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NONLINEAR EVOLUTION EQUATIONS AND THEIR STATIONARY REDUCTIONS

1. Introduction

In [5] Bogoyavlenskiĭ and Novikov pointed out that each stationary flow of the KdV hierarchy (of order $2m+1$) is Lagrangian, so can be written as an m-degrees of freedom Hamiltonian system by using a generalised Legendre transformation. They proved that this Hamiltonian system is completely integrable. In [4] it was shown that these stationary flows are bi-Hamiltonian, by exploiting the Miura map between the KdV and MKdV hierarchies, which induces a diffeomorphism between the phase spaces of the stationary flows. This was unsatisfactory, in that it gave no clear relationship between the two Hamiltonian structures of the PDE and those of the stationary flow. This was remedied in [1] which used the Miura map after first 'reversing the roles of x and t'. These $x-t$ role reversals had previously been considered in a different context in [4, 5]. In the current paper we give a systematic construction of these Poisson brackets, starting from the related spectral problem. This answers a number of the questions raised in [4].

In this paper we present our ideas in the context of the KdV hierarchy, so Sec. 2 contains all the relevant background formulae. Sec. 3 gives some background material on stationary flows and $x-t$ reversed equations. We introduce the spectral problem for the $x-t$ reversed KdV hierarchy in Sec. 4 and systematically derive the Hamiltonian structures and isospectral flows for the simplest (KdV) example. This is a general framework which applies even when the Lagrangian is degenerate and the Legendre transformation is invalid, since we do not rely on the use of canonical co-ordinates. A more complete treatment and further examples can be found in [3].

2. The KdV and ZS Hierarchies

The KdV hierarchy can be represented in zero curvature form as follows:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ \frac{1}{4}\lambda - u & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv U\Psi,$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{t_m}=\begin{pmatrix}-\frac{1}{2}\alpha_{(m)x} & \alpha_{(m)}\\ -\frac{1}{2}\alpha_{(m)xx}-\left(u-\frac{1}{4}\lambda\right)\alpha_{(m)} & \frac{1}{2}\alpha_{(m)x}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\equiv V_{(m)}\Psi, \tag{1}$$

which leads to:

$$u_{t_m}=(\mathbf{B}_1-\lambda\mathbf{B}_0)\,\alpha_{(m)}\equiv\left(\frac{1}{2}\partial^3+u\partial+\partial u-\frac{1}{2}\lambda\partial\right)\alpha_{(m)}, \tag{2}$$

where $\partial=\frac{\partial}{\partial x}$. The isospectral flow (iso) is given in bi-Hamiltonian form. The first three flows correspond to:

$$\begin{aligned}
V_{(0)}&=\begin{pmatrix}0 & 1\\ \frac{1}{4}\lambda-u & 0\end{pmatrix},\\
V_{(1)}&=\begin{pmatrix}-u_x & \lambda+2u\\ \frac{1}{4}\lambda^2-\frac{1}{2}\lambda u-u_{xx}-2u^2 & u_x\end{pmatrix},\\
V_{(2)}&=\begin{pmatrix}-\lambda u_x-u_{xxx}-6uu_x & \lambda^2+2\lambda u+2u_{xx}+6u^2\\ C_{(2)} & \lambda u_x+u_{xxx}+6uu_x\end{pmatrix},
\end{aligned} \tag{3}$$

where

$$C_{(2)}=\frac{1}{4}\lambda^3-\frac{1}{2}\lambda^2u-\frac{1}{2}\left(u_{xx}+u^2\right)\lambda-u_{xxxx}-8uu_{xx}-6u_x^2-6u^3,$$

giving:

$$\begin{aligned}
u_{t_0}&=u_x,\\
u_{t_1}&=u_{xxx}+6uu_x,\\
u_{t_2}&=u_{xxxxx}+10uu_{xxx}+20u_xu_{xx}+30u^2u_x.
\end{aligned} \tag{4}$$

2.1 The Miura map.

The Miura map and its generalisations will be important in what follows. The original Miura map (with parameter) is:

$$u=M[v]\equiv-v_x-v^2+\frac{1}{4}\lambda, \tag{5}$$

giving

$$M'(-\frac{1}{2}\partial)(M')^\dagger=\frac{1}{2}\partial^3+2u\partial+u_x-\frac{1}{2}\lambda\partial\equiv\mathbf{B}_1-\lambda\mathbf{B}_0. \tag{6}$$

