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A NEW VALUATION ON POLYHEDRAL CONES

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We define a new family of valuations on polyhedral cones valued in the space of bounded polyhedra.

Introduction

Let C be a polyhedral cone in a finite dimensional Euclidean space V over $\mathbb{R}$ and let $\mathcal{F}(C)$ be the set of all faces in C. A classical result in polyhedra geometry, known as the Brianchon–Gram–Sommerville relation (see, for example, [She67] and [Sch17]), reveals the relationship between the indicators of the cone C and those of the angle cones $A(F,C)$ (see Subsection 1.1) in correspondence to all the faces F in C:

$$\sum_{F\in\mathcal{F}(C)} (-1)^{\dim C-\dim F}[A(F,C)] = [-\operatorname{relint} C].$$

If we let C^* denote the dual cone of C, composed of the vectors with nonnegative scalar product with the elements of C, then applying the above identity to C^* we obtain

$$\sum_{F\in\mathcal{F}(C)} (-1)^{\dim F-\dim F_0}[F^*] = [-\operatorname{relint} C^*],$$

where F_0 is the minimal face of C. Indeed, the result is a consequence of a natural bijection between the faces of C and C^* and some natural properties of angle cones of dual cones all listed in Lemma 1.2.

This motivates us to define a function with two variables that evaluates on both the angle cones and the dual cones of the faces. For a cone C we let

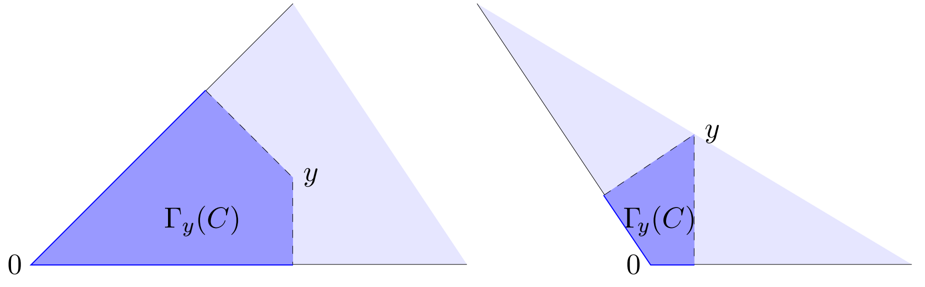
$$\Gamma(C,x,y) = \sum_{F\in\mathcal{F}(C)} (-1)^{\dim F}[A(F,C)](x)[F^*](y).$$

Furthermore, we define a function Γ_y that sends the indicator $[C](x)$ to the function $\Gamma(C,x,x-y)$. Our main theorem is the following.

Ключевые слова: polyhedral cones, Brianchon–Gram–Sommerville relation, bounded polyhedron.

Theorem 0.1 (cf. Theorem 2.7.). *For any $y \in V$, Γ_y is a valuation from the algebra of polyhedral cones to the algebra of bounded polyhedra.*

If C is a cone in $\mathbb{R}^2$ and y belongs to the interior of $C \cap C^*$, the function $\Gamma_y(C)$ is the characteristic function of $C \cap (y - \operatorname{relint} C^*)$.



The function Γ_y (or its close cousin, discussed in Subsection 2.3) was first introduced by Arthur [Art81] (see also Subsection 1.7 of [LW09]) in his work on the Arthur–Selberg trace formula on locally symmetric spaces. Arthur studied it for simplicial cones that arise as chambers associated with parabolic subgroups of a reductive algebraic group. In this context, Γ_y allows us to decompose the domain of integration (the Siegel domain) into manageable pieces. Our work can be viewed as a generalization of Arthur's construction to arbitrary polyhedral cones. It will be used in the second author's work [Zyd] that aims at extending part of Arthur's work to a more general class of integrals (period integrals).

§1. Preliminaries

We begin with introducing the basic notation in the algebra of polyhedra. Let V be a fixed finite dimensional Euclidean space over $\mathbb{R}$. Let $\langle \cdot, \cdot \rangle$ be the scalar product on it. By a hyperplane in V we always mean a hyperplane through the point 0. For $y \in V \setminus \{0\}$, denote by H_y the hyperplane in V perpendicular to y, and by

$$H_y^+ = \{v \in V \mid \langle v, y \rangle \geqslant 0\}, \quad H_y^- = -H_y^+$$

the closed halfspaces determined by H_y. In the following, a halfspace always means a set H^+ of the form H_y^+. A *cone* (respectively, *polyhedron*) is an intersection of a finite number of halfspaces (respectively, translates of halfspaces). In particular, a cone is always nonempty as 0 is an element of every cone.

