing on the interval $[0, \pi]$, for $t \in [cn^{-1/3}, \pi]$ and for large n we find

$$g'(t) \leqslant g'(cn^{-1/3}) \leqslant -\widetilde{c} n^{1/3} < 0.$$

Applying Proposition 2.1 on $[cn^{-1/3}, \pi]$, we obtain

$$\left| \int_{cn^{-1/3}}^{\pi} e^{ig(t)} \mathrm{d}t \right| \ll n^{-1/3}.$$

(4) If $k/n \in (\alpha_0 + n^{-2/3}, \alpha_0^{-1} - n^{-2/3})$, then the equation $g'(t) = 0$ has exactly one solution t_+ on $(0, \pi)$. A direct computation shows that

$$|g''(t_+)| = k\left(\frac{k}{n} - \alpha_0\right)^{1/2}\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/2}.$$

We choose $\eta = \eta(n) > 0$ whose exact value is to be specified later. We split the integral

$$\int_0^\pi e^{ig(t)}\mathrm{d}t = \int_0^{t_+-\eta} e^{ig(t)}\mathrm{d}t + \int_{t_+-\eta}^{t++\eta} e^{ig(t)}\mathrm{d}t + \int_{t_++\eta}^{\pi} e^{ig(t)}\mathrm{d}t$$

and notice that

$$\left|\int_{t_+-\eta}^{t_++\eta} e^{ig(t)}\mathrm{d}t\right| \leqslant 2\eta.$$

The remaining integrals are treated via Proposition 2.1. Since g' is monotone decreasing on $(0, \pi)$ and $g'(0) = n\alpha_0^{-1} - k$, we have

$$\left|\int_0^{t_+-\eta} e^{ig(t)}\mathrm{d}t\right| \ll \max\left\{\frac{1}{|g'(0)|}, \frac{1}{|g'(t_+ - \eta)|}\right\}$$
$$= \max\left\{\frac{1}{n\alpha_0^{-1} - k}, \frac{1}{|g'(t_+ - \eta)|}\right\}.$$

As before, we assume that k is closer to $\alpha_0^{-1}n$ so that

$$(k - n\alpha_0)^{-1} \leqslant (n\alpha_0^{-1} - k)^{-1}$$

and we seek a suitable lower bound for $|g'(t_+ - \eta)|$. This is achieved as follows. First we use the mean value theorem for integrals to see that there is $s = s(n) \in (t_+ - \eta,\, t_+)$ with

$$|g'(t_+ - \eta)| = \left|\int_{t_+-\eta}^{t+} g''(t)\mathrm{d}t\right| = |g''(s)|\eta.$$

By the mean value theorem for differentiation, there exists also

$$u = u(n) \in (s, t_+)$$

such that

$$\frac{|g''(t_+) - g''(s)|}{|g''(t_+)|} = \frac{(t_+ - s)|g'''(u)|}{|g''(t_+)|} \ll \frac{\eta n}{k\left(\frac{k}{n} - \alpha_0\right)^{1/2}\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/2}}$$
$$\ll \frac{\eta}{\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/2}},$$

where we have used, for the last inequality, that

$$k/n \in \left(\alpha_0 + n^{-2/3}, \alpha_0^{-1} - n^{-2/3}\right)$$

is bounded from below. We also have made use of the assumption that k/n is closer to α_0^{-1} than α_0, which implies that $\left(\frac{k}{n} - \alpha_0\right)^{-1/2}$ is bounded by a constant. In particular assuming

$$\eta = \frac{1}{2\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/4} n^{1/2}},$$

we have $\eta \leqslant \frac{1}{2n^{1/3}}$ and

$$|g''(s)| \geqslant |g''(t_+)| \cdot \left|1 - \frac{|g''(t_+) - g''(s)|}{|g''(t_+)|}\right|$$
$$\gg |g''(t_+)|.$$

In summary, we find

$$\left|\int_0^{t_+-\eta} e^{ig(t)}\mathrm{d}t\right| \ll \frac{1}{n\alpha_0^{-1} - k} + \frac{1}{\eta|g''(t_+)|}$$
$$\ll \frac{1}{n\alpha_0^{-1} - k} + \frac{1}{\eta k\left(\frac{k}{n} - \alpha_0\right)^{1/2}\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/2}}$$
$$\ll \frac{1}{n\alpha_0^{-1} - k} + \frac{1}{\eta n\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/2}}$$
$$\ll \frac{1}{n\alpha_0^{-1} - k} + \frac{1}{n^{1/2}\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/4}}.$$

