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SOME INEQUALITIES BETWEEN THE BEST SIMULTANEOUS APPROXIMATION AND THE MODULUS OF CONTINUITY IN A WEIGHTED BERGMAN SPACE

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Abstract: Some inequalities between the best simultaneous approximation of functions and their intermediate derivatives, and the modulus of continuity in a weighted Bergman space are obtained. When the weight function is $\gamma(\rho) = \rho^\alpha$, $\alpha > 0$, some sharp inequalities between the best simultaneous approximation and an m th order modulus of continuity averaged with the given weight are proved. For a specific class of functions, the upper bound of the best simultaneous approximation in the space B_{2,γ_1} , $\gamma_1(\rho) = \rho^\alpha$, $\alpha > 0$, is found. Exact values of several n -widths are calculated for the classes of functions $W_p^{(r)}(\omega_m, q)$.

Keywords: The best simultaneous approximation, Modulus of continuity, Upper bound, n -widths.

1. Introduction

Extremal problems of polynomial approximation of functions in a Bergman space were studied, for example, in [8, 13–15]. Here, we will continue our research in this direction and study the simultaneous approximation of functions and their intermediate derivatives in a weighted Bergman space based on the works [4–6, 10]. Note that the problem of simultaneous approximation of periodic functions and their intermediate derivatives by trigonometric polynomials in the uniform metric was studied by Garkavi [1]. In the case of entire functions, this problem was studied by Timan [12].

To solve the problem, we first will prove an analog of Ligun’s inequality [2].

Let us introduce the necessary definitions and notation to formulate our results. Let

$$U := \{z \in \mathbb{C} : |z| < 1\}$$

be the unit disk in \mathbb{C} , and let $\mathcal{A}(U)$ be the set of functions analytic in the disk U . Denote by $B_{2,\gamma}$ the weighted Bergman space of analytic functions $f \in \mathcal{A}(U)$ such that [8]

$$\|f\|_{2,\gamma} := \left(\frac{1}{2\pi} \iint_{(U)} |f(z)|^2 \gamma(|z|) d\sigma \right)^{1/2} < \infty, \quad (1.1)$$

$d\sigma$ is an area element, $\gamma := \gamma(|z|)$ is a nonnegative measurable function that is not identically zero, and the integral is understood in the Lebesgue sense. It is obvious, that the norm (1.1) can be written in the form

$$\|f\|_{2,\gamma} = \left(\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \rho \gamma(\rho) |f(\rho e^{it})|^2 d\rho dt \right)^{1/2}.$$

In the particular case of $\gamma \equiv 1$, $B_q := B_{q,1}$ is the usual Bergman space. The m th order modulus of continuity in $B_{2,\gamma}$ is defined as

$$\begin{aligned} \omega_m(f, t)_{2,\gamma} &= \sup \{ \|\Delta_m(f, \cdot, \cdot, h)\|_{2,\gamma} : |h| \leq t \} = \\ &= \sup \left\{ \left(\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \rho \gamma(\rho) |\Delta_m(f; \rho, u, h)|^2 d\rho du \right)^{1/2} : |h| \leq t \right\}, \end{aligned}$$

where

$$\Delta_m(f; \rho, u, h) = \sum_{k=0}^m (-1)^k C_m^k f(\rho e^{i(u+kh)}).$$

Let \mathcal{P}_n be the set of complex polynomials of order at most n . Consider the best approximation of functions $f \in B_{2,\gamma}$:

$$E_{n-1}(f)_{2,\gamma} = \inf \{ \|f - p_{n-1}\|_{2,\gamma} : p_{n-1} \in \mathcal{P}_{n-1} \}$$

Denote by $\mathcal{B}_{2,\gamma}^{(r)}$ and $\mathcal{B}_2^{(r)}$, $r \in \mathbb{N}$ the class of functions $f \in \mathcal{A}(U)$ whose r th order derivatives

$$f^{(r)}(z) = d^r f / dz^r$$

belong to the spaces $B_{2,\gamma}$ and B_2 , respectively. Define

$$\alpha_{n,r} = n(n-1) \cdots (n-r+1), \quad n > r.$$

It is well known [7, 8] that the best approximation of functions

$$f = \sum_{k=0}^{\infty} c_k(f) z^k \in B_{2,\gamma}$$

is equal to

$$\begin{aligned} E_{n-1}(f)_{2,\gamma} &= \left(\sum_{k=n}^{\infty} |c_k(f)|^2 \int_0^1 \rho^{2k+1} \gamma(\rho) d\rho \right)^{1/2}, \\ E_{n-s-1}(f^{(s)})_{2,\gamma} &= \left(\sum_{k=n}^{\infty} |c_k(f)|^2 \alpha_{k,s}^2 \int_0^1 \rho^{2(k-s)+1} \gamma(\rho) d\rho \right)^{1/2}, \end{aligned} \quad (1.2)$$

and the modulus of continuity of $f \in B_{2,\gamma}$ is

$$\omega_m(f^{(r)}, t)_{2,\gamma} = 2^{m/2} \sup_{|h| \leq t} \left\{ \sum_{k=r}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 (1 - \cos(k-r)h)^m \int_0^1 \rho^{2(k-r)+1} \gamma(\rho) d\rho \right\}^{1/2}. \quad (1.3)$$