Each Hamiltonian functional $\mathcal{H}[u]$, is pulled back to a functional (an equivalence class of functionals) of v:

$$\tilde{\mathcal{H}}[v]=\mathcal{H}[M[v]], \tag{7}$$

and we have the mapping of Hamiltonian vector fields:

$$v_t = -\frac{1}{2}\partial\delta_v\tilde{\mathcal{H}} \mapsto u_t = (\mathbf{B}_1 - \lambda\mathbf{B}_0)\delta_u\mathcal{H}. \tag{8}$$

The Miura map can be used to build an infinite family of commuting Hamiltonians. We introduce the asymptotic expansion:

$$v = -\frac{1}{2}k + uk^{-1} + \sum_{i=2}^{\infty} v_i k^{-i}, \quad \lambda = k^2, \tag{9}$$

which starts as:

$$\begin{gathered} v_2 = u_x, \quad v_3 = u^2 + u_{xx}, \\ v_4 = \left(2u^2 + u_{xx}\right)_x, \quad v_5 = 2u^3 - u_x^2 + \left(u_{xxx} + 6uu_x\right)_x, \end{gathered} \tag{10}$$

and satisfies (for $i \geq 1$):

$$v_{i+1} = v_{ix} + \sum_{j=0}^{i} v_j v_{i-j}, \tag{11}$$

so, at each level, v_{i+1} can be solved in terms of a *local* functional of $v_1, \cdots, v_i$. It is easy to prove that the even coefficients v_{2i} are exact derivatives, so that:

$$\delta_u v = k^{-1}\left(1 + 2u\lambda^{-1} + \left(2u_{xx} + 6u^2\right)\lambda^{-2} + \ldots\right). \tag{12}$$

Since:

$$(\mathbf{B}_1 - \lambda\mathbf{B}_0)\,\delta_u v = -\frac{1}{2}M'\partial\delta_v v = 0, \tag{13}$$

we may define $\alpha = k\delta_u v$, and the m^{th} flow by:

$$u_{t_m} = (\mathbf{B}_1 - \lambda\mathbf{B}_0)\,\alpha_{(m)}, \quad \alpha_{(m)} = (\lambda^m\alpha)_+ . \tag{14}$$

This is the image of the vector field:

$$v_{t_m} = -\frac{1}{2}\partial\,\beta_{(m)}, \quad \beta_{(m)} = (M')^\dagger\alpha_{(m)}\,|_{u=M[v]} . \tag{15}$$

The first two flows and their 'modifications' are:

$$\begin{gathered} \alpha_{(0)} = 1, \quad \beta_{(0)} = -2v \Rightarrow \quad u_{t_0} = u_x, \quad v_{t_0} = v_x, \\ \left.\begin{aligned} \alpha_{(1)} &= \lambda + 2u, \\ \beta_{(1)} &= -3\lambda v + 4v^3 - 2v_{xx} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} u_{t_1} &= u_{xxx} + 6uu_x, \\ v_{t_1} &= v_{xxx} - 6v^2v_x + \tfrac{3}{2}\lambda v_x. \end{aligned}\right. \end{gathered} \tag{16}$$

For each flow (mkdv-m), $-\frac{1}{2}\beta_{(m)}$ is an expression in v and its $x-$derivatives. By substituting (9) into (15) and equating coefficients of k, we obtain an infinite sequence of conservation laws, obeyed by each member of the hierarchy:

$$\mathcal{H}_{nt_m} = \mathcal{F}_{nm\,x}. \tag{17}$$

The *fluxes* $\mathcal{F}_{nm}$ play the role of conserved densities when we consider the $x - t_m$ reversal in Sec. 4.

2.2. Zero curvature : 2 × 2 matrices.

Consider the pair of matrix equations:

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \tag{18}$$

with

$$U = \begin{pmatrix} w & q \\ r & -w \end{pmatrix}, \quad V = \begin{pmatrix} \gamma & \alpha \\ \beta & -\gamma \end{pmatrix}, \tag{19}$$

where q, r, w are potential functions (dependent on λ in some specified way) and V is to be determined algorithmically for some classes of $\lambda-$dependence. The integrability conditions:

$$U_t = (\partial - adU)V \tag{20}$$

are written

$$\sum_i u_{it}\,\mathbf{e}_i = \left(\sum_{i,j} B_{ij} e_i \otimes e_j\right)\left(\sum_k v_k^* e_k^*\right) = \sum_{i,j}\left(B_{ij} v_j^*\right) e_i, \tag{21}$$

with e_i^* the dual basis of e_i , so that $B_{ij} e_i \otimes e_j$ is a $(2,0)$ tensor. In components,

$$u_{it} = -B_{ij} v_j^* \ , \quad v_i^* = g_{ij} v_j \ , \quad g_{ij} = tr(e_i e_j). \tag{22}$$