We associate a polyhedron P with the *indicator function* $[P]$. Let $\mathcal{P}(V)$ be the $\mathbb{R}$-vector space spanned by the indicator functions of all polyhedra $P \subset V$. Let $\mathcal{PC}(V)$ be the vector space spanned by the indicator functions of cones. Finally let $\mathcal{P}_{bd}(V)$ be the vector space spanned by the indicator functions of bounded polyhedra. It turns out that $\mathcal{P}(V), \mathcal{PC}(V)$, and $\mathcal{P}_{bd}(V)$ possess a natural algebra structure, see [Bar08] for a reference.

Let W be an $\mathbb{R}$-vector space. A linear transformation $T : \mathcal{P}(V) \to W$ is called a *valuation*. For example, the Euler characteristic $\chi : \mathcal{P}(V) \to \mathbb{R}$ such that $\chi([P]) = 1$ for all nonempty polyhedrons P is a valuation. The following theorem is a useful tool to decide whether a map on $\mathcal{PC}(V)$ is a valuation. It was originally stated in [Gro78, Theorem 2] for polytopes but it continues to be true in the following form for cones.

Theorem 1.1. *A map $T : \mathcal{PC}(V) \to W$ is a valuation if and only if for all $y \in V \setminus \{0\}$ and all cones C we have*

$$T([C]) = T([C \cap H_y^+]) + T([C \cap H_y^-]) - T([C \cap H_y]).$$

We will mostly work with cones. A *face* of a cone is a cone of the form $C \cap H_y$ where $C \subset H_y^+$. In this definition, we allow $y = 0$ so that C is always its own face. Denote by $\mathcal{F}(C)$ the set of all faces of C. Note that $\mathcal{F}(C)$ is a singleton $\{C\}$ if and only if C is a subspace of V. Let $F_0 = F_0(C)$ be the minimal face of a cone C. It is a vector subspace of V. We call C *pointed* if F_0 is reduced to the origin. For each face F in $\mathcal{F}(C)$, let V_F be the subspace of V spanned by F and let d_F be its dimension. Define $\varepsilon_F = (-1)^{d_F}$ and $\varepsilon_C^F = (-1)^{d_C - d_F}$. We recall the Euler characteristic formula

$$\sum_{F \in \mathcal{F}(C)} \varepsilon_C^F = \begin{cases} 1 & \text{if } C \text{ is a subspace of } V, \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

In the following we will usually use C to denote a cone and use E, F, G to denote its faces.

1.1. Angle cones and dual cones. Let C be a cone. Let $\operatorname{relint} C$ be the largest open subset of V_C contained in C. Define the *dual cone* C^* of C as

$$C^* = \{v \in V \mid \langle v, y\rangle \geqslant 0, \ \forall\, y \in C\}.$$

Also define the *angle cone* as

$$A(F, C) = \{a(x - z) \mid a > 0, \ x \in C, z \in \operatorname{relint} F\}.$$

The *Minkowski sum* of two cones C_1, C_2 is defined to be

$$C_1 + C_2 = \{x + y \mid x \in C_1, \ y \in C_2\}.$$

We state some well-known properties of the angle cone and the dual cone.

Lemma 1.2. *Let C be a cone, F_0 the minimal face of C and F an arbitrary face of C.*

(1) $A(F_0, C) = C$, $A(C, C) = V_C$ *and* $(F_0)^* = F_0^{\perp}$.

(2) (See [Bar08, Theorem 4.13]) *C is the Minkowski sum of F_0 and some pointed cone C'. In particular, $C^* = (F_0 + C')^* = F_0^* \cap (C')^*$.*

(3) *The map*

$$\begin{aligned} \{\, G \in \mathcal{F}(C) : G \supset F \,\} &\to \mathcal{F}(A(F,C)) \\ G &\mapsto A(F,G) \end{aligned}$$

is an inclusion-preserving bijection. Furthermore, for any face $G \in \mathcal{F}(C)$ containing F, we have $A(A(F,G), A(F,C)) = A(G,C)$.