Denote by

$$\mu_s(\gamma) = \int_0^1 \gamma(\rho) \rho^s d\rho, \quad s = 0, 1, 2, \dots \quad (1.4)$$

the moments of order s of the weight function $\gamma(\rho)$ on $[0, 1]$. According to notation (1.4), we write equalities (1.2) and (1.3) in compact form:

$$\begin{aligned} E_{n-1}(f)_{2,\gamma} &= \left(\sum_{k=n}^{\infty} |c_k(f)|^2 \mu_{2k+1}(\gamma) \right)^{1/2}, \\ E_{n-s-1}(f^{(s)})_{2,\gamma} &= \left(\sum_{k=n}^{\infty} |c_k(f)|^2 \alpha_{k,s}^2 \mu_{2(k-s)+1}(\gamma) \right)^{1/2}, \\ \omega_m(f^{(r)}, t)_{2,\gamma} &= 2^{m/2} \sup_{|h| \leq t} \left\{ \sum_{k=r}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 (1 - \cos(k-r)h)^m \mu_{2(k-r)+1}(\gamma) \right\}^{1/2}. \end{aligned} \quad (1.5)$$

2. Analog of Ligun's inequality

For compact statement of the results, we introduce the following extremal characteristic:

$$\mathcal{H}_{m,n,r,s,p}(q, \gamma, h) = \sup_{f \in \mathcal{B}_{2,\gamma}^{(r)}} \frac{2^{m/2} E_{n-s-1}(f^{(s)})_{2,\gamma}}{\left(\int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma} q(t) dt\right)^{1/p}},$$

where $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r \geq s$, $0 < p < 2$, $0 < h \leq \pi/(n - r)$, and $q(t)$ is a real, nonnegative, measurable weight function that is not identically zero on $[0, h]$.

Theorem 1. *Let $k, m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}_+$, $k > n > r \geq s$, $0 < p < 2$, $0 < h \leq \pi/(n - r)$, and let $q(t)$ be a nonnegative, measurable function that is not identically zero on $[0, h]$. Then*

$$\frac{1}{\mathcal{L}_{n,r,s,p}(q, \gamma, h)} \leq \mathcal{H}_{m,n,r,s,p}(q, \gamma, h) \leq \frac{1}{\inf_{n \leq k < \infty} \mathcal{L}_{k,r,s,p}(q, \gamma, h)}, \tag{2.1}$$

where

$$\mathcal{L}_{k,r,s,p}(q, \gamma, h) = \frac{\alpha_{k,r}}{\alpha_{k,s}} \left(\frac{\mu_{2(k-r)+1}(\gamma)}{\mu_{2(k-s)+1}(\gamma)}\right)^{1/2} \left(\int_0^h (1 - \cos(k - r)t)^{mp/2} q(t) dt\right)^{1/p}.$$

P r o o f. Consider the simplified variant of Minkowski's inequality [3, p. 104]:

$$\left(\int_0^h \left(\sum_{k=n}^{\infty} |g_k(t)|^2\right)^{p/2} dt\right)^{1/p} \geq \left(\sum_{k=n}^{\infty} \left(\int_0^h |g_k(t)|^p dt\right)^{2/p}\right)^{1/2}, \tag{2.2}$$

which is hold for all $0 < p \leq 2$ and $h \in \mathbb{R}_+$. Setting

$$g_k = f_k q^{1/p} \quad (0 < p \leq 2)$$

in (2.2), we get

$$\left(\int_0^h \left(\sum_{k=n}^{\infty} |f_k(t)|^2\right)^{p/2} q(t) dt\right)^{1/p} \geq \left(\sum_{k=n}^{\infty} \left(\int_0^h |f_k(t)|^p q(t) dt\right)^{2/p}\right)^{1/2}. \tag{2.3}$$