A similar argument applies to $\int_{t_++\eta}^{\pi} e^{ig(t)}\mathrm{d}t$. We finally obtain

$$\left|\int_0^{\pi} e^{ig(t)}\mathrm{d}t\right| \ll \eta + \frac{1}{n\alpha_0^{-1} - k} + \frac{1}{n^{1/2}\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/4}}$$

$$
\begin{aligned}
&\ll \frac{1}{n\alpha_0^{-1}-k}+\frac{1}{n^{1/2}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/4}}\\
&\ll \frac{1}{n^{1/2}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/4}}\left(1+\frac{1}{n^{1/2}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{3/4}}\right)\\
&\ll \frac{1}{n^{1/2}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/4}}\left(1+\frac{1}{n^{1/2}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{3/4}}\right)\\
&\ll \frac{1}{n^{1/2}\left(\alpha_0^{-1}-\frac{k}{n}\right)^{1/4}},
\end{aligned}
$$

which completes the proof. □

4.4. On the l^p_A-norms of φ^n. More generally, the l^p_A-norms of φ^n (where $\varphi=\varphi_\lambda$) are defined for $1\leqslant p<\infty$ by

$$
\|\varphi^n\|^p_{l^p_A} := \|\widehat{\varphi^n}\|^p_{l^p} := \sum_{k\geqslant 0} |\widehat{\varphi^n}(k)|^p,
$$

and for $p=\infty$ by $\|\varphi^n\|_{l^\infty_A} := \sup_{k\geqslant 0}|\widehat{\varphi^n}(k)|$. Below we will see that the above Proposition 4.4 yields asymptotically sharp upper estimates on

$$
\|\varphi^n\|_{l^p_A} \ \text{ for } p\in[1,\infty],
$$

as $n\to\infty$. The study of the l^p-norms of $\widehat{\varphi^n}$ was probably initiated by J.-P. Kahane [28] who was interested in the case of $p=1$. Kahane's motivation [28, Theorem 1] was to generalize a theorem by Z. K. Leibenson [30], which is a special case of a theorem (see [42, Theorem 4.1.3]) about homomorphisms of group algebras due to P. T. Cohen. Let $\theta:\mathbb{R}\to\mathbb{R}$ be a continuous, nonconstant and 2π-periodic function. A. Beurling and H. Helson [5] proved that if $\|\widehat{e^{in\theta}}\|_{l^1}=\mathcal{O}(1)$, $n\in\mathbb{Z}$, then θ is affine. Kahane proved that:

(1) if θ is piecewise linear, then $\|\widehat{e^{in\theta}}\|_{l^1}=\mathcal{O}(\log(|n|))$, [28, Theorem III], and
(2) if θ is analytic, then $\|\widehat{e^{in\theta}}\|_{l^1}\approx\sqrt{|n|}$, [28, Theorem V].

Writing $\varphi(e^{it})$ as $e^{i\theta(t)}$ for $t\in(-\pi,\pi]$, we deduce from (2) that

$$
\|\widehat{\varphi^n}\|_{l^1}\sim c_1\sqrt{n},\quad n\to\infty.
$$

The precise value c_1 of the limit

$$
\lim_{n\to\infty} n^{-1/2}\|\widehat{\varphi^n}\|_{l^1}
$$

was computed in [20]. A discussion on l^p-norms for $p \in [1,\infty]$ occurred in [7], where the asymptotic behavior

$$\|\widehat{\varphi^n}\|_{l^p} \approx n^{\frac{2-p}{2p}} \text{ for } p \in [1,2] \tag{4.14}$$

was derived. The discussion in [7] is more general and again is motivated by investigating the boundedness of the composition operator C_φ. To extend the result of [7] to the whole interval $p \in [1,\infty]$, we first split the sum

$$\sum_{k\geqslant 0} |\widehat{\varphi^n}(k)|^p$$

according to the regions of Proposition 4.4 and get (see [47] for details)