(4) *The map*

$$\begin{aligned} \{\, G \in \mathcal{F}(C) : G \supset F \,\} &\to \mathcal{F}(A(F,C)^*) \\ G &\mapsto A(G,C)^* \end{aligned}$$

is an inclusion-reversing bijection. Moreover, for any face $G \in \mathcal{F}(C)$ containing F, we have $A(A(G,C)^*, A(F,C)^*) = A(F,G)^*$.

(5) (See [Bar08, Theorem 5.3]) *There exists a unique valuation*

$$D : \mathcal{PC}(V) \to \mathcal{PC}(V)$$

such that $D([C]) = [C^*]$ for all cones C.

We have the following classical result, usually referred to as the Brianchon–Gram–Sommerville relation [She67] (see also [Sch17, Theorem 3.1)]

Theorem 1.3. *The following identities hold*

$$\sum_{E \in \mathcal{F}(C)} \varepsilon_C^E [A(E,C)] = [-\operatorname{relint} C]$$

and

$$\sum_{E \in \mathcal{F}(C)} \varepsilon_E^{F_0} [E^*] = [-\operatorname{relint} C^*].$$

For all $F \in \mathcal{F}(C)$, we also have

$$\sum_{E \supset F} \varepsilon_C^E [\operatorname{relint} A(E,C)] = [-A(F,C)]$$

and

$$\sum_{E \in \mathcal{F}(C)} \varepsilon_E^{F_0} [\operatorname{relint} E^*] = [-C^*].$$

Proof. Only the second and fourth relations are not mentioned in the references above. The second follows from the first and the fourth from the third by replacing C with its dual cone C^*. To get the exact form stated in the theorem, one applies item (4) of Lemma 1.2. □

§2. Main results

By Theorem 1.3, both

$$\sum_{E\in\mathcal{F}(C)} \varepsilon_C^E[A(E,C)] \quad \text{and} \quad \sum_{E\in\mathcal{F}(C)} \varepsilon_E^{F_0}[E^*]$$

are valuations. This motivates us to extend them to a new function with two variables, called the Γ function. We will see in the next subsection that a one parameter family of valuations can be derived from the Γ function.

2.1. The Γ function.

Definition 2.1. For any cone C and any points $x, y \in V$, let

$$\Gamma(C,x,y) = \sum_{F\in\mathcal{F}(C)} \varepsilon_F[A(F,C)](x)[F^*](y).$$

We begin with computing the value of $\Gamma(C,x,y)$ for some special $x, y \in V$. When the context is clear, we abbreviate $\Gamma(C,x,y)$ as $\Gamma(x,y)$.

Lemma 2.2. (1) *If $y \in C^*$ or if $x \in -\operatorname{relint} C$, then*

$$\Gamma(x,y) = \varepsilon_C[-\operatorname{relint} C](x)[C^*](y).$$

(2) *If $y \in -\operatorname{relint} C^*$ or if $x \in C$, then $\Gamma(x,y) = \varepsilon_{F_0}[C](x)[-\operatorname{relint} C^*](y)$.*
(3) *If C is a vector subspace of V, then $\Gamma(x,y) = \varepsilon_C[C](x)[C^\perp](y)$.*
(4) *$\Gamma(x,y) = 0$ unless $x \in V_C$ and $y \in F_0^\perp$.*

Proof. The first two identities are immediate consequences of the first two identities in Theorem 1.3 and Lemma 1.2 (3)–(4). The last two identities follow from Lemma 1.2 (1)–(2). □

We say that the cone C is *nondegenerate* if it is pointed and

$$d_C = \dim V_C = \dim V.$$

Proposition 2.3. *Suppose C is nondegenerate. If $\langle x,y\rangle > 0$, then $\Gamma(x,y) = 0$.*

Proof. Suppose $\langle x,y\rangle > 0$. We may assume that we are not in any of the first three cases considered in Lemma 2.2, because in these cases the result is clear. Moreover, without changing the value of $\Gamma(x,y)$ or the fact that $\langle x,y\rangle > 0$ we may assume that x and y are in general position with respect to C, that is to say, both x and y do not belong to any space V_F or $V_F^\perp$ for $F \in \mathcal{F}(C)\setminus\{\{0\},C\}$.