From (1.3) with respect to (2.3), we get

$$\begin{aligned} & \left\{ \int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma} q(t) dt \right\}^{1/p} = \left\{ \int_0^h (\omega_m^2(f^{(r)}, t)_{2,\gamma})^{p/2} q(t) dt \right\}^{1/p} \\ & \geq \left\{ \int_0^h \left(2^m \sum_{k=n}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 (1 - \cos(k - r)t)^m \mu_{2(k-r)+1}(\gamma)\right)^{p/2} q(t) dt \right\}^{1/p} \\ & \geq \left\{ \sum_{k=n}^{\infty} \left[2^{mp/2} \alpha_{k,r}^p |c_k(f)|^p \int_0^h (1 - \cos(k - r)t)^{mp/2} (\mu_{2(k-r)+1}(\gamma))^{p/2} q(t) dt\right]^{2/p} \right\}^{1/2} \\ & = 2^{m/2} \left\{ \sum_{k=n}^{\infty} |c_k(f)|^2 \mu_{2(k-r)+1}(\gamma) \left[\alpha_{k,r}^p \int_0^h (1 - \cos(k - r)t)^{mp/2} q(t) dt\right]^{2/p} \right\}^{1/2} \\ & = 2^{m/2} \left\{ \sum_{k=n}^{\infty} |c_k(f)|^2 \alpha_{k,s}^2 \mu_{2(k-s)+1}(\gamma) \mu_{2(k-r)+1}(\gamma) (\mu_{2(k-s)+1}(\gamma))^{-1} \right\} \end{aligned}$$

$$\begin{aligned} & \left[\left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \int_0^h (1 - \cos(k-r)t)^{mp/2} q(t) dt \right]^{2/p} \Bigg\}^{1/2} \\ & \geq 2^{m/2} \inf_{n \leq k < \infty} \left\{ \frac{\alpha_{k,r}}{\alpha_{k,s}} \left(\frac{\mu_{2(k-r)+1}(\gamma)}{\mu_{2(k-s)+1}(\gamma)} \right)^{1/2} \left(\int_0^h (1 - \cos(k-r)t)^{mp/2} q(t) dt \right)^{1/p} \right\} \\ & \times \left(\sum_{k=n}^{\infty} |c_k(f)|^2 \alpha_{k,s}^2 \mu_{2(k-s)+1}(\gamma) \right)^{1/2} = 2^{m/2} E_{n-s-1}(f^{(s)})_{2,\gamma} \inf_{n \leq k < \infty} \mathcal{L}_{k,r,s,p}(q, \gamma, h), \end{aligned}$$

and this yields the inequality

$$\frac{2^{m/2} E_{n-s-1}(f^{(s)})_{2,\gamma}}{\left(\int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma} q(t) dt \right)^{1/p}} \leq \frac{1}{\inf_{n \leq k < \infty} \mathcal{L}_{k,r,s,p}(q, \gamma, h)} \tag{2.4}$$

or

$$\mathcal{K}_{m,n,r,s,p}(q, \gamma, h) \leq \frac{1}{\inf_{n \leq k < \infty} \mathcal{L}_{k,r,s,p}(q, \gamma, h)}. \tag{2.5}$$

To estimate the value in (2.1) from below, consider the function

$$f_0(z) = z^n \in \mathcal{B}_{2,\gamma}^{(r)}.$$

Simple calculation leads to the following relations:

$$\begin{aligned} E_{n-s-1}(f_0^{(s)})_{2,\gamma} &= \alpha_{n,s} \left(\int_0^1 \rho^{2(n-s)+1} \gamma(\rho) d\rho \right)^{1/2} = \alpha_{n,s} (\mu_{2(n-s)+1}(\gamma))^{1/2}, \\ \omega_m^2(f_0^{(r)}, t)_{2,\gamma} &= 2^m \alpha_{n,r}^2 (1 - \cos(n-r)t)^m \int_0^1 \rho^{2(n-r)+1} \gamma(\rho) d\rho \\ &= 2^m \alpha_{n,r}^2 (1 - \cos(n-r)t)^m \mu_{2(n-r)+1}(\gamma), \end{aligned}$$

using which, we get the lower estimate

$$\begin{aligned} \mathcal{K}_{m,n,r,p}(q, \gamma, h) &\geq \frac{2^{m/2} E_{n-s-1}(f_0^{(s)})_{2,\gamma}}{\left(\int_0^h \omega_m^p(f_0^{(r)}, t)_{2,\gamma} q(t) dt \right)^{1/p}} \\ &= \frac{2^{m/2} \alpha_{n,s} (\mu_{2(n-s)+1}(\gamma))^{1/2}}{\left(2^{mp/2} \alpha_{n,r}^p (\mu_{2(n-r)+1}(\gamma))^{p/2} \int_0^h (1 - \cos(n-r)t)^{mp/2} q(t) dt \right)^{1/p}} = \frac{1}{\mathcal{L}_{n,r,s,p}(q, \gamma, h)}. \end{aligned} \tag{2.6}$$