$$\|\varphi^n\|_{l_p^A} \ll \begin{cases} n^{\frac{2-p}{2p}} & \text{if } p \in [1,4), \\ \left(\frac{\log n}{n}\right)^{\frac{1}{4}} & \text{if } p = 4, \\ n^{\frac{1-p}{3p}} & \text{if } p \in (4,\infty], \end{cases}$$

with constants depending only on p and λ. It was shown in [47] that these upper bounds are asymptotically sharp as $n \to \infty$. For $p \in [1,4)$ this can be seen directly from Lemma 3.10. Indeed, keeping the notation of this lemma, we get

$$\begin{aligned} \|\varphi^n\|_{l_p^A} &> \sum_{k\in E_n} |\widehat{\varphi^n}(k)|^p \\ &\gg |E_n| n^{-p/2} \\ &\gg n^{\frac{2-p}{2}}. \end{aligned}$$

The proofs of the sharpness of the remaining upper bounds for $p = 4$ and $p \in (4,\infty]$ are more delicate and require an asymptotic expansion of $\widehat{\varphi^n}(k)$, which is uniform in k as it approaches one of the end-points $\alpha_0 n$, $\alpha_0^{-1} n$, see [47].

4.5. $F(\lambda,\dots,\lambda)$ **grows as** $\sqrt{n}$**.** The important Proposition 4.4 suggests that the slowest decay of the Fourier coefficients $\widehat{\varphi_\lambda^n}(k)$ is of order $\mathcal{O}(n^{-1/3})$ and occurs for $k = \lfloor \alpha_0 n \rfloor$ or $k = \lfloor \alpha_0^{-1} n \rfloor$ and more generally when

$$k/n \in \left[\alpha_0 - n^{-2/3}, \alpha_0 + n^{-2/3}\right) \cup \left(\alpha_0^{-1} - n^{-2/3}, \alpha_0^{-1} + n^{-2/3}\right].$$

This is confirmed by a careful asymptotic analysis of these coefficients (see [8]). To sum up we know from Proposition 4.4 that:

(1) if $\alpha \in (0,\alpha_0)$ and $k \notin \left[\alpha n,\, \alpha^{-1} n\right]$, then $\widehat{\varphi_\lambda^n}(k)$ decays exponentially in n,
(2) if $\beta \in (\alpha_0, 1)$ and $k \in \left[\beta n,\, \beta^{-1} n\right]$, then $\widehat{\varphi_\lambda^n}(k) = \mathcal{O}(n^{-1/2})$,
(3) if $k = \lfloor \alpha_0 n \rfloor$ or $k = \lfloor \alpha_0^{-1} n \rfloor$, then $\widehat{\varphi_\lambda^n}(k) = \mathcal{O}(n^{-1/3})$.

In order to show that $F(\lambda,\dots,\lambda)\gg\sqrt{n}$, a first natural question arises: is there a linear combination of the coefficients $\widehat{\varphi^n_\lambda}(k)$ whose l^∞ norm is bounded from above by $1/\sqrt{n}$? Using the stationary phase method, the second author and O. Szehr first observed that for $k=\lfloor\alpha_0 n\rfloor$ or $k=\lfloor\alpha_0^{-1}n\rfloor$ the decay of the sequence

$$\left(\widehat{\varphi^n_\lambda}(k)-\widehat{\varphi^n_\lambda}(k-2)\right)_{k\geqslant 2}$$

is of order $\mathcal{O}\left(n^{-2/3}\right)$. This observation led them to replacing the function φ^n_λ by $(1-z^2)\varphi^n_\lambda$ in Lemma 4.2. Clearly, we have

$$\widehat{(1-z^2)\varphi^n_\lambda}(k)=\widehat{\varphi^n_\lambda}(k)-\widehat{\varphi^n_\lambda}(k-2),\quad k\geqslant 2.$$

This linear combination of the coefficients $\widehat{\varphi^n_\lambda}(k)$ speeds up their decay at and near the boundaries $\alpha_0 n$, $\alpha_0^{-1}n$, and turns out to answer positively the above question, see Proposition 4.6 below.

4.5.1. *The second analog of Bourgain's Lemma.* The lemma below was originally stated in [46, (3.2)]. It is actually a modification of Lemma 4.2 where the finite Blaschke product φ associated with the sequence $(\lambda_1,\dots,\lambda_n)$ is replaced by the weighted Blaschke product $(1-z^2)\varphi$.