The simplest way of calculating this is to check the trace form $tr(U_t V)$ since:

$$tr(U_t V) = \mathbf{u}_t \cdot \mathbf{v}^*. \tag{23}$$

For our example,

$$\mathbf{u} = (q, r, w)^T,\ \mathbf{v} = (\alpha, \beta, \gamma)^T,\ \mathbf{v}^* = (\beta, \alpha, 2\gamma)^T = -\delta h \tag{24}$$

and

$$B_{ij} = \begin{pmatrix} 0 & -\partial + 2w & -q \\ -\partial - 2w & 0 & r \\ q & -r & -\frac{1}{2} \end{pmatrix}. \tag{25}$$

Example 2.1. The linear pencil : $w = \lambda + w_0$, $q = q_0$, $r = r_0$.

In this case we immediately have the bi-Hamiltonian formulation:

$$u_t = (\mathbf{B}_1 - \lambda \mathbf{B}_0)\delta h, \tag{26}$$

written explicitly as:

$$\begin{pmatrix} q_0 \\ r_0 \\ w_0 \end{pmatrix}_t = \\ = \left[\begin{pmatrix} 0 & -\partial + 2w_0 & -q_0 \\ -\partial - 2w_0 & 0 & r_0 \\ q_0 & -r_0 & -\frac{1}{2} \end{pmatrix} - \lambda \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \beta \\ \alpha \\ 2\gamma \end{pmatrix}. \tag{27}$$

3. Stationary Flows : Canonical Co-ordinates

In this section we briefly review what is known about stationary flows of the KdV hierarchy. Bogoyavlenskii and Novikov [5] used a generalised Legendre transformation to write the stationary flows in Hamiltonian form and then proved complete integrability. In [4] the Miura map was used to construct a second Hamiltonian structure and in [1] the equations were viewed with x and t reversing their roles.

3.1. Generalised Lagrangians.

We consider a (generalised) Lagrangian:

$$\mathcal{L}(q^{(0)}, q^{(i)} \cdots, q^{(n)}), \quad q^{(i)} = \frac{d^i q}{dx^i}, \; n \neq 1, \tag{28}$$

where $\mathcal{L}$ is non-degenerate ($\frac{\partial^2 \mathcal{L}}{\partial q^{(n)2}} \neq 0$). The corresponding Euler-Lagrange equations are:

$$\sum_{i=0}^{n} (-\partial)^i \frac{\partial \mathcal{L}}{\partial q^{(i)}} = 0. \tag{29}$$

Canonical co-ordinates can be defined as:

$$q_i = q^{(i-1)}, \quad i = 1, \cdots, n, \tag{30}$$

$$p_n = \frac{\partial \mathcal{L}}{\partial q^{(n)}}, \tag{31}$$

$$p_i = \frac{\partial \mathcal{L}}{\partial q^{(i)}} - \dot{p}_{i+1}, \quad i = 1, \cdots, n-1. \tag{32}$$

The Euler-Lagrange equations (29) then take canonical Hamiltonian form with:

$$h = \sum_{i=1}^{n-1} q_{i+1}p_i + q_{nx}p_n - \mathcal{L}(q_1, \cdots, q_n, q_{nx}), \tag{33}$$

where q_{nx} is removed by inverting the non-degenerate Legendre transformation [2].

3.2. The KdV hierarchy.

The m^{th} stationary flow in the KdV hierarchy is defined by

$$u_{t_m} = \frac{1}{2}\partial\delta_u \mathcal{H}_{m+1} = 0,\ m \geq 1, \tag{34}$$

which is in the form of a generalised Lagrangian equation:

$$\delta_u \mathcal{L}_{m+1} = 0, \quad \mathcal{L}_{m+1} = \mathcal{H}_{m+1} - c_m u. \tag{35}$$