Put $C_y = H_y \cap C$ and $C_y^+ = H_y^+ \cap C$. Since $y \notin C^* \cup (-\operatorname{relint} C^*)$, C_y^+ is a nondegenerate cone in V and C_y is its facet. Let $\mathcal{F}(C,y)$ be the subset of $\mathcal{F}(C)$ composed of faces F of dimension at least 2 such that $\dim(F\cap H_y) = \dim F - 1 \geqslant 1$. The map

$$F \in \mathcal{F}(C,y) \mapsto F \cap H_y$$

is a bijection between $\mathcal{F}(C,y)$ and $\mathcal{F}(C_y)\setminus\{\{0\}\}$.

We assume that y is in general position with respect to C so $[F^*](y)\neq 0$ for a face F of C means that $F\in\mathcal{F}(C_y^+)$ and $F\notin\mathcal{F}(C,y)$. Hence

$$\Gamma(C,x,y)=\sum_{F\in\mathcal{F}(C)}\varepsilon_F[A(F,C)](x)[F^*](y)=\sum_{F\in\mathcal{F}(C_y^+)\setminus\mathcal{F}(C,y)}\varepsilon_F[A(F,C_y^+)](x).$$

The rest of the faces in C_y^+ have nontrivial intersection with H_y. So they are either $\{0\}$, or those in bijection with faces in $\mathcal{F}(C,y)$. Applying the first identity of Theorem 1.3 to C_y^+, we obtain

$$\begin{aligned}&\varepsilon_C[-\mathrm{relint}\, C_y^+](x)\\&\quad=\sum_{F\in\mathcal{F}(C_y^+)}\varepsilon_F[A(F,C_y^+)](x)\\&\quad=[C_y^+](x)+\sum_{F\in\mathcal{F}(C,y)}\varepsilon_F\left([A(F\cap C_y^+,C_y^+)](x)-[A(F\cap H_y,C_y^+)](x)\right)+\Gamma(C,x,y).\end{aligned}$$

The fact that $x\notin -\mathrm{relint}\, C$ implies that $[-\mathrm{relint}\, C_y^+](x)=0$ and the fact that $x\notin C$ implies that $[C_y^+](x)=0$. We obtain hence

$$\Gamma(C,x,y)=\sum_{F\in\mathcal{F}(C,y)}\varepsilon_F\left([A(F\cap H_y,C_y^+)](x)-[A(F\cap C_y^+,C_y^+)](x)\right).$$

Since $\langle x,y\rangle>0$, it follows that $[A(C_y,C_y^+)](x)=[A(C_y^+,C_y^+)](x)=1$. For all other faces $F\in\mathcal{F}(C,y)\setminus C$, we have

$$[A(F\cap H_y,C_y^+)](x)=[A(F\cap C_y^+,C_y^+)](x),$$

which implies $\Gamma(C,x,y)=0$. □

2.2. The one parameter family of valuations.

Definition 2.4. Let C be a cone in V. For $y\in V$, let $\Gamma_y([C])(x)=\Gamma(C,x,x-y)$ where $x\in V$.

First we show that $\Gamma_y([C])$ is supported in a disk of radius $\frac{\|y\|}{2}$ in V. For $v\in V$ and $r\in\mathbb{R}_{\geqslant 0}$, let $B(v,r)=\{w\in V\mid \|w-v\|\leqslant r\}$.

Lemma 2.5. *$\Gamma_y([C])(x)=0$ unless $\langle x,x-y\rangle\leqslant 0$.*

Proof. By Lemma 2.2 (2), there exists a pointed cone C' such that $C=F_0+C'$, where C' is in the orthogonal complement of the subspace F_0. Furthermore, every face F of C can be written as F_0+F', where F' is a face in C'. Let x' and y' be orthogonal projections of x and y onto $F_0^{\perp}$. We have $[A(F_0+F',F_0+$