Lemma 4.5. *For any sequence $(\lambda_1,\dots,\lambda_n)\in\mathbb{D}^n$, we have*

$$F\left(\lambda_1,\dots,\lambda_n\right)\geqslant\frac{1}{\|(1-z^2)\varphi\|_{l^\infty_A}}-\prod_{i=1}^n|\lambda_i|,$$

where $\varphi(z)=\prod_{i=1}^n\frac{z-\lambda_i}{1-\overline{\lambda_i}z}$.

Proof. The proof is an adaptation of that of Lemma 4.2. We keep the same notation and the same definitions for the functions h and g. Instead of considering their scalar product we compute the scalar product of z^2h and $(1-z^2)\varphi$:

$$\left\langle z^2h,(1-z^2)\varphi\right\rangle=\left\langle(z^2-1)h,\varphi\right\rangle=\left\langle(z^2-1)g,1\right\rangle=-g(0)=(-1)^{n-1}.$$

Applying the same Hölder's inequality and observing that $\left\|z^2h\right\|_W=\|h\|_W$, we conclude that

$$1\leqslant\left\|z^2h\right\|_W\left\|(1-z^2)\varphi\right\|_{l^\infty_A}=\|h\|_W\left\|(1-z^2)\varphi\right\|_{l^\infty_A}.$$

It follows that any candidate function h in the definition of F satisfies

$$\|h\|_W\geqslant\frac{1}{\|(1-z^2)\varphi\|_{l^\infty_A}}.\qquad\square$$

4.5.2. *Upper bounds on* $\left|\widehat{\varphi_\lambda^n}(k) - \widehat{\varphi_\lambda^n}(k-2)\right|$. It is easily verified that for any fixed k the coefficients $(1-z^2)\widehat{\varphi_\lambda^n}(k)$ decay exponentially when n grows large. The interesting behavior, which is relevant for the l_A^∞-norm, therefore occurs when $k = k(n)$ is a sequence. Our goal in this section is to state asymptotically sharp upper bounds on the coefficients $\widehat{(1-z^2)\varphi_\lambda^n}(k)$ as $n \to \infty$, when k belongs to the "critical n-dependent intervals" highlighted by Proposition 4.4. This is the content of [46, Section 6], which makes use of standard tools from asymptotic analysis previously developed in [45, 47] to determine the asymptotic growth of the Taylor coefficients $\widehat{\varphi_\lambda^n}(k)$ both with respect to k and to n. An application of Proposition 4.4, item (1) (or equivalently of [45, Proposition 6, item (1)]) shows that if $\alpha \in (0, \alpha_0)$ and $k \notin \left[\alpha n,\, \alpha^{-1} n\right]$, then $\widehat{\varphi_\lambda^n}(k)$ decays exponentially in n. To obtain a sharp upper bound on $\widehat{(1-z^2)\varphi_\lambda^n}(k)$ for $k \in \left[\beta n,\, \beta^{-1} n\right]$ and $\beta \in (\alpha_0, 1)$, we rely on the *method of stationary phase* (see Theorem 2.4) as we did to determine an asymptotic formula for $a_{n,k}$, see the proof of Lemma 3.10 above. When k gets close to one of the boundaries $\alpha_0 n$, $\alpha_0^{-1} n$, the situation turns out to be much more delicate: the asymptotic behavior of $\widehat{(1-z^2)\varphi_\lambda^n}(k)$ is described in terms of the *Airy function* and we rely on a *uniform* version of the method of stationary phase/ steepest descents as is introduced in [12]. We refer to [46, Section 6] for a proof of the proposition below, which achieves the above-mentioned goal.

Proposition 4.6. *Suppose that* $\lambda \in (0,1)$, $\varphi = \varphi_\lambda = \frac{z-\lambda}{1-\lambda z}$, $n \geqslant 1$, *and* $k \geqslant 0$. *Set* $\alpha_0 := \frac{1-\lambda}{1+\lambda}$ *and choose fixed* $\alpha \in (0, \alpha_0)$ *and* $\beta \in (\alpha_0, 1)$. *The following assertions hold depending on the region to which* k *belongs.*

(1) *If* $k/n \leqslant \alpha$, *then* $|\widehat{(1-z^2)\varphi^n}(k)|$ *decays exponentially and uniformly over* k *as* n *tends to* ∞. *Similarly, if* $k/n \geqslant \alpha^{-1}$, *then* $|\widehat{(1-z^2)\varphi^n}(k)|$ *decays exponentially and uniformly over* k *as* n *tends to* ∞.