It is possible to write these equations as an m-degrees of freedom canonical Hamiltonian system. If we take c_m as a dynamical variable then the Poisson bracket is degenerate, with c_m as Casimir:

$$\{f, g\} = \frac{\partial f}{\partial x_i} J_{ij} \frac{\partial g}{\partial x_j} = (\nabla f)^T J(\nabla g), \quad J = \begin{pmatrix} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

where I_m is the $m \times m$ unit matrix. Bogoyavlenskii and Novikov [5] showed that this Hamiltonian system is completely integrable with first integrals given by the fluxes of the Hamiltonians $\mathcal{H}_1, \cdots, \mathcal{H}_m$:

$$\begin{array}{ccccl} 0 & = & \mathcal{H}_{0t_m} & = & \mathcal{F}_{0mx} \quad \text{the equation}, \\ 0 & = & \mathcal{H}_{kt_m} & = & \mathcal{F}_{kmx} \quad k = 1, \cdots, m. \end{array} \tag{37}$$

The equation $\mathcal{F}_{0m} = c_m$ is an ODE of order $2m$. This is used to eliminate higher derivatives from the fluxes $\mathcal{F}_{km}$, which can then be written as functions of the variables (q_i, p_i, c_m): $\mathcal{F}_{km} = h_k(q_i, p_i, c_m)$, $k = 1, \cdots, m$. The later fluxes become functionally dependent upon $h_1, \cdots, h_m$. Since $\{\mathcal{H}_i, \mathcal{H}_m\} = 0$ for all elements of the KdV hierarchy, the stationary manifold is invariant under the action of the flows. The first component of each of the commuting Hamiltonian flows of the stationary flow are precisely

$$u_{t_k} = \frac{1}{2}\partial\, \delta\, \mathcal{H}_{k+1},\ k = 1, \cdots, m-1, \tag{38}$$

when written in terms of the stationary manifold co-ordinates, since $q_1 = u$.

The Miura map between the KdV and MKdV equations induces a diffeomorphism between corresponding stationary manifolds. The fluxes/Hamiltonians for the stationary flows are calculated directly from:

$$v_{t_m} = -\frac{1}{2}\tilde{c}_{mx} \tag{39}$$

with $\tilde{c}_m$ given as a differential polynomial in v. Thus, substituting the expansion for v into $\tilde{c}_m$, we directly construct the fluxes as coefficients in the k-expansion of $\tilde{c}_m$. Furthermore, it is possible to use the Miura map to construct the second Hamiltonian structure for the stationary flows.

Example 3.1. (The stationary KdV and MKdV equations)

The stationary KdV equation (which defines c_1) is:

$$2u_{xx} + 6u^2 = c_1, \tag{40}$$

with Lagrangian $\mathcal{L}$ given by $\mathcal{L} = 2u^3 - u_x^2 - c_1 u$ and canonical coordinates:

$$q = u, \quad p = -2u_x, \quad c_1, \tag{41}$$

c_1 being the Casimir of the degenerate canonical bracket. The Legendre transformation gives

$$h_{\mathcal{L}} = c_1 q - 2q^3 - \frac{1}{4}p^2, \tag{42}$$

which is the flux of $\mathcal{H}_1 = u^2$.

The Miura map induces the diffeomorphism M given by:

$$\begin{aligned} q &= \frac{1}{4}\lambda - \frac{1}{2}\tilde{p} - \tilde{q}^2, \\ p &= -\tilde{c}_1 + 4\tilde{q}^3 + 2\tilde{q}\tilde{p} - 3\lambda\tilde{q}, \\ c_1 &= \frac{1}{2}\tilde{p}^2 - 2\tilde{q}^4 + 2\tilde{q}\tilde{c}_1 + 3\lambda\tilde{q}^2 + \frac{3}{8}\lambda^2, \end{aligned} \tag{43}$$

relating (40) to the stationary MKdV equation:

$$-2v_{xx} + 4v^3 - 3\lambda v = \tilde{c}_1, \tag{44}$$

which has Lagrangian $\mathcal{L} = v_x^2 + v^4 - \tilde{c}_1 v - \frac{3}{2}\lambda v^2$ and canonical variables:

$$\tilde{q} = v, \quad \tilde{p} = 2v_x, \quad \tilde{c}_1. \tag{45}$$

The canonical Poisson bracket between the modified variables:

$$\{\tilde{q}, \tilde{p}\} = 1, \quad \{\tilde{q}, \tilde{c}_1\} = \{\tilde{p}, \tilde{c}_1\} = 0 \tag{46}$$

induces:

$$\{q,p\} = -2q - \lambda, \quad \{q,c_1\} = -p, \quad \{p,c_1\} = -2c_1 + 12q^2. \qquad (47)$$

This is a non-canonical Poisson bracket for the stationary KdV equation. In fact it is a one parameter family of Poisson brackets: the coefficients of λ is just the canonical bracket, while the remaining part is the *second* bracket. This proves not only that each coefficient is independently a Poisson bracket, but also that they are compatible.

