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# A DISCRETE LIMIT THEOREM FOR THE MELLIN TRANSFORMS OF THE RIEMANN ZETA-FUNCTION

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**Аннотация**

A discrete limit theorem for the Mellin transforms of the Riemann zeta-function is proven.

## 1 Introduction

As usual, denote by $\zeta(s)$, $s = \sigma + it$, the Riemann zeta-function. In analytic number theory, the modified Mellin transforms $\mathcal{Z}_k(s)$ of powers of the function $\zeta(s)$ are considered. For $k \geq 0$ and $\sigma \geq \sigma_0(k) > 1$, the function $\mathcal{Z}_k(s)$ is defined by

$$\mathcal{Z}_k(s) = \int_1^\infty \left|\zeta\Big(\frac{1}{2} + ix\Big)\right|^{2k} x^{-s} dx.$$

In view of the Mellin inversion formula, the function $\mathcal{Z}_k(s)$ is very useful for the investigations of power moments

$$\int_0^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^{2k} dt$$

of the Riemann zeta-function.

This paper is devoted to the asymptotic behavior of $\mathcal{Z}_1(s)$.The basic properties of the function $\mathcal{Z}_1(s)$ were obtained in [7]. In view of the well-known estimate $I_1(T) = O(T \log T)$, the integral defining $\mathcal{Z}_1(s)$ converges absolutely for $\sigma > 1$. Therefore, the function $\mathcal{Z}_1(s)$ is analytic in the half-plane $\{s \in \mathbb{C} : \sigma > 1\}$. In [7], the function $\mathcal{Z}_1(s)$ has been meromorphically continued to the half-plane $\{s \in \mathbb{C} : \sigma > -\frac{3}{4}\}$. Moreover, it was obtained that the point $s = 1$ is a pole of order two with residue $2\gamma_0 - \log 2\pi$, where $\gamma_0$ is the Euler constant. M. Jutila in [8] continued meromorphically the function $\mathcal{Z}_1(s)$ to the whole complex plane with possible poles of order at most two at the points $s = -m$, $m \in \mathbb{N}$. Recently, M. Lukkarinen proved in [12] that $\mathcal{Z}_1(s)$ has simple poles only at $s = -(2m-1)$, $m \in \mathbb{N}$. She also found the formulae for the residues at the above points.

In the paper [7], also the first estimates and mean square estimates for the function $\mathcal{Z}_1(s)$ have been obtained.
The idea of application of probabilistic methods in the theory of the Riemann zeta-function belongs to H. Bohr and B. Jessen [3], [4]. Later, Bohr-Jessen's theory was developed by many authors, for history and results, see [11].
Let $meas\{A\}$ denote the Lebesgue measure of a measurable set $A \in \mathbb{R}$, and let $\mathcal{B}(S)$ stand for the class of Borel sets of the space $S$. The first probabilistic limit theorem for the function $\mathcal{Z}_1(s)$ has been obtained in [10].

THEOREM 1. *Let $\sigma > \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_\sigma$ such that the probability measure*

$$\frac{1}{T} meas\{t \in [0,T] : \mathcal{Z}_1(\sigma + it) \in A\},\ A \in \mathcal{B}(\mathbb{C}),$$

*converges weakly to $P_\sigma$ as $T \to \infty$.*

Theorem 1 is of so called continuous character, because the imaginary part $t$ in $\mathcal{Z}_1(\sigma + it)$ varies continuosly in $[0,T]$.
Also, a discrete limit theorem for the function $\mathcal{Z}_1(s)$ is known [1]. In this theorem, the imaginary part $t$ in $\mathcal{Z}_1(\sigma + it)$ in the definition of the probability measure takes values in an arithmetical progression. Let, for $N \in \mathbb{N} \cup \{0\}$,

$$\mu_N(...) = \frac{1}{N+1} \sum_{\substack{0 \le m \le N \\ ...}} 1,$$

where in place of dots a condition satisfied by $m$ is to be written. Let $h > 0$ be a fixed number.

THEOREM 2. *[1] Let $\sigma > \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $P_\sigma$ such that the probability measure*

$$\mu_N(\mathcal{Z}_1(\sigma + imh) \in A),\ A \in \mathcal{B}(\mathbb{C}),$$

*converges weakly to $P_\sigma$ as $N \to \infty$.*

The aim of this note is to obtain a discrete limit theorem in the space of analytic functions for $\mathcal{Z}_1(s)$. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Denote by $H(D)$ the space of analytic functions on $D$ equipped with topology of uniform convergence on compacta.