(2) *If* $k/n \in (\alpha, \alpha_0 - n^{-2/3}] \cup [\alpha_0^{-1} + n^{-2/3}, \alpha^{-1})$, *then we have the following asymptotic growth estimate*:

$$\left|\widehat{(1-z^2)\varphi^n}(k)\right| \ll \frac{\left(\min\left((\alpha_0 - k/n),\, (k/n - \alpha_0^{-1})\right)\right)^{1/4}}{n^{1/2}} \times \exp\left(-\frac{2}{3} n \left(\min\left((\alpha_0 - k/n),\, (k/n - \alpha_0^{-1})\right)\right)^{3/2}\right).$$

(3) *If* $k/n \in [\alpha_0 - n^{-2/3}, \alpha_0 + n^{-2/3}] \cup [\alpha_0^{-1} - n^{-2/3}, \alpha_0^{-1} + n^{-2/3}]$, *then*

$$\left|\widehat{(1-z^2)\varphi_\lambda^n}(k)\right| \ll \frac{1}{n^{2/3}}.$$

(4) *If* $k/n \in [\alpha_0 + n^{-2/3}, \alpha_0^{-1} - n^{-2/3}]$, *then*

$$\left|\widehat{(1 - z^2)\varphi^n}(k)\right| \ll \frac{\left(\frac{k}{n} - \alpha_0\right)^{1/4}\left(\alpha_0^{-1} - \frac{k}{n}\right)^{1/4}}{n^{1/2}}.$$

In particular,

$$\left\|(1 - z^2)\varphi^n\right\|_{l^\infty_A} \ll \frac{1}{n^{1/2}}.$$

Estimate (4.12) follows directly from Lemma 4.5 combined with Proposition 4.6.

4.5.3. *Operator interpretation.* In this paragraph we state the main result from [46] (see [46, Theorem 6]). It exhibits a sequence of explicit $n \times n$ *Toeplitz matrices* T_λ with *singleton spectrum* $\{\lambda\} \in \mathbb{D} \setminus \{0\}$ and an n-dimensional Banach space E on which T_λ acts such that

$$|\lambda|^n \left\|T_\lambda^{-1}\right\| \gg \sqrt{n}\, \|T_\lambda\|^{\,n-1},$$

which can be seen as an interpretation of (4.12) from the point of view of operator theory (see [46, Section 4] for a detailed discussion). Given $\lambda \in \mathbb{D}\setminus\{0\}$ and $n \geqslant 1$, let E be the n-dimensional Banach space of rational functions of degree at most n whose poles are located at $1/\bar{\lambda}$, equipped with the norm $\|\cdot\|_{l^\infty_A}$. The space E coincides with the so-called model space associated with the finite Blaschke product

$$(\varphi_\lambda(z))^n = \left(\frac{z - \lambda}{1 - \bar{\lambda}z}\right)^n.$$

A natural orthonormal basis for E (with respect to the scalar product $\langle \cdot, \cdot \rangle$) is the Malmquist–Walsh basis $\mathcal{B} = \{e_j\}_{j=1,\dots,n}$ given by (see [35, p. 117])

$$e_j(z) := \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda}z}\left(\frac{z - \lambda}{1 - \bar{\lambda}z}\right)^{j-1}, \quad j = 1, \dots, n.$$

In other words, given $\lambda \in \mathbb{D} \setminus \{0\}$, the Toeplitz matrix T_λ is an *explicit counterexample to Schäffer's conjecture.*