The variable c_1 is a Casimir for the degenerate canonical bracket with its level surfaces being symplectic leaves. The symplectic leaves of second Poisson bracket are different, being the level surfaces of $h_{\mathcal{L}}$, while c_1 now generates the stationary KdV equation.

Example 3.2. The stationary fifth order KdV equation

Here:

$$u_{t_2} = \frac{1}{2}\partial\delta\mathcal{H}_3 = 0, \quad \mathcal{H}_3 = u_{xx}^2 - 10uu_x^2 + 5u^4. \qquad (48)$$

The Lagrangian $\mathcal{L}_3 = u_{xx}^2 - 10uu_x^2 + 5u^4 - c_2u$ gives canonical variables:

$$\begin{aligned} &q_1 = u, \quad q_2 = u_x, \quad p_2 = 2u_{xx}, \quad p_1 = -2u_{xxx} - 20uu_x, \\ &c_2 = 2u_{xxxx} + 20uu_{xx} + 10u_x^2 + 20u^3, \end{aligned} \qquad (49)$$

with c_2 being a Casimir function of the degenerate canonical bracket. The Legendre transformation gives rise to the Hamiltonian:

$$h_{\mathcal{L}} = \frac{1}{4}p_2^2 + q_2p_1 + 10q_1q_2^2 - 5q_1^4 + q_1c_2, \qquad (50)$$

which is just the flux of $\mathcal{H}_1 = u^2$ when written in terms of (49). The commuting Hamiltonian is the flux of $\mathcal{H}_2 = 2u^3 - u_x^2$:

$$h_2 = -\frac{1}{4}p_1^2 - q_1p_2^2 + (q_2^2 - 10q_1^3)p_2 - 4q_1q_2p_1 - 10q_1^2q_2^2 - 24q_1^5 + (\frac{1}{2}p_2 + 3q_1^2)c_2. \qquad (51)$$

The corresponding flow is:

$$q_{1s} = -\frac{1}{2}p_1 - 4q_1q_2, \qquad (52)$$

$$q_{2s} = -2q_1p_2 + q_2^2 - 10q_1^3 + \frac{1}{2}c_2, \qquad (53)$$

$$p_{1s} = p_2^2 + 30q_1^2p_2 + 4q_2p_1 + 20q_1q_2^2 + 120q_1^4 - 6c_2q_1, \qquad (54)$$

$$p_{2s} = -2q_2p_2 + 4q_1p_1 + 20q_1^2q_2. \qquad (55)$$

Using the definition (49), the q_1 $(= u)$ component is just the KdV equation:

$$u_s = u_{xxx} + 6uu_x. \tag{56}$$

An alternative route is to treat the KdV equation as a Noether symmetry of Lagrangian $\mathcal{L}_2$ with h_2 being the corresponding constant of motion. The Miura map induces a diffeomorphism between the 5 dimensional spaces of the stationary 5th order KdV and MKdV equations, with which it is possible to construct a second Hamiltonian structure for this hierarchy [4]. This will be constructed in another way in the next section.

3.3. Reversing the Role of x and t.

Each member of the KdV hierarchy is bi-Hamiltonian, in both the PDE and stationary flow cases. However, the latter case required a Legendre transformation and a Miura map, so showed no clear relationship between the Hamiltonian structures of the PDE and those of the stationary flow. This deficit was overcome in [1] where they took t as the 'spatial' variable and x as the evolution parameter. In so doing, the KdV equation becomes a system of three equations in x, with a 3×3 matrix Poisson bracket, *which reduces to the stationary case when the potential functions are independent of t*

With the same canonical variables as before, the KdV equation can be written *as a PDE* in the form:

$$\begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2}p \\ 6q^2 - c_1 \\ 2q_t \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2\partial_t \end{pmatrix} \begin{pmatrix} c_1 - 6q^2 \\ -\frac{1}{2}p \\ q \end{pmatrix}, \tag{57}$$

where this last vector is just $\delta_q(c_1 q - 2q^3 - \frac{1}{4}p^2)$. Similarly the MKdV equation (see (16)) can be written:

$$\begin{pmatrix} \tilde{q} \\ \tilde{p} \\ \tilde{c}_1 \end{pmatrix}_x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2\partial_t \end{pmatrix} \begin{pmatrix} \tilde{c}_1 - 4\tilde{q}^3 + 3\lambda\tilde{q} \\ \frac{1}{2}\tilde{p} \\ \tilde{q} \end{pmatrix} = \tilde{\mathbf{B}}\delta_{\tilde{q}}\tilde{h}, \tag{58}$$

where $\tilde{h} = \tilde{q}\tilde{c}_1 - \tilde{q}^4 + \frac{1}{4}\tilde{p}^2 + \frac{3}{2}\lambda\tilde{q}^2$.