THEOREM 3. *On $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure $P$ such that the probability measure*

$$\mu_N(\mathcal{Z}_1(s + imh) \in A), A \in \mathcal{B}(H(D)),$$

*converges weakly to $P$ as $N \to \infty$.*

## 2 A limit theorem on a torus

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane. Define, for $a > 1$,

$$\Omega_a = \prod_{u \in [1,a]} \gamma_u,$$

where $\gamma_u = \gamma$ for all $u \in [1, a]$. Since $\gamma$ is a compact, by the Tikhonov theorem the torus $\Omega_a$ is a compact topological Abelian group.

THEOREM 4. *On $(\Omega_a, \mathcal{B}(Q_a))$ there exists a probability measure $Q_a$ such that the probability measure*

$$Q_{N,a}(A) \stackrel{\text{def}}{=} \mu_N((u^{imh} : u \in [1, a]) \in A), A \in \mathcal{B}(\Omega_a),$$

*converges weakly to $Q_a$ as $N \to \infty$.*

*Proof.* Let $\mathbb{Z}$ denote the set of all integers. The dual group of $\Omega$ is

$$\oplus_{u \in [1,a]} \mathbb{Z}_u,$$

where $\mathbb{Z}_u = \mathbb{Z}$ for all $u \in [1, a]$. An element $\underline{k} = \{k_u : u \in [1, a]\} \in \oplus_{u \in [1,a]} \mathbb{Z}_u$, where only a finite number of integers $k_u$ are distinct from zero, acts on $\Omega_a$ by

$$\underline{x} \to \underline{x}^{\underline{k}} = \prod_{u \in [1,a]} x_u^{k_u},$$

where $\underline{x} = \{x_u : x_u \in \gamma, u \in [1, a]\}$. Therefore, the Fourier transform $g_N(\underline{k})$ of the probability measure $Q_N$ is

$$g_N(\underline{k}) = \int_\Omega \prod_{u \in [1,a]} x_u^{k_u} dQ_{N,a} = \frac{1}{N+1} \sum_{m=0}^{N} \prod_{u \in [1,a]} e^{imhk_u \log u}$$

$$= \frac{1}{N+1} \sum_{m=0}^{N} \exp\{imh \sum_{u \in [1,a]} k_u \log u\},$$

where only a finite number of integers $k_u$ are non-zero. Since

$$\exp\{ih \sum_{u \in [1,a]} k_u \log u\} = 1$$

if and only if there exists $r \in \mathbb{Z}$ such that

$$\sum_{u \in [1,a]} k_u \log u = \frac{2\pi r}{h},$$

hence we deduce that

$$g_N(\underline{k}) = \begin{cases} 1 & \text{if } \sum\limits_{u\in[1,a]} k_u \log u = \frac{2\pi r}{h} \text{ for some } r \in \mathbb{Z}, \\ \frac{1}{N+1} \frac{1-\exp\left\{i(N+1)h \sum\limits_{u\in[1,a]} k_u \log u\right\}}{1-\exp\left\{ih \sum\limits_{u\in[1,a]} k_u \log u\right\}} & \text{otherwise,} \end{cases}$$

Therefore,

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \sum_{u\in[1,a]} k_u \log u = \frac{2\pi r}{h} \text{ for some } r \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, by the continuity theorem for probability measures on locally compact topological groups, see, for example, [6], Theorem 1.4.2, we obtain that the probability measure $Q_{N,a}$ converges weakly to a probability measure $Q_a$ with the Fourier transform given by the right-band side of (1). The theorem is proved.

# 3 Limit theorem for integral over a finite interval

For fixed $\sigma_1 > \frac{1}{2}$, and $x > y \geq 1$, define

$$v(x,y) = \exp\left\{-\left(\frac{x}{y}\right)^{\sigma_1}\right\}.$$

In this section, we will prove a discrete limit theorem in the space of analytic functions for the function

$$\mathcal{Z}_{1,a,y}(s) = \int_1^a \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 v(x,y)x^{-s}dx,$$

with finite fixed $a > 1$.
Divide the interval $[1,a]$ by points $1 = x_0 < x_1 < ... < x_n = a$ into $n$ subintervals of the same length $\frac{a-1}{n}$. In each interval $[x_{j-1}, x_j]$, we take a point $\xi_j$ and define the sum

$$S_{1,n,a,y}(s) = \sum_{j=1}^n \left|\zeta\left(\frac{1}{2}+i\xi_j\right)\right|^2 v(\xi_j,y)\xi_j^{-s} \triangle x_j$$

with $\triangle x_j = x_j - x_{j-1}$.
Define

$$P_{N,n,a,y}(A) = \mu_N(S_{1,n,a,y}(s+imh) \in A),\ A \in \mathcal{B}(H(D)).$$

LEMMA **1.** *On $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure $P_{n,a,y}$ such that the measure $P_{N,n,a,y}$ converges weakly to $P_{n,a,y}$ as $N \to \infty$.*