$C')](x) = 1$ if and only if $[A(F',C')](x') = 1$. Similarly, $[(F_0 + F')^*](y) = [F_0^\perp \cap (F')^*](y) = 1$ if and only if $[F'^*](y') = 1$. Hence

$$\Gamma_y([C])(x) = \sum_{F' \in \mathcal{F}(C')} \varepsilon_{F_0+F'}[A(F_0 + F', F_0 + C')](x)[(F_0 + F')^*](x - y)$$
$$= \varepsilon_{F_0}\Gamma_{y'}([C'])(x').$$

Since $\langle x, x-y\rangle = \langle x', x'-y'\rangle$, it suffices to prove the claim for pointed cones. This immediately follows from Lemma 2.2 (4) and Proposition 2.3. □

Corollary 2.6. *$\Gamma_y([C])(x)$ vanishes unless $x \in B\Big(\frac{y}{2}, \frac{\|y\|}{2}\Big)$.*

Proof. Suppose $x \notin B\Big(\frac{y}{2}, \frac{\|y\|}{2}\Big)$. That means that $\|x - y/2\| > 1/2\|y\|$. Expanding it we get

$$\|x\|^2 - \langle x, y\rangle = \langle x, x - y\rangle > 0$$

which by Lemma 2.5 implies $\Gamma_y([C])(x) = 0$. □

We are ready to prove our main theorem.

Theorem 2.7. *Let $y \in V$. Γ_y extends to a valuation on $\mathcal{PC}(V)$ with values in $\mathcal{P}_{bd}(V)$.*

Proof. Corollary 2.6 implies that $\Gamma_y([C]) \in \mathcal{P}_{bd}(V)$ for all cones C.

To show that Γ_y is a valuation, by Theorem 1.1 it suffices to show that for all cones C and all hyperplanes H and the corresponding halfspaces H^+ and H^- we have

$$\Gamma_y([C]) = \Gamma_y([C \cap H^+]) + \Gamma_y([C \cap H^-]) - \Gamma_y([C \cap H]).$$

Fix a hyperplane H. For all $F \in \mathcal{F}(C)$, set

$$F^+ = F \cap H^+, \quad F^- = F \cap H^-, \quad F_H = F \cap H.$$

We partition $\mathcal{F}(C)$ into the following four disjoint subsets.

(1) $\mathcal{F}_1(C) = \{F \in \mathcal{F}(C) \mid F \cap \operatorname{relint} H^+ \neq \varnothing,\ F \cap \operatorname{relint} H^- \neq \varnothing\}$. Note then that $\mathcal{F}_1(C)$ can be defined alternatively as

$$\mathcal{F}_1(C) = \{F \in \mathcal{F}(C) \mid F^+ \neq F_H,\ F^- \neq F_H\}.$$

In particular, if $F \in \mathcal{F}_1(C)$, we have $\dim F = \dim F^+ = \dim F^- = 1 + \dim F_H$.

(2) $\mathcal{F}_2(C) = \{F \in \mathcal{F}(C) \setminus \mathcal{F}_1(C) \mid F \cap \operatorname{relint} H^+ \neq \varnothing\}$. Note then that $\mathcal{F}_2(C)$ can be defined alternatively as

$$\mathcal{F}_2(C) = \{F \in \mathcal{F}(C) \mid F^+ \neq F_H,\ F^+ = F\} = \{F \in \mathcal{F}(C) \mid F^+ \neq F_H,\ F^- = F_H\}.$$

(3) $\mathcal{F}_3(C) = \{F \in \mathcal{F}(C) \setminus \mathcal{F}_1(C) \mid F \cap \operatorname{relint} H^- \neq \varnothing\}$. Note then that $\mathcal{F}_3(C)$ can be defined alternatively as

$$\mathcal{F}_3(C)=\{F\in\mathcal{F}(C) \mid F^-\neq F_H,\ F^-=F\}=\{F\in\mathcal{F}(C) \mid F^-\neq F_H,\ F^+=F_H\}.$$

(4) $\mathcal{F}_4(C) = \{F \in \mathcal{F}(C) \mid F = F_H\}$.

We identify C_H with a face in the cones C^+ and C^-. We have then the following disjoint union decompositions ($\sqcup$ denoting disjoint union).