The Miura map (miura1) is only changed by $-2\tilde{q}_t$ in the definition of c_1. The above bracket for the MKdV equation is then transformed to:

$$\begin{pmatrix} 0 & -2q & \partial_t - p \\ 2q & -2\partial_t & 12q^2 - 2c_1 \\ \partial_t + p & 2c_1 - 12q^2 & 4(q\partial_t + \partial_t q) \end{pmatrix} - \lambda \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2\partial_t \end{pmatrix}. \tag{59}$$

These define Poisson brackets for the *full* KdV equation written in terms of the stationary manifold co-ordinates and clearly reduce to those of the stationary equations when the potential functions are independent of t.

These $x - t$ reversal flows and Poisson brackets have also been considered in [2, 4].

4. The Spectral Problem

Whilst the above argument gives the Poisson brackets of the stationary equations as restrictions of those of the PDE, we still need to fit stationary flows onto our general scheme of relating Hamiltonian structures to spectral problems. In this paper we present this in the context of the KdV hierarchy, referring to [3] for further details and examples.

In order to consider the t_m–stationary flow we just reverse the roles of U (which equals $V_{(0)}$) and $V_{(m)}$, which is now playing the role of the spectral problem:

$$V_{(m)x} = U_\xi - [V_{(m)}, U], \tag{60}$$

where $\xi = t_m$. It should be noted that $V_{(m)}$ depends upon $u, u_x, \dots, u_{x\cdots x}$, where the last term has $2m$ derivatives, so can be written in terms of any set of co-ordinates on the stationary manifold. For comparison of results we choose the canonical co-ordinates.

4.1. The KdV equation.

The spectral problem is

$$\Psi_\xi = \mathcal{U}\Psi, \quad \text{where} \quad \mathcal{U} \equiv V_{(1)} = \begin{pmatrix} \frac{1}{2}p & \lambda + 2q \\ \frac{1}{4}\lambda^2 - \frac{1}{2}\lambda q + q^2 - \frac{1}{2}c_1 & -\frac{1}{2}p \end{pmatrix}. \tag{61}$$

We are interested in the hierarchy rather than just the KdV flow, so consider the time evolution:

$$\Psi_\tau = \mathcal{V}\Psi \equiv \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{62}$$

where A, B and C are functions of q, p, c_1, their ξ–derivatives and λ.

In terms of $\mathcal{U}$ and $\mathcal{V}$ the integrability conditions are just:

$$\mathcal{U}_\tau = \mathcal{V}_\xi - [\mathcal{U}, \mathcal{V}]. \tag{63}$$

We are interested in the equation for $\mathbf{q} = (q, p, c_1)^T$ and have:

$$\mathcal{U}_\tau = \frac{D\mathcal{U}}{D\mathbf{q}} \cdot \mathbf{q}_\tau, \tag{64}$$

where $\frac{D\mathcal{U}}{D\mathbf{q}}$ is the Fréchet derivative. If we represent $\mathcal{U} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ by the vector $\mathbf{b} = (b, c, a)^T$, and $\mathcal{J}_\varphi$ is the Fréchet derivative of the map $\varphi : \mathbf{q} \mapsto \mathbf{b}$, then

$$\mathbf{b}_\tau = \mathcal{J}_\varphi \, \mathbf{q}_\tau , \text{ where } \mathcal{J}_\varphi = \begin{pmatrix} 2 & 0 & 0 \\ 2q - \frac{1}{2}\lambda & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, \tag{65}$$

whilst from (25) we have:

$$\mathbf{b}_{\tau_{\hat{h}}} \equiv \begin{pmatrix} b \\ c \\ a \end{pmatrix}_{\tau_{\hat{h}}} = \begin{pmatrix} 0 & -\partial + 2a & -b \\ -\partial - 2a & 0 & c \\ b & -c & -\frac{1}{2}\partial \end{pmatrix} \begin{pmatrix} C \\ B \\ 2A \end{pmatrix} \equiv \tilde{\mathbf{B}} \delta_b \hat{h}, \tag{66}$$