*Proof.* Consider the function $h_{n,a,y} : \Omega_a \to H(D)$ given by the formula

$$h_{n,a,y}(f) = \sum_{j=1}^{n} \Big|\zeta\Big(\frac{1}{2} + i\xi_j\Big)\Big|^2 v(\xi_j, y)\xi_j^{-s} f(\xi_j) \triangle x_j,\ f \in \Omega_a.$$

Then the function $h_{n,a,y}$ is continuous, and

$$h_{n,a,y}(x^{-imh}) = \sum_{j=1}^{n} \Big|\zeta\Big(\frac{1}{2} + i\xi_j\Big)\Big|^2 v(\xi_j, y)\xi_j^{-s-imh} \triangle x_j = S_{1,n,a,y}(s + imh).$$

Thus, $P_{N,n,a,y} = Q_N h_{n,a,y}^{-1}$, and the lemma follows from Theorem 4 and Theorem 5.1 of [2], by using the continuity of the function $h_{n,a,y}$. The limit measure $P_{n,a,y} = Qh_{n,a,y}^{-1}$, where $Q$ is the limit measure in Theorem 4.

LEMMA **2.** *Let $K$ be a compact subset of the strip $D$. Then the relation*

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s\in K} |S_{1,n,a,y}(s + imh) - \mathcal{Z}_{1,a,y}(s + imh)| = 0$$

*holds.*

*Proof.* Let $L$ be a simple closed contour lying in $D$ and enclosing the set $K$. Then, by the integral Cauchy formula

$$S_{1,n,a,y}(s + imh) - \mathcal{Z}_{1,a,y}(s + imh) = \frac{1}{2\pi i} \int_L \frac{S_{1,n,a,y}(z + imh) - \mathcal{Z}_{1,a,y}(z + imh)}{z - s} dz.$$

Hence,

$$\sup_{s\in K} |S_{1,n,a,y}(s+imh) - \mathcal{Z}_{1,a,y}(s+imh)| \ll \frac{1}{\delta} \int_L |S_{1,n,a,y}(z+imh) - \mathcal{Z}_{1,a,y}(z+imh)||dz|,$$

where $\delta$ is the distance of $L$ from the set $K$. Therefore,

$$\frac{1}{N+1} \sum_{m=0}^{N} \sup_{s\in K} |S_{1,n,a,y}(s + imh) - \mathcal{Z}_{1,a,y}(s + imh)|$$

$$\ll \frac{1}{\delta} \int_L |dz| \frac{1}{N+1} \sum_{m=0}^{N} |S_{1,n,a,y}(z + imh) - \mathcal{Z}_{1,a,y}(z + imh)|. \tag{2}$$

Clearly, we have

$$|S_{1,n,a,y}(\sigma + imh) - \mathcal{Z}_{1,a,y}(\sigma + imh)|^2$$
$$= S_{1,n,a,y}(\sigma + imh)\overline{S_{1,n,a,y}(\sigma + imh)} - S_{1,n,a,y}(\sigma + imh)\overline{\mathcal{Z}_{1,a,y}(\sigma + imh)}$$
$$-\overline{S_{1,n,a,y}(\sigma + imh)}\mathcal{Z}_{1,a,y}(\sigma + imh) + \mathcal{Z}_{1,a,y}(\sigma + imh)\overline{\mathcal{Z}_{1,a,y}(\sigma + imh)}.$$

It is not difficult to see that

$$\frac{1}{N+1}\sum_{m=0}^{N} S_{1,n,a,y}(\sigma+imh)\overline{S_{1,n,a,y}(\sigma+imh)}$$

$$=\sum_{j=1}^{n}\Big|\zeta\Big(\frac{1}{2}+i\xi_j\Big)\Big|^4 v^2(\xi_j,y)\xi_j^{-2\sigma}(\triangle x_j)^2$$

$$+\sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n}\Big|\zeta\Big(\frac{1}{2}+i\xi_j\Big)\Big|^2\Big|\zeta\Big(\frac{1}{2}+i\xi_k\Big)\Big|^2 v(\xi_j,y)v(\xi_k,y)(\xi_j,\xi_k)^{-\sigma}$$

$$\times\frac{1-\Big(\frac{\xi_k}{\xi_j}\Big)^{i(N+1)h}}{(N+1)\Big(1-\Big(\frac{\xi_k}{\xi_j}\Big)^{ih}\Big)}\triangle x_j\triangle x_k.$$