Comparing the upper estimate (2.5) and the lower estimate (2.6), we obtain the required two-sided inequality (2.1). This completes the proof of Theorem 1. \square

Corollary 1. *The following two-sided inequality holds for $\gamma_1(\rho) = \rho^\alpha$, $\alpha \geq 0$, in Theorem 1:*

$$\frac{1}{\mathcal{G}_{n,r,s,p,\alpha}(q, h)} \leq \mathcal{K}_{m,n,r,s,p}(q, \gamma_1, h) \leq \frac{1}{\inf_{n \leq k < \infty} \mathcal{G}_{k,r,s,p,\alpha}(q, h)}, \tag{2.7}$$

where

$$\mathcal{G}_{k,r,s,p,\alpha}(q, h) = \frac{\alpha_{k,r}}{\alpha_{k,s}} \left(\frac{2(k-s+1) + \alpha}{2(k-r+1) + \alpha} \right)^{1/2} \left(\int_0^h (1 - \cos(k-r)t)^{mp/2} q(t) dt \right)^{1/p}. \tag{2.8}$$

The following problem naturally arises from (2.7): to find an exact upper bound for the extremal characteristic

$$\mathcal{H}_{m,n,r,s,p}(q, \gamma_1, h) = \sup_{f \in \mathcal{B}_{2,\gamma_1}^{(r)}} \frac{2^{m/2} E_{n-s-1}(f^{(s)})_{2,\gamma_1}}{\left(\int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma_1} q(t) dt \right)^{1/p}},$$

where $m, n \in \mathbb{N}$, $r, s \in \mathbb{Z}_+$, $n > r \geq s$, $0 < p < 2$, $0 < h \leq \pi/(n - r)$, $\gamma_1(\rho) = \rho^\alpha$, and $\alpha \geq 0$.

Theorem 2. *Let a weight function $q(t)$, $t \in [0, h]$, be continuous and differentiable on the interval. If the differential inequality*

$$\left(\sum_{l=s}^{r-1} \frac{p}{k-l} - \frac{2p(r-s)}{[2(k-r+1)+\alpha](2(k-s+1)+\alpha)} - \frac{1}{k-r} \right) q(t) - \frac{1}{k-r} t q'(t) \geq 0 \quad (2.9)$$

holds for all $k \in \mathbb{N}$, $r, s \in \mathbb{Z}_+$, $k > n > r \geq s$, $0 < p \leq 2$, and $\alpha \geq 0$, then the following equality holds for all $m, n \in \mathbb{N}$ and $0 < h \leq \pi/(n - r)$:

$$\mathcal{H}_{m,n,r,s,p}(q, \gamma_1, h) = \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\frac{2(n-r+1)+\alpha}{2(n-s+1)+\alpha} \right)^{1/2} \left(\int_0^h (1 - \cos(n-r)t)^{mp/2} q(t) dt \right)^{1/p}. \quad (2.10)$$

P r o o f. To prove equality (2.10), it suffices to show that the following equality holds in (2.7):

$$\inf_{n \leq k < \infty} \mathcal{G}_{k,r,s,p,\alpha}(q, h) = \mathcal{G}_{n,r,s,p,\alpha}(q, h). \quad (2.11)$$

We should note that a similar problem of finding a lower bound in (2.11) for some specific weights for $p = 2$ was considered in [2]. In the general case, this problem was studied in [9], where it was proved that, if the weight function $q \in C^{(1)}[0, h]$ for $1/r < p \leq 2$, $r \geq 1$, and $0 < t \leq h$ satisfies the differential equation

$$(rp - 1)q(t) - tq'(t) \geq 0,$$

then (2.11) holds.

Let us now show that, under all constrains on the parameters k, r, s, m, p, α , and h in Theorem 2, the function

$$\psi(k) = \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha} \right)^{p/2} \int_0^h (1 - \cos(k-r)t)^{mp/2} q(t) dt \quad (2.12)$$

increases for $n \leq k < \infty$. Indeed, differentiating (2.12) and using the identity

$$\frac{d}{dk} (1 - \cos(k-r)t)^{mp/2} = \frac{t}{k-r} \frac{d}{dt} (1 - \cos(k-r)t)^{mp/2},$$