where $\partial \equiv \frac{\partial}{\partial \xi}$ and $(C, B, 2A) = -\left(\frac{\delta \hat{h}}{\delta b}, \frac{\delta \hat{h}}{\delta c}, \frac{\delta \hat{h}}{\delta a}\right)$. This is a general formula for *any* time evolution. In our case (a, b, c), and therefore (A, B, C), depend upon λ. The infinite hierarchies of Hamiltonians are built out of 'universal' Hamiltonians ((9) for the original KdV hierarchy). We will call these $h(q, p, c_1)$ and $\tilde{h}(b, c, a)$ and they are related by $h(q, p, c_1) = \tilde{h} \circ \varphi(\mathbf{q})$. The corresponding gradients are related by:

$$\delta_q h = \mathcal{J}_\varphi^\dagger \cdot \delta_b \tilde{h} \quad \Rightarrow \quad \begin{pmatrix} \delta_q h \\ \delta_p h \\ \delta_{c_1} h \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{2}\lambda - 2q & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} C \\ B \\ 2A \end{pmatrix}, \tag{67}$$

and the Hamiltonian structures in the $\mathbf{q}$ and $\mathbf{b}$ spaces by:

$$\tilde{\mathbf{B}} = \mathcal{J}_\varphi \mathbf{B} \mathcal{J}_\varphi^\dagger . \tag{68}$$

In the present case it is possible to invert $\mathcal{J}_\varphi$ to obtain:

$$\mathcal{J}_\varphi^{-1} \, \tilde{\mathbf{B}} \, (\mathcal{J}_\varphi^{-1})^\dagger = \mathbf{B}, \tag{69}$$

giving

$$\mathbf{B} = \begin{pmatrix} 0 & -2q - \lambda & -p + \partial \\ 2q + \lambda & -2\partial & 12q^2 - 2c_1 \\ p + \partial & -12q^2 + 2c_1 & (4q - \lambda)\,\partial + \partial\,(4q - \lambda) \end{pmatrix}. \tag{70}$$

As previously mentioned, this is related to the simple Hamiltonian structure of (mkdv-xt) through the Miura map. This is the simplest way of proving that $\mathbf{B}$ is indeed Hamiltonian.

To calculate the hierarchy of Hamiltonians we repeat the argument used with the KdV hierarchy in the usual co-ordinates. Since $\tilde{c}_1$ is a *flux* in the usual MKdV language, it is now a conserved density under τ-evolutions. Furthermore, it is a Casimir function of the Poisson bracket given in (mkdv-xt), so generates the bi-Hamiltonian ladder (59):

$$\tilde{\mathbf{B}}\delta_{\tilde{q}}\tilde{c}_1 = 0 \quad \Rightarrow \quad \mathbf{B}\delta_q\tilde{c}_1 = M'\tilde{\mathbf{B}}(M')^\dagger\delta_q\tilde{c}_1 = M'\tilde{\mathbf{B}}\delta_{\tilde{q}}\tilde{c}_1 = 0, \quad (71)$$

where $\mathbf{B}$ is given by (skdv-pb). The asymptotic series for $\tilde{c}_1$ can be derived from the *known* series (9) by substitution into the formula (c1tilde), replacing x-derivatives of u by q, p, c_1 and their ξ-derivatives. Explicitly, we have:

$$\begin{aligned}
\tilde{q} =& -\frac{1}{2}k + qk^{-1} - \frac{1}{2}pk^{-2} + (\frac{1}{2}c_1 - 2q^2)k^{-3} + (q_\xi + qp)k^{-4} + \cdots \\
\tilde{c}_1 =& -2v_{xx} + 4v^3 - 3\lambda v \\
=& k^3 - c_1k^{-1} - 2q_\xi k^{-2} + (\frac{1}{2}p^2 + 4q^3 - 2c_1q + p_\xi)k^{-3} + \cdots
\end{aligned} \quad (72)$$