Since

$$\lim_{n\to\infty}\sum_{j=1}^{n}\Big|\zeta\Big(\frac{1}{2}+i\xi_j\Big)\Big|^4 v^2(\xi_j,y)\xi_j^{-2\sigma}\triangle x_j=\int_1^a\Big|\zeta\Big(\frac{1}{2}+ix\Big)\Big|^4 v^2(x,y)x^{-2\sigma}dx<\infty,$$

we have that

$$\sum_{j=1}^{n}\Big|\zeta\Big(\frac{1}{2}+i\xi_j\Big)\Big|^4 v^2(\xi_j,y)\xi_j^{-2\sigma}(\triangle x_j)^2$$

$$=\frac{1}{n}\Big(\int_1^a\Big|\zeta\Big(\frac{1}{2}+ix\Big)\Big|^4 v^2(x,y)x^{-2\sigma}dx+o(1)\Big)=o(1) \tag{3}$$

as $n\to\infty$. On the other hand,

$$\sum_{\substack{j=1\\ j\neq k}}^{n}\sum_{k=1}^{n}\Big|\zeta\Big(\frac{1}{2}+i\xi_j\Big)\Big|^2\Big|\zeta\Big(\frac{1}{2}+i\xi_k\Big)\Big|^2 v(\xi_j,y)v(\xi_k,y)(\xi_j\xi_k)^{-\sigma}$$

$$\times\frac{1-\Big(\frac{\xi_k}{\xi_j}\Big)^{i(N+1)h}}{(N+1)\Big(1-\Big(\frac{\xi_k}{\xi_j}\Big)^{ih}\Big)}\triangle x_j\triangle x_k=o(1) \tag{4}$$

as $N\to\infty$.
Similarly

$$\frac{1}{N+1}\sum_{m=0}^{N}\mathcal{Z}_{1,a,y}(\sigma+imh)\overline{\mathcal{Z}_{1,a,y}(\sigma+imh)}$$

$$=\frac{1}{N+1}\sum_{m=0}^{N}\Big(\int_1^a\Big|\zeta\Big(\frac{1}{2}+iu\Big)\Big|^2 v(u,y)u^{-\sigma-imh}du\int_1^a\Big|\zeta\Big(\frac{1}{2}+ix\Big)\Big|^2 v(x,y)x^{-\sigma+imh}dx\Big)$$

$$= \int_1^a \int_{\substack{1 \\ u=x}}^a \left|\zeta\Big(\frac{1}{2}+iu\Big)\right|^2 \left|\zeta\Big(\frac{1}{2}+ix\Big)\right|^2 v(u,y)v(x,y)(ux)^{-\sigma}dudx$$

$$+ \int_1^a \int_{\substack{1 \\ u\neq x}}^a \left|\zeta\Big(\frac{1}{2}+iu\Big)\right|^2 \left|\zeta\Big(\frac{1}{2}+ix\Big)\right|^2 v(u,y)v(x,y)(ux)^{-\sigma}$$

$$\times \frac{1-\left(\frac{x}{u}\right)^{i(N+1)h}}{(N+1)\left(1-\left(\frac{x}{u}\right)^{ih}\right)} dudx = o(1) \tag{5}$$

as $N \to \infty$.

In the same way we find that

$$\frac{1}{N+1}\sum_{m=0}^{N} S_{1,n,a,y}(\sigma+imh)\overline{\mathcal{Z}_{1,a,y}(\sigma+imh)} = o(1) \tag{6}$$

and

$$\frac{1}{N+1}\sum_{m=0}^{N} \overline{S_{1,n,a,y}(\sigma+imh)}\mathcal{Z}_{1,a,y}(\sigma+imh) = o(1) \tag{7}$$

as $N \to \infty$. From (3)-(7), we have that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}|S_{1,n,a,y}(z+imh)-\mathcal{Z}_{1,a,y}(z+imh)| = 0.$$

This and (2) prove the lemma.

Let $\{K_l : l \in \mathbb{N}\}$ be a sequence of compact subsets of $D$ such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l$ [5]. For $f, g \in H(D)$, define

$$\rho(f,g) = \sum_{l=1}^{\infty} 2^{-l}\frac{\rho_l(f,g)}{1+\rho_l(f,g)},$$

where

$$\rho_l(f,g) = \sup_{s\in K_l}|f(s)-g(s)|.$$

Then it is easily seen that $\rho(f,g)$ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta.