we obtain

$$\begin{aligned} \psi'(k) &= \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \sum_{l=s}^{r-1} \frac{p}{k-l} \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha} \right)^{p/2} \int_0^h (1 - \cos(k-r)t)^{mp/2} q(t) dt \\ &+ \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \frac{p}{2} \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha} \right)^{p/2-1} \frac{4s-4r}{[2(k-r+1)+\alpha]^2} \int_0^h (1 - \cos(k-r)t)^{mp/2} q(t) dt \\ &+ \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha} \right)^{p/2} \int_0^h \frac{d}{dk} (1 - \cos(k-r)t)^{mp/2} q(t) dt \end{aligned}$$

$$\begin{aligned}
 &= \int_0^h (1 - \cos(k - r)t)^{mp/2} q(t) dt \left\{ \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \sum_{l=s}^{r-1} \frac{p}{k-l} \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha} \right)^{p/2} \right. \\
 &\quad \left. - \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \frac{2p(r-s)}{[2(k-r+1)+\alpha](2(k-s+1)+\alpha)} \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha} \right)^{p/2} \right\} \\
 &\quad + \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha} \right)^{p/2} \int_0^h \frac{t}{k-r} \frac{d}{dt} (1 - \cos(k-r)t)^{mp/2} q(t) dt \\
 &= \left(\frac{\alpha_{k,r}}{\alpha_{k,s}} \right)^p \left(\frac{2(k-s+1)+\alpha}{2(k-r+1)+\alpha} \right)^{p/2} \left\{ \frac{h}{k-r} (1 - \cos(k-r)h)^{mp/2} q(h) + \int_0^h (1 - \cos(k-r)t)^{mp/2} \right. \\
 &\quad \left. \times \left[\left(\sum_{l=s}^{r-1} \frac{p}{k-l} - \frac{2p(r-s)}{[2(k-r+1)+\alpha](2(k-s+1)+\alpha)} - \frac{1}{k-r} \right) q(t) - \frac{1}{k-r} t q'(t) \right] dt \right\}.
 \end{aligned}$$

This relation and condition (2.9) imply that $\psi(k) > 0$, $k \geq n > r \geq s$, and we obtain equality (2.10). Theorem 2 is proved. \square

Denote by $W_p^{(r)}(\omega_m, q)$ ($r \in \mathbb{Z}_+$, $0 < p \leq 2$) the set of functions $f \in \mathcal{B}_{2,\gamma_1}^{(r)}$ whose r th derivatives $f^{(r)}$ satisfy the following condition for all $0 < h \leq \pi/(n-r)$ and $n > r$:

$$\int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma_1} q(t) dt \leq 1.$$

Since, for $f \in \mathcal{B}_{2,\gamma_1}^{(r)}$, its intermediate derivatives $f^{(s)}$ ($1 \leq s \leq r-1$) also belong to L_2 , the behavior of the value $E_{n-s-1}(f^{(s)})_2$ for some classes $\mathfrak{M}^{(r)} \subset \mathcal{B}_{2,\gamma_1}^{(r)}$, $n > r \geq s$, $n \in \mathbb{N}$, and $r, s \in \mathbb{Z}_+$, is of interest. More precisely, it is required to find the value

$$\mathcal{A}_{n,s}(\mathfrak{M}^{(r)}) := \sup \{ E_{n-s-1}(f^{(s)})_{2,\gamma_1} : f \in \mathfrak{M}^{(r)} \}.$$

Corollary 2. *The following equality holds for all $n \in \mathbb{N}$, $n > r \geq s$, $0 < p \leq 2$, and $0 < h \leq \pi/(n-r)$:*

$$\mathcal{A}_{n,s}(W_p^{(r)}(\omega_m, q)) := \sup \{ E_{n-s-1}(f^{(s)})_{2,\gamma_1} : f \in W_p^{(r)}(\omega_m, q) \} = \frac{1}{2^{m/2} \mathcal{G}_{n,r,s,p,\alpha}(q, h)}. \tag{2.13}$$

Moreover, there is a function $g_0 \in W_p^{(r)}(\omega_m, q)$ on which the upper bound in (2.13) is attained.

P r o o f. Assuming that $\gamma = \gamma_1(\rho) = \rho^\alpha$ in (2.4), with respect to (2.8), we can write

$$E_{n-s-1}(f^{(s)})_{2,\gamma_1} \leq \frac{\left(\int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma_1} q(t) dt \right)^{1/p}}{2^{m/2} \inf_{n \leq k < \infty} \mathcal{L}_{k,r,s,p}(q, \gamma_1, h)} = \frac{\left(\int_0^h \omega_m^p(f^{(r)}, t)_{2,\gamma_1} q(t) dt \right)^{1/p}}{2^{m/2} \inf_{n \leq k < \infty} \mathcal{G}_{k,r,s,p,\alpha}(q, h)}.$$