- $\mathcal{F}(C^+) = \bigsqcup_{F\in\mathcal{F}_1(C)}\{F^+\} \sqcup \bigsqcup_{F\in\mathcal{F}_2(C)}\{F^+\} \sqcup \mathcal{F}(C_H)$.
- $\mathcal{F}(C^-) = \bigsqcup_{F\in\mathcal{F}_1(C)}\{F^-\} \sqcup \bigsqcup_{F\in\mathcal{F}_3(C)}\{F^-\} \sqcup \mathcal{F}(C_H)$.
- $\mathcal{F}(C_H) = \bigsqcup_{F\in\mathcal{F}_1(C)}\{F_H\} \sqcup \bigsqcup_{F\in\mathcal{F}_4(C)}\{F\}$.

Let $F \in \mathcal{F}_1(C)$. By Lemma 1.2 (5) and the fact that $[F] = [F^+]+[F^-]-[F_H]$, we have

$$[F^*] = [(F^+)^*] + [(F^-)^*] - [F_H^*].$$

This shows that $[A(F,C)](x)[F^*](x-y)$ equals

$$[A(F,C)](x)[(F^+)^*](x-y)+[A(F,C)](x)[(F^-)^*](x-y)-[A(F,C)](x)[F_H^*](x-y).$$

Moreover, $[A(F,C)] = [A(F^+,C^+)] = [A(F^-,C^-)]$, and it is not hard to see that

$$A(F,C) \cap H^+ = A(F_H,C^+),$$
$$A(F,C) \cap H^- = A(F_H,C^-),$$
$$A(F,C) \cap H = A(F_H,C_H).$$

Hence

$$[A(F,C)] = [A(F_H,C^+)] + [A(F_H,C^-)] - [A(F_H,C_H)].$$

On the other hand, if $F \in \mathcal{F}_2(C)$, we clearly have

$$[A(F,C)](x)[F^*](x-y) = [A(F^+,C^+)](x)[(F^+)^*](x-y),$$

and if $F \in \mathcal{F}_3(C)$, we have

$$[A(F,C)](x)[F^*](x-y) = [A(F^-,C^-)](x)[(F^-)^*](x-y).$$

These properties show that the difference of

$$\Gamma_y([C])(x)$$

with

$$\Gamma_y([C^+])(x) + \Gamma_y([C^-])(x) - \Gamma_y([C_H])(x)$$

equals the sum over $F \in \mathcal{F}_4(C)$ of $\varepsilon_F[F^*](x-y)$ times

$$[A(F,C)] - [A(F,C^+)] - [A(F,C^-)] + [A(F,C_H)] \tag{2.1}$$

evaluated at x. It remains to observe that for $F \in \mathcal{F}_4(C)$ we have

$$A(F,C) \cap H^+ = A(F,C^+),$$
$$A(F,C) \cap H^- = A(F,C^-),$$
$$A(F,C) \cap H = A(F,C_H)$$

which shows that (2.1) is zero in this case. □

2.3. A variation: from $[C]$ to $[\operatorname{relint} C]$. Besides the indicator function on cones, sometimes it is more convenient to work with indicator functions on relatively open cones. As mentioned in the introduction, the motivation for the valuation Γ_y comes from the construction of Arthur [Art81]. Generalizing the definition of Γ introduced by Arthur faithfully, we should have rather studied the sum

$$\sum_F \varepsilon_F[\operatorname{relint} A(F,C)](x)[\operatorname{relint} F^*](x-y), \quad \text{for } x,y \in V.$$

There is a close relationship of this expression with the valuation Γ_y. To define a variant of the valuation Γ_y for open cones, we need the following lemma, which follows easily from Theorem 1.1.

Lemma 2.8. *Let $\phi : \mathcal{PC}(V) \to W$ be a valuation, where W is a vector space. Then, $\phi' : \mathcal{PC}(V) \to W$ defined as $\phi'([C]) := \varepsilon_C \phi([\operatorname{relint} C])$ is a valuation.*

By the above lemma, the map

$$\Gamma'_y : \mathcal{PC}(V) \to \mathcal{P}_{bd}(V)$$
$$[C] \mapsto \varepsilon_C \Gamma_y([\operatorname{relint} C])$$

is a valuation. The following can be regarded as a reciprocity formula for Γ'_y.