THEOREM 5. *On $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure $P_{a,y}$ such that the probability measure*

$$P_{N,a,y}(A) \stackrel{\text{def}}{=} \mu_N(\mathcal{Z}_{1,a,y}(s+imh) \in A), A \in \mathcal{B}(H(D)),$$

*converges weakly to $P_{a,y}$ as $N \to \infty$.*

*Proof.* On a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$, define a random variable $\theta_N$ by

$$\mathbb{P}(\theta_N = hm) = \frac{1}{N+1}, \ m = 0, 1, ..., N.$$

Let

$$U_{N,n,a,y}(s) = S_{1,n,a,y}(s + i\theta_N).$$

Then Lemma 1 implies the relation

$$U_{N,n,a,y}(s) \xrightarrow[N\to\infty]{\mathcal{D}} U_{n,a,y}(s), \tag{8}$$

where $U_{n,a,y}(s)$ is a $H(D)$ - valued random variable having the distribution $P_{n,a,y}$, where $P_{n,a,y}$ is the limit measure in Lemma 1.
We will prove that the family of probability measures $\{P_{n,a,y} : n \in \mathbb{N}\}$ is tight. For $M_l > 0$,

$$\mathbb{P}(\sup_{s\in K_l} |U_{N,n,a,y}| > M_l) \le \frac{1}{M_l(N+1)} \sum_{m=0}^{N} \sup_{s\in K_l} |S_{1,n,a,y}(s+imh)|.$$

This and Lemma 2 show that

$$\limsup_{N\to\infty} \mathbb{P}(\sup_{s\in K_l} |U_{N,n,a,y}(s)| > M_l)$$

$$\le \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M_l(N+1)} \sum_{m=0}^{N} \sup_{s\in K_l} |S_{1,n,a,y}(s+imh)|$$

$$\le \sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{M_l(N+1)} \sum_{m=0}^{N} \sup_{s\in K_l} |S_{1,n,a,y}(s+imh) - \mathcal{Z}_{1,a,y}(s+imh)|$$

$$+ \limsup_{N\to\infty} \frac{1}{M_l(N+1)} \sum_{m=0}^{N} \sup_{s\in K_l} |\mathcal{Z}_{1,a,y}(s+imh)|$$

$$\ll \frac{1}{M_l} + \limsup_{N\to\infty} \frac{1}{M_l(N+1)} \sum_{m=0}^{N} \sup_{s\in K_l} \mathcal{Z}_{1,y}(\sigma) \le \frac{R_l}{M_l} \tag{9}$$

with $R_l < \infty$, since the integral $\mathcal{Z}_{1,y}(s)$ converges absolutely for $\sigma > \frac{1}{2}$, see [1], Lemma 8, therefore, uniformly on compact subsets of $D$. Now we take $M_l = M_{l,\varepsilon} =$

$R_l 2^l \varepsilon^{-1}$, where $\varepsilon > 0$ is an arbitrary number. Then (8), (9) and Lemma 4 yield, for all $n \in \mathbb{N}$, the relation

$$\mathbb{P}(\sup_{s\in K_l} |U_{n,a,y}(s)| > M_l) \le \frac{\varepsilon}{2^l},\ l \in \mathbb{N}. \tag{10}$$

Define

$$K_\varepsilon = \{g \in H(D) : \sup_{s\in K_l} |g(s)| \le M_l,\ l \in \mathbb{N}\}.$$

Then $K_\varepsilon$ is a compact subset of the space $H(D)$, and, in view of (10)

$$\mathbb{P}(U_{n,a,y}(s) \in K_\varepsilon) \ge 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$, or, by the definition of the random element $U_{n,a,y}(s)$,

$$P_{n,a,y}(K_\varepsilon) \ge 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This shows that the family $\{P_{n,a,y:n\in\mathbb{N}}\}$ is tight. Hence, by the Prokhorov theorem [2], the family $\{P_{n,a,y} : n \in \mathbb{N}\}$ is relatively compact. Therefore, there exists a sequence $\{P_{n_k,a,y}\} \subset \{P_{n,a,y}\}$ such that $\{P_{n_k,a,y}\}$ converges weakly to a certain probability measure $\{P_{a,y}\}$ on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. Thus,

$$U_{n_k,a,y}(s) \xrightarrow[k\to\infty]{\mathcal{D}} P_{a,y}. \tag{11}$$

Define

$$Y_{N,a,y}(s) = \mathcal{Z}_{1,a,y}(s + i\theta_N).$$

Then, by Lemma 2, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}(\rho(U_{N,n,a,y}(s), Y_{N,a,y}(s)) \ge \varepsilon)$$

$$= \lim_{n\to\infty} \limsup_{N\to\infty} \mu_N(\rho(S_{1,n,a,y}(s + imh), \mathcal{Z}_{1,a,y}(s + imh)) \ge \varepsilon)$$

$$\le \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{m=0}^{N} \rho(S_{1,n,a,y}(s + imh), \mathcal{Z}_{1,a,y}(s + imh)) = 0.$$

This, (8), (11) and Theorem 4.2 of [2] give the relation

$$Y_{N,a,y}(s) \xrightarrow[N\to\infty]{\mathcal{D}} P_{a,y}.$$

Therefore, the measure $P_{N,a,y}$ converges weakly to $P_{a,y}$ as $N \to \infty$. The theorem is proved.