Using equality (2.11) and the definition of the class $W_p^{(r)}(\omega_m, q)$, we get

$$E_{n-s-1}(f^{(s)})_{2,\gamma_1} \leq \frac{1}{2^{m/2} \mathcal{G}_{n,r,s,p,\alpha}(q, h)}. \tag{2.14}$$

From (2.14), it follows the upper estimate of the value on the left-hand side of (2.13):

$$\mathcal{A}_{n,s}(W_p^{(r)}(\omega_m; q, \Phi)) \leq \frac{1}{2^{m/2} \mathcal{G}_{n,r,s,p,\alpha}(q, h)}. \tag{2.15}$$

To obtain the lower estimate for this value, consider the function

$$g_0(z) = \frac{\sqrt{2(n-r+1)+\alpha}}{2^{m/2}\alpha_{n,r}} \left(\int_0^h (1-\cos(n-r)t)^{mp/2} q(t) dt \right)^{-1/p} z^n$$

and show that g_0 belongs to $W_p^{(r)}(\omega_m, q)$. Differentiating this function r times, we obtain

$$g_0^{(r)}(z) = \sqrt{\frac{2(n-r+1)+\alpha}{2^m}} \left(\int_0^h (1-\cos(n-r)t)^{mp/2} q(t) dt \right)^{-1/p} z^{n-r}.$$

Using this equality and formulas (1.3), we get

$$\omega_m \left(g_0^{(r)}, t \right)_{2,\gamma_1} = \frac{[1-\cos(n-r)t]^{m/2}}{\left(\int_0^h (1-\cos(n-r)t)^{mp/2} q(t) dt \right)^{1/p}}.$$

Raising both sides of this inequality to a power p ($0 < p \leq 2$), multiplying them by the weight function $q(t)$, and integrating with respect to t from 0 to h , we obtain

$$\int_0^h \omega_m^p(g_0^{(r)}, t)_{2,\gamma_1} q(t) dt = 1$$

or, equivalently,

$$\left(\int_0^h \omega_m^p(g_0^{(r)}, t)_{2,\gamma_1} q(t) dt \right)^{1/p} = 1.$$

Thus, the inclusion $g_0 \in W_p^{(r)}(\omega_m, q)$ is proved.

Since the relation

$$g_0^{(s)}(z) = \sqrt{\frac{2(n-r+1)+\alpha}{2^m}} \frac{\alpha_{n,s}}{\alpha_{n,r}} \left(\int_0^h (1-\cos(n-r)t)^{mp/2} q(t) dt \right)^{-1/p} z^{n-s}$$

holds for all $0 \leq s \leq r < n$, $n \in \mathbb{N}$, and $r, s \in \mathbb{Z}_+$, according to (1.5), we have

$$\begin{aligned} E_{n-s-1} \left(g_0^{(s)} \right)_{2,\gamma_1} &= \frac{1}{2^{m/2}} \frac{\alpha_{n,s}}{\alpha_{n,r}} \sqrt{\frac{2(n-r+1)+\alpha}{2(n-s+1)+\alpha}} \left(\int_0^h [1-\cos(n-r)t]^{mp} q(t) dt \right)^{-1/p} \\ &= \frac{1}{2^{m/2} \mathcal{G}_{n,r,s,p,\alpha}(q, h)}. \end{aligned}$$

Using this equality, we obtain the lower estimate

$$\sup \{ E_{n-s-1}(f^{(s)})_{2,\gamma_1} : f \in W_p^{(r)}(\Omega_m, q) \} \geq E_{n-s-1}(g_0^{(s)})_{2,\gamma_1} = \frac{1}{2^{m/2} \mathcal{G}_{n,r,s,p,\alpha}(q, h)}. \quad (2.16)$$

Comparing the upper estimate (2.15) and the lower estimate (2.16), we obtain the required equality (2.13). \square

3. Exact values of n -widths for the classes $W_p^{(r)}(\omega_m, q)$ ($r \in \mathbb{Z}_+$, $0 < p \leq 2$)

Recall definitions and notation needed in what follows. Let X be a Banach space, let S be the unit ball in X , let $\Lambda_n \subset X$ be an n -dimensional subspace, let $\Lambda^n \subset X$ be a subspace of codimension n , let $\mathcal{L} : X \rightarrow \Lambda_n$ be a continuous linear operator, let $\mathcal{L}^\perp : X \rightarrow \Lambda_n$ be a continuous linear projection operator, and let \mathfrak{M} be a convex centrally symmetric subset of X . The quantities