Proposition 2.9. *Let $x, y \in V$. Then*

$$\Gamma'_y([-\operatorname{relint} C])(x) = \sum_{F \in \mathcal{F}(C)} \varepsilon_F^{F_0(C)}[\operatorname{relint} A(F,C)](x)[\operatorname{relint} F^*](x-y).$$

Proof. Since $[C] = \sum_{F\in\mathcal{F}(C)}[\operatorname{relint} F]$ and $[\operatorname{relint} C] = \sum_{F\in\mathcal{F}(C)} \varepsilon_C^F[F]$, using the fact that Γ'_y is a valuation we obtain

$$\begin{aligned}\Gamma'_y([-\operatorname{relint} C]) &= \sum_F \varepsilon_C^F \Gamma'_y([-F]) = \varepsilon_C \sum_F \Gamma_y([-\operatorname{relint} F]) \\ &= \varepsilon_C \Gamma_y([-C]) = \sum_F \varepsilon_C^F[-A(F,C)][-F^*](\cdot - y) \\ &\overset{(1)}{=} \sum_F \varepsilon_C^F \Big(\sum_{G \supset F} \varepsilon_C^G[\operatorname{relint} A(G,C)]\Big)\Big(\sum_{E \subset F} \varepsilon_E^{F_0}[\operatorname{relint}(E^*)](\cdot - y)\Big)\end{aligned}$$

$$= \sum_{E\subset G} \varepsilon_C^G \varepsilon_E^{F_0} [\text{relint } A(G,C)][\text{relint } E^*](\,\cdot\, - y) \sum_{E\subset F\subset G} \varepsilon_C^F$$

$$\overset{(2)}{=} \sum_{F} \varepsilon_F^{F_0} [\text{relint } A(F,C)][\text{relint } F^*](\,\cdot\, - y).$$

Here equation (1) follows from the third and fourth identities in Theorem 1.3 as well as Lemma 1.2 (3)–(4). Equation (2) follows from the fact that $\sum_{E\subset F\subset G} \varepsilon_C^F$ is zero unless $E = G = F$ by the Euler characteristic formula (1.1). □

We end by discussing the functions $\Gamma_0([C])$ and $\Gamma'_0([C])$, i.e., when $y = 0$. In what follows, if we let $V_C^\perp$ be the orthogonal complement in V of the linear span V_C of a cone C, then for $x \in V$ we denote by x_C and x^C its respective orthogonal projections to V_C and $V_C^\perp$.

Proposition 2.10. *We have the following identities for all $x \in V$.*

(1)

$$\sum_{F\in\mathcal{F}(C)} \varepsilon_C^F [A(F,C)](x)[F^*](x) = \begin{cases} 1 & \textit{if } C \textit{ is a subspace of } V \textit{ and } x = 0, \\ 0 & \textit{else.} \end{cases}$$

(2)

$$\sum_{F\in\mathcal{F}(C)} \varepsilon_C^F [\text{relint } A(F,C)](x)[\text{relint } F^*](x) = \begin{cases} 1 & \textit{if } C \textit{ is a subspace of } V \textit{ and } x=0, \\ 0 & \textit{else.} \end{cases}$$

(3)

$$\sum_{F\in\mathcal{F}(C)} \varepsilon_C^F [A(F,C)^*](x^F)[F](x_F) = \begin{cases} 1 & \textit{if } C \textit{ is a subspace of } V, \\ 0 & \textit{else.} \end{cases}$$

(4)

$$\sum_{F\in\mathcal{F}(C)} \varepsilon_C^F [\text{relint } A(F,C)^*](x^F)[\text{relint } F](x_F) = \begin{cases} 1 & \textit{if } C \textit{ is a subspace of } V, \\ 0 & \textit{else.} \end{cases}$$

Proof. Corollary 2.6, with $y = 0$, immediately yields the first two identities given the Euler relation (1.1). Applying to the first identity the valuation that takes characteristic functions of cones to characteristic functions of their duals yields then the third identity. The last one is finally an easy consequence of the third one.

□

Remark 2.11. The above proposition, or an essential part thereof, was proved by Schneider [Sch17, Theorem 1.1] (later proved for general polyhedra in [HK18]). We provide thus an alternative proof of Schneider's result.

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