# 4 Limit theorem for absolutely convergent integral

This section is devoted to discrete limit theorem in the space of analytic functions for the function

$$\mathcal{Z}_{1,y}(s) = \int_1^\infty \left|\zeta\Big(\frac{1}{2}+ix\Big)\right|^2 v(x,y)x^{-s}dx.$$

The above integral absolutely converges for $\sigma > \frac{1}{2}$ (Lemma 8, [1]).

THEOREM 6. *On $(H(D), \mathcal{B}(H(D)))$, there exists a probability measure $P_y$ such that the probability measure*

$$P_{N,y}(A) \stackrel{\text{def}}{=} \mu_N(\mathcal{Z}_{1,y}(s+imh) \in A), A \in \mathcal{B}(H(D)),$$

*converges weakly to $P_y$ as $N \to \infty$.*

*Proof.* Let $\theta_N$ be the random variable defined in Theorem 5. Define $H(D)$ - valued random element $Y_{N,a,y}(s)$ by the formula

$$Y_{N,a,y}(s) = \mathcal{Z}_{1,a,y}(s+i\theta_N).$$

Then, by Theorem 5 we have that

$$Y_{N,a,y}(s) \xrightarrow[N\to\infty]{\mathcal{D}} Y_{a,y}(s), \tag{12}$$

where $Y_{a,y}(s)$ is an $H(D)$-valued random element having the distribution $P_{a,y}$. First, we will show that the family of probability measures $\{P_{a,y} : a > 1\}$ is tight for fixed $y$. For arbitrary $M_l > 0$,

$$\mathbb{P}(\sup_{s\in K_l}|Y_{N,a,y}(s)| > M_l) \le \frac{1}{M_l(N+1)}\sum_{m=0}^{N}\sup_{s\in K_l}|\mathcal{Z}_{1,a,y}(s+imh)|. \tag{13}$$

The integral defining $\mathcal{Z}_{1,y}(s)$ converges absolutely for $\sigma > \frac{1}{2}$, therefore, uniformly on compact subsets of $D$. Thus, by (13),

$$\limsup_{N\to\infty}\mathbb{P}(\sup_{s\in K_l}|Y_{N,a,y}(s)| > M_l) \le \frac{R_l}{M_l},$$

where

$$R_l = \sup_{s\in K_l}\int_1^\infty \left|\zeta\Big(\frac{1}{2}+ix\Big)\right|^2 v(x,y)x^{-\sigma}dx < \infty.$$

Now let $M_l = M_{l,\varepsilon} = R_l 2^l \varepsilon^{-1}$, where $\varepsilon$ is an arbitrary positive number. Then, in view of (12), Theorem 5.1 of [2] and (13), we find that, for all $a > 1$,

$$\mathbb{P}(\sup_{s\in K_n}|Y_{a,y}(s)| > M_n) \le \frac{\varepsilon}{2^n}.n \in \mathbb{N}, \tag{14}$$

The set $H_\varepsilon = \{g \in H(D) : \sup_{s\in K_n} |g(s)| \le M_n, n \in \mathbb{N}\}$ is compact in the space $H(D)$. Moreover, (14) shows that

$$\mathbb{P}(Y_{a,y}(s) \in H_\varepsilon) \ge 1 - \varepsilon$$

for all $a > 1$, or, by the definition of the random element $Y_{a,y}$

$$P_{a,y}(H_\varepsilon) \ge 1 - \varepsilon$$

for all $a > 1$. So, we proved that the family of probability measures $\{P_{a,y} : a > 1\}$ is tight, therefore, by the Prokhorov theorem it is relatively compact. Thus, there exists a subsequence $\{P_{a_k,y}\} \subset \{P_{a,y}\}$ such that $P_{a_k,y}$ converges weakly to some probability measure $P_{a,y}$ on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. In other words,

$$Y_{a_k,y}(s) \xrightarrow[k\to\infty]{\mathcal{D}} P_y. \tag{15}$$

Moreover, the uniform convergence on compact subsets of $D$ of $\mathcal{Z}_{1,a,y}(s)$ to $\mathcal{Z}_{1,y}(s)$ as $a \to \infty$ implies, for every $\varepsilon > 0$,

$$\lim_{a\to\infty} \limsup_{N\to\infty} \mu_N(\rho(\mathcal{Z}_{1,y}(s + imh), \mathcal{Z}_{1,a,y}(s + imh)) \ge \varepsilon)$$

$$\le \lim_{a\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon(N+1)} \sum_{m=0}^{N} \rho(\mathcal{Z}_{1,y}(s + imh), \mathcal{Z}_{1,a,y}(s + imh)) = 0,$$

where $\rho$ is the metric in $H(D)$ defined by terms of the sequence $\{K_n\}$. Hence, setting

$$Y_{N,y}(s) = \mathcal{Z}_{1,y}(s + i\theta_N),$$

we find that

$$\lim_{a\to\infty} \limsup_{N\to\infty} \mathbb{P}(\rho(Y_{N,a,y}(s), Y_{N,y}(s)) \ge \varepsilon) = 0. \tag{16}$$

This, (12), (15) and Theorem 4.2 of [2] complete the proof of the theorem.