$$\begin{aligned} b_n(\mathfrak{M}, X) &= \sup \{ \sup \{ \varepsilon > 0; \varepsilon S \cap \Lambda_{n+1} \subset \mathfrak{M} \} : \Lambda_{n+1} \subset X \}, \\ d_n(\mathfrak{M}, X) &= \inf \{ \sup \{ \inf \{ \|f - g\|_X : g \in \Lambda_n \} : f \in \mathfrak{M} \} : \Lambda_n \subset X \}, \\ \delta_n(\mathfrak{M}, X) &= \inf \{ \inf \{ \sup \{ \|f - \mathcal{L}f\|_X : f \in \mathfrak{M} \} : \mathcal{L}X \subset \Lambda_n \} : \Lambda_n \subset X \}, \\ d^n(\mathfrak{M}, X) &= \inf \{ \sup \{ \|f\|_X : f \in \mathfrak{M} \cap \Lambda^n \} : \Lambda^n \subset X \}, \\ \Pi_n(\mathfrak{M}, X) &= \inf \{ \inf \{ \sup \{ \|f - \mathcal{L}^\perp f\|_X : f \in \mathfrak{M} \} : \mathcal{L}^\perp X \subset \Lambda_n \} : \Lambda_n \subset X \} \end{aligned}$$

are called the *Bernstein*, *Kolmogorov*, *linear*, *Gelfand*, and *projection n -widths* of a subset \mathfrak{M} in the space X , respectively. These n -widths are monotone in n and related as follows in a Hilbert space X (see, e.g., [3, 11]):

$$b_n(\mathfrak{M}, X) \leq d^n(\mathfrak{M}, X) \leq d_n(\mathfrak{M}, X) = \delta_n(\mathfrak{M}, X) = \Pi_n(\mathfrak{M}, X). \quad (3.1)$$

For an arbitrary subset $\mathfrak{M} \subset X$, we set

$$E_{n-1}(\mathfrak{M})_X := \sup \{ E_{n-1}(f)_2 : f \in \mathfrak{M} \}.$$

Theorem 3. *The following equalities hold for all $m, n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$, and $0 \leq h \leq \pi/(n-r)$:*

$$\begin{aligned} \lambda_n(W_p^{(r)}(\omega_m, q), B_{2, \gamma_1}) &= E_{n-1}(W_p^{(r)}(\omega_m, q), B_{2, \gamma_1}) \\ &= \frac{1}{2^{m/2} \alpha_{n,r}} \sqrt{\frac{2(n-r+1) + \alpha}{2(n+1) + \alpha}} \left(\int_0^h [1 - \cos(n-r)t]^{mp} q(t) dt \right)^{-1/p}, \end{aligned} \quad (3.2)$$

where $\lambda_n(\cdot)$ is any of the n -widths $b_n(\cdot)$, $d_n(\cdot)$, $d^n(\cdot)$, $\delta_n(\cdot)$, and $\Pi_n(\cdot)$.

P r o o f. We obtain the upper estimates of all n -widths for the class $W_p^{(r)}(\omega_m, q)$ with $s = 0$ from (2.14) since

$$\begin{aligned} E_{n-1}(W_p^{(r)}(\omega_m, q))_{2, \gamma_1} &= \sup \{ E_{n-1}(f)_{2, \gamma_1} : f \in W_p^{(r)}(\omega_m, q) \} \\ &\leq \frac{1}{2^{m/2} \alpha_{n,r}} \sqrt{\frac{2(n-r+1) + \alpha}{2(n+1) + \alpha}} \left(\int_0^h [1 - \cos(n-r)t]^{mp} q(t) dt \right)^{-1/p}. \end{aligned}$$

Using relations (3.1) between the n -widths, we obtain the upper estimate in (3.2):

$$\begin{aligned} \lambda_n(W_p^{(r)}(\omega_m, q)) &\leq E_{n-1}(W_p^{(r)}(\omega_m, q))_{2, \gamma_1} \\ &\leq \frac{1}{2^{m/2} \alpha_{n,r}} \sqrt{\frac{2(n-r+1) + \alpha}{2(n+1) + \alpha}} \left(\int_0^h [1 - \cos(n-r)t]^{mp} q(t) dt \right)^{-1/p}. \end{aligned} \quad (3.3)$$