Theorem 4.7. *For any fixed* $\lambda \in \mathbb{D} \setminus \{0\}$ *the upper triangular* $n \times n$ *Toeplitz matrix*

$$T_\lambda = \begin{pmatrix} \lambda & 1 - |\lambda|^2 & -\bar{\lambda}(1 - |\lambda|^2) & \dots & (-\bar{\lambda})^{n-2}(1 - |\lambda|^2) \\ 0 & \lambda & 1 - |\lambda|^2 & \ddots & \vdots \\ 0 & \ddots & \lambda & \ddots & -\bar{\lambda}(1 - |\lambda|^2) \\ \vdots & \ddots & \ddots & \ddots & 1 - |\lambda|^2 \\ 0 & \dots & 0 & 0 & \lambda \end{pmatrix}$$

acting on $(E, \|\cdot\|_{l^\infty_A})$ *with respect to the basis* $\mathcal{B}$ *satisfies* $\|T_\lambda\|_* \leqslant 1$,

$$\left\|\det(T_\lambda)T_\lambda^{-1}\right\|_* = F(\lambda,\dots,\lambda), \tag{4.15}$$

and

$$K_n \geqslant \left\|\det(T_\lambda)T_\lambda^{-1}\right\|_* \gg \sqrt{n},$$

where $\|\cdot\|_*$ *is the operator norm induced by*

$$\|R\|_{l^\infty_A} := \left\|\sum_{j=1}^n x_j e_j\right\|_{l^\infty_A}, \quad R \in E,\ (x_j)_{j=1}^n \in \mathbb{C}^n.$$

Proof. Indeed, assuming without loss of generality that $\lambda \in (0,1)$, we first observe that T_λ is the matrix of the backward shift operator $S^*\colon E \to E$, $S^*f = (f - f(0))/z$ with respect to the Malmquist–Walsh basis $(e_j)_{j=1}^n$ of E. In particular we have $\|T_\lambda\|_* \leqslant 1$. Second, it can be checked that the simple test vector $X_0 = (0,\dots,0,-1,0,1)$ (i.e., the rational function $e_n(z) - e_{n-2}(z)$) already achieves the estimate in the above theorem:

$$\|T_\lambda^{-1}\cdot X_0^\top\|_{l^\infty_A} \geqslant c(\lambda)\cdot|\lambda|^{-n}\sqrt{n}\cdot\|X_0^\top\|_{l^\infty_A},$$

where $X_0^\top$ is the transpose of X_0 and $c(\lambda) > 0$ depends only on λ. To obtain the above lower estimate we observe that

$$e_n - e_{n-2} = \frac{(1-\lambda^2)^{1/2}}{1-\lambda z}\left(\varphi_\lambda^2 - 1\right)\varphi_\lambda^{n-3} = \frac{(1-\lambda^2)^{3/2}}{(1-\lambda z)^3}\left(z^2-1\right)\varphi_\lambda^{n-3}.$$

As a consequence we get

$$\begin{aligned}\|e_n - e_{n-2}\|_{l^\infty_A} &\leqslant (1-\lambda^2)^{3/2}\|(1-\lambda z)^{-3}\|_{W^+}\|\left(1-z^2\right)\varphi_\lambda^{n-3}\|_{l^\infty_A}\\ &\ll \|\left(1-z^2\right)\varphi_\lambda^{n-3}\|_{l^\infty_A} \ll \frac{1}{\sqrt{n}},\end{aligned}$$

where the last inequality is a direct application of Proposition 4.6, and therefore $\|X_0^\top\|_{l^\infty_A} \ll \frac{1}{\sqrt{n}}$. To evaluate $\|T_\lambda^{-1}\cdot X_0^\top\|_{l^\infty_A}$ from below we put

$$f = (S^*)^{-1}(e_n - e_{n-2}),\ \ f \in E,$$

which means $(f - f(0))/z = e_n - e_{n-2}$. We get

$$f = z(e_n - e_{n-2}) + f(0) = (1-\lambda^2)^{3/2}\frac{z\left(z^2-1\right)(z-\lambda)^{n-3}}{(1-\lambda z)^n} + f(0),$$

where $f(0) = (-1)^{n+1}\frac{(1-\lambda^2)^{3/2}}{\lambda^n}$, since $f(\infty) = 0$.

In particular,

$$\|T_\lambda^{-1}\cdot X_0^\top\|_{l^\infty_A} = \|f\|_{l^\infty_A} \geqslant |f(0)| \geqslant \frac{(1-\lambda^2)^{3/2}}{\lambda^n},$$

and thus

$$\frac{\|T_\lambda^{-1}\cdot X_0^\top\|_{l_A^\infty}}{\|X_0^\top\|_{l_A^\infty}} \gg \frac{\sqrt{n}}{\lambda^n}. \qquad \square$$

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