# 5 Approximation of $\mathcal{Z}_1(s)$ by $\mathcal{Z}_{1,y}(s)$

To pass from the function $\mathcal{Z}_{1,y}(s)$ to $\mathcal{Z}_1(s)$, we need an approximation in the mean of $\mathcal{Z}_1(s)$ by $\mathcal{Z}_{1,y}(s)$.

THEOREM 7. *Let $K$ be a compact subset of the strip $D$. Then*

$$\lim_{y\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s\in K} |\mathcal{Z}_1(s + imh) - \mathcal{Z}_{1,y}(s + imh)| = 0.$$

*Proof.* Let

$$l_y(s) = \frac{\sigma_1}{s}\Gamma\Big(\frac{s}{\sigma_1}\Big)y^s.$$

For $s \in K$, suppose that $\frac{1}{2} + \theta_1 \le \sigma \le 1 - \theta_2, \theta_1 > 0, \theta_2 > 0$. We put $\sigma_2 = \frac{1}{2} + \frac{\theta_1}{2}$. Then we have that

$$\mathcal{Z}_{1,y}(s) = \frac{1}{2\pi i}\int_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty} \mathcal{Z}_1(s+z)l_y(z)\frac{dz}{z}$$

$$+\mathcal{Z}_1(s) + R_y(s), \tag{17}$$

where

$$R_y(s) = \mathrm{Res}_{z=1-s} Z_1(s+z)l_y(z)z^{-1}.$$

Let $L$ be a simple closed contour lying in $D$ and enclosing the set $K$, and let $|L|$ be the length of $L$. Then the integral Cauchy formula shows that

$$\sup_{s\in K}|\mathcal{Z}_1(s+imh) - \mathcal{Z}_{1,y}(s+imh)| \le \frac{1}{2\pi\delta}\int\limits_L |\mathcal{Z}_1(z+imh) - \mathcal{Z}_{1,y}(z+imh)||dz|,$$

where $\delta$ is the distance of $L$ from the set $K$. Hence,

$$\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}|\mathcal{Z}_1(s+imh) - \mathcal{Z}_{1,y}(s+imh)|$$

$$\ll \frac{1}{N\delta}\int\limits_L |dz|\sum_{m=0}^{N}|\mathcal{Z}_1(\mathrm{Re}z + imh + i\mathrm{Im}z) - \mathcal{Z}_{1,y}(\mathrm{Re}z + imh + i\mathrm{Im}z)|$$

$$\ll \frac{|L|}{N\delta}\sup_{s\in L}\sum_{m=0}^{N}|\mathcal{Z}_1(\sigma+imh+it) - \mathcal{Z}_{1,y}(\sigma+imh+it)|. \tag{18}$$

From (17) we have

$$\mathcal{Z}_1(\sigma+imh+it) - \mathcal{Z}_{1,y}(\sigma+imh+it) = -R_y(\sigma+imh+it)$$

$$+O\Big(\int_{-\infty}^{\infty}|\mathcal{Z}_1(\sigma_2+imh+it+i\tau)|l_y(\sigma_2-\sigma+i\tau)|d\tau\Big).$$

This gives

$$\frac{1}{N}\sum_{m=0}^{N}|\mathcal{Z}_1(\sigma+imh+it) - \mathcal{Z}_{1,y}(\sigma+imh+it)|$$

$$\ll \frac{1}{N}\sum_{m=0}^{N}|R_y(\sigma+imh+it)|$$

$$+\int_{-\infty}^{\infty}|l_y(\sigma_2-\sigma+i\tau)|\Big(\frac{1}{N}\sum_{m=0}^{N}|\mathcal{Z}_1(\sigma_2+imh+it+i\tau)|\Big)d\tau. \tag{19}$$

We always can choose $L$ to be bounded. Then $t$ is bounded, we obtain that

$$\frac{1}{N}\sum_{m=0}^{N}|\mathcal{Z}_1(\sigma_2+imh+it+i\tau)|$$

$$\ll\Big(\frac{1}{N}\sum_{m=0}^{N}|\mathcal{Z}_1(\sigma_2+imh+it+i\tau)|^2\Big)^{\frac{1}{2}}\ll 1+|\tau|. \tag{20}$$