To obtain the lower estimate on the right-hand side of (3.2) for all n -widths in the $(n+1)$ -dimensional subspace of complex algebraic polynomials

$$\mathcal{P}_{n+1} = \left\{ p_n(z) : p_n(z) = \sum_{k=0}^n a_k z^k, a_k \in \mathbb{C} \right\},$$

we introduce the ball

$$\mathbb{B}_{n+1} := \left\{ p_n(z) \in \mathcal{P}_n : \|p_n\| \leq \frac{1}{2^{m/2}\alpha_{n,r}} \sqrt{\frac{2(n-r+1)+\alpha}{2(n+1)+\alpha}} \left(\int_0^h [1 - \cos(n-r)t]^{mp} q(t) dt \right)^{-1/p} \right\},$$

where $n > r$, $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, and show that $\mathbb{B}_{n+1} \subset W_p^{(r)}(\omega_m, q)$. Indeed, for all $p_n(z) \in \mathbb{B}_{n+1}$, from (1.3), we write

$$\begin{aligned} \omega_m^2(p_n^{(r)}, t)_{2,\gamma_1} &= 2^m \sum_{k=r}^\infty \frac{\alpha_{k,r}^2 |a_k(f)|^2}{2(k-r+1)+\alpha} (1 - \cos(k-r)h)^m \\ &\leq 2^m \max_{r \leq k \leq n} \{ \alpha_{k,r}^2 (1 - \cos(k-r)h)^m \} \sum_{k=r}^\infty \frac{|a_k(f)|^2}{2(k-r+1)+\alpha}. \end{aligned} \tag{3.4}$$

We have to prove that

$$\max_{r \leq k \leq n} \{ \alpha_{k,r}^2 (1 - \cos(k-r)h)^m \} = \alpha_{n,r}^2 (1 - \cos(n-r)h)^m, \quad 0 \leq h \leq \pi/(n-r).$$

Consider the function

$$\varphi(k) = \alpha_{k,r}^2 (1 - \cos(k-r)h)^m, \quad r \leq k \leq n, \quad 0 \leq h \leq \pi/(n-r).$$

We will show that the function $\varphi(k)$ is monotone increasing for all accepted values k and h . To this end, it suffices to show that $\varphi'(k) > 0$. In fact

$$\varphi'(k) = 2\alpha_{k,r}^2 \sum_{l=0}^{r-1} \frac{1}{k-l} (1 - \cos(k-r)h)^m + mh\alpha_{k,r}^2 \sin(k-r)h (1 - \cos(k-r)h)^{m-1} \geq 0.$$

Hence, we can write (3.4) in the form

$$\begin{aligned} \omega_m^2(p^{(r)}, t)_{2,\gamma_1} &\leq 2^m \alpha_{n,r}^2 (1 - \cos(n-r)h)^m \sum_{k=r}^\infty \frac{|a_k(f)|^2}{2(k-r+1)+\alpha} \\ &\leq 2^m \alpha_{n,r}^2 (1 - \cos(n-r)h)^m \sum_{k=0}^\infty \frac{|a_k(f)|^2}{2(k-r+1)+\alpha} = 2^m \alpha_{n,r}^2 (1 - \cos(n-r)h)^m \|p_n\|_{2,\gamma_1}^2. \end{aligned} \tag{3.5}$$

From (3.5), we have

$$\omega_m(p^{(r)}, t)_{2,\gamma_1} \leq 2^{m/2} \alpha_{n,r} (1 - \cos(n-r)h)^{m/2} \|p_n\|_{2,\gamma_1}.$$

Raising both sides of this inequality to a power p ($0 < p \leq 2$), multiplying them by the weight function $q(t)$, and integrating with respect to t from 0 to h , we obtain

$$\int_0^h \omega_m^p(p^{(r)}, t)_{2,\gamma_1} q(t) dt \leq 2^{mp/2} \alpha_{n,r}^p \|p_n\|_{2,\gamma_1}^p \int_0^h (1 - \cos(n-r)h)^{mp/2} q(t) dt \leq 1$$

for all $p_n \in \mathbb{B}_{n+1}$. It follows that $\mathbb{B}_{n+1} \subset W_p^{(r)}(\omega_m, q)$. Then, according to the definition of the Bernstein n -width and (3.1), we can write the following lower estimate for all above listed n -widths:

$$\begin{aligned} \lambda_n(W_p^{(r)}(\omega_m, q), B_{2,\gamma_1}) &\geq b_n(W_p^{(r)}(\omega_m, q), B_{2,\gamma_1}) \geq b_n(\mathbb{B}_{n+1}, B_{2,\gamma_1}) \\ &\geq \frac{1}{2^{m/2}\alpha_{n,r}} \sqrt{\frac{2(n-r+1)+\alpha}{2(n+1)+\alpha}} \left(\int_0^h [1 - \cos(n-r)t]^{mp} q(t) dt \right)^{-1/p}. \end{aligned} \tag{3.6}$$

Comparing the upper estimate (3.3) and the lower estimate in (3.6), we obtain the required equality (3.2). Theorem 3 is proved. \square

4. Conclusion

Upper and lower estimates have been proven for extremal characteristics in a weighted Bergman space. In the case of a power function considered instead of a general weight, the values of n -widths have been calculated for a specific class of functions.

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