Moreover, simple calculations show that

$$\frac{1}{N}\sum_{m=0}^{N}|R_y(\sigma+imh+it)|=o(1), \tag{21}$$

as $N\to\infty$.
Now suppose that, for $s\in L$,

$$\sigma\geq\frac{1}{2}+\frac{3\theta_1}{4},$$

and $\delta\geq\frac{\theta_1}{4}$. Then estimates (18)-(21) yield, as $N\to\infty$,

$$\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}|\mathcal{Z}_1(s+imh)-\mathcal{Z}_{1,y}(s+imh)|$$

$$\ll\sup_{s\in L}\int_{-\infty}^{\infty}|l_y(\sigma_2-\sigma+i\tau)|(1+|\tau|)d\tau+o(1)$$

$$\ll\sup_{\sigma\ll-\frac{\theta_1}{4}}\int_{-\infty}^{\infty}|l_y(\sigma+it)|(1+|t|)dt+o(1).$$

Since

$$\lim_{y\to\infty}\sup_{\sigma\leq-\frac{\theta_1}{4}}\int_{-\infty}^{\infty}|l_y(\sigma+it)|(1+|t|)dt=0,$$

hence the theorem follows.

# 6 Proof of the main theorem

Let $Y_y(s)$ be an $H(D)$- valued random element having the distribution $P_y$. Define

$$Y_{N,y}(s)=\mathcal{Z}_{1,y}(s+i\theta_N).$$

Then by Theorem 6 we have that

$$Y_{N,y}(s)\xrightarrow[N\to\infty]{\mathcal{D}}Y_y(s). \tag{22}$$

For $M_l > 0$,

$$\mathbb{P}(\sup_{s\in K_l}|Y_{1,N,y}(s)| > M_l) \le \frac{1}{M_l(N+1)}\sum_{m=0}^{N}\sup_{s\in K_l}|\mathcal{Z}_{1,y}(s+imh)|. \tag{23}$$

The integral defining $\mathcal{Z}_{1,y}(s)$ converges absolutely for $\sigma > \frac{1}{2}$, therefore uniformly on compact subsets of $D$. Thus, by (23),

$$\limsup_{N\to\infty}\mathbb{P}(\sup_{s\in K_l}|Y_{N,y}(s)| > M_l) \le \frac{R_l}{M_l},$$

where $R_l < \infty$. Let, for arbitrary $\varepsilon > 0$, $M_l = R_l 2^l \varepsilon^{-1}$. Then, using (22) and Theorem 5.1 of [2], we deduce that

$$\mathbb{P}(\sup_{s\in K_n}|Y_y(s)| > M_l) \le \frac{\varepsilon}{2^l}, l \in \mathbb{N}. \tag{24}$$

By the compactness principle, the set $H_\varepsilon = \{f \in H(D) : \sup_{s\in K_l}|f(s)| \le M_l : l \in \mathbb{N}\}$ is compact in $H(D)$, and by (24)

$$\mathbb{P}(Y_y(s) \in H_\varepsilon) \ge 1 - \varepsilon$$

for all $y \ge 1$, or

$$P_y(H_\varepsilon) \ge 1 - \varepsilon$$

for all $y \ge 1$. So, we proved that the family of probability measures $\{P_y\}$ is tight. Moreover, Theorem 7 implies

$$\lim_{n\to\infty}\limsup_{N\to\infty}\mu_N(\rho(\mathcal{Z}_1(s+imh), \mathcal{Z}_{1,y}(s+imh)) \ge \varepsilon)$$

$$\le \lim_{y\to\infty}\limsup_{N\to\infty}\frac{1}{\varepsilon(N+1)}\sum_{m=0}^{N}\rho(\mathcal{Z}_1(s+imh), \mathcal{Z}_{1,y}(s+imh)) = 0.$$

Hence, taking $Y_N(s) = \mathcal{Z}_1(s+i\theta_N)$, we find that

$$\lim_{y\to\infty}\limsup_{N\to\infty}\mathbb{P}(\rho(Y_N(s), Y_{N,y}(s)) \ge \varepsilon) = 0. \tag{25}$$

The tightness of the family $\{P_y\}$ gives its relative compactness. Let $\{P_{y_k}\} \subset \{P_y\}$ be such that $P_{y_k}$ converges weakly, say, to $P$ as $k \to \infty$. Then

$$Y_{y_k} \xrightarrow[k\to\infty]{D} P, \tag{26}$$

and the theorem is a result of (22),(25), (26) and Theorem 4.2 of [2].

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