The Hamiltonians:

$$h_{-1} = \frac{1}{2}c_1, \quad h_0 = -\frac{1}{4}p^2 + qc_1 - 2q^3, \quad h_1 = \frac{1}{4}c_1^2 + q_\xi p, \quad (73)$$

and second Hamiltonian structure, generate:

$$\begin{aligned}
\begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_{\tau_{-1}} &= \begin{pmatrix} -\frac{1}{2}p \\ 6q^2 - c_1 \\ 2q_\xi \end{pmatrix}, \\
\begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_{\tau_0} &= \begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_\xi, \\
\begin{pmatrix} q \\ p \\ c_1 \end{pmatrix}_{\tau_1} &= \begin{pmatrix} \frac{1}{2}c_{1\xi} - 2qq_\xi - \frac{1}{2}pc_1 \\ -2q_{\xi\xi} - 2qp_\xi + 6c_1q^2 - c_1^2 \\ (4c_1q - 4q^3 - \frac{1}{2}p^2 - p_\xi)_\xi \end{pmatrix}.
\end{aligned} \quad (74)$$

In general we have:

$$\mathbf{q}_{\tau_r} = (\mathbf{B}_1 - \lambda\mathbf{B}_0)\delta_q h_{(r)} = \mathbf{B}_1\delta h_r, \quad h_{(r)} = \lambda^{r+1}h_{-1} + \cdots + h_r. \quad (75)$$

The time evolution Ψ_{τ_r} is given explicitly in terms of the gradients of Hamiltonians:

$$\begin{aligned} A_{(r)} &= -\delta_p h_{(r)}, \quad B_{(r)} = 2\delta_{c_1} h_{(r)}, \\ C_{(r)} &= -\frac{1}{2}\delta_q h_{(r)} + (\frac{1}{2}\lambda - 2q)\delta_{c_1} h_{(r)}. \end{aligned} \tag{76}$$

Thus h_{-1} generates the KdV equation and h_0 the 'translational flow', telling us that $\tau_0 = \xi$. We can use the τ_{-1} flow to write p, c_1 and q_ξ in terms of q and its τ_{-1} derivatives after which:

$$q_{\tau_1} = q_{xxxxx} + 10qq_{xxx} + 20q_x q_{xx} + 30q^2 q_x, \tag{77}$$

where we have written $\tau_{-1} = x$. Thus, the hierarchy is just the KdV hierarchy in disguise and thus inherits all the properties of the usual hierarchy. However, without the τ_{-1} flow, this hierarchy shows no hint of being reducible from the (q, p, c_1) space to the q space.

Remark 4.1. The τ_1 flow of (74) is (up to changes of notation and conventions) just equation (2.9) of [4]. In that paper the authors erroneously labelled the next 'time variable' also as x and said, "... there is no obvious way to identify this nonlinear system with any of the known KdV hierarchy."

If we were just interested in the hierarchy of PDEs, there is no particular advantage in using these (more complicated) coordinates. However, by setting all ξ−derivatives to be zero, the Hamiltonian pair (70) immediately reduces to the known pair (47) of the stationary reduction.

4.2. Higher order flows.

The KdV equation itself is deceptively simple, since the map $\varphi : \mathbf{q} \mapsto \mathbf{b}$ is invertible, so the systems (btau) and (jphi-1) are equivalent. However, when $m \geq 2$, the stationary manifold of the m^{th} member of the KdV hierarchy is of dimension $2m + 1$. Thus, whilst $\mathbf{b}$ is a 3−vector, q is a $(2m + 1)$−vector and $\mathcal{J}_\varphi$ *projects* $\mathbf{q}_\tau$ onto $\mathbf{b}_\tau$: $\mathbf{b}_\tau = \mathcal{J}_\varphi \mathbf{q}_\tau$, since $\mathcal{J}_\varphi$ is a $3 \times (2m + 1)$ matrix. Thus, the simple formula (jphi-1) is no longer possible. Nevertheless, it *is* possible to modify the procedure to find $\mathbf{B}_0$ and $\mathbf{B}_1$ so that:

$$\mathbf{q}_{\tau_r} = (\mathbf{B}_1 - \lambda \mathbf{B}_0)\, \delta_q h_{(r)}, \tag{78}$$

where $h_{(r)} = \sum_{i=0}^{r+m} \lambda^{r+m-i} h_{i-m}$, with h_k being easily calculated from the relevant fluxes. When $m = 2$, we have:

$$\mathbf{B}_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\partial \end{pmatrix}, \tag{79}$$

$$\mathbf{B}_1 = \begin{pmatrix} 0 & -\frac{1}{2} & 4q_1 & 0 & 2q_2 \\ \frac{1}{2} & 0 & -4q_2 & -6q_1 & p_2 \\ -4q_1 & 4q_2 & -2\partial & 60q_1^2 + 2p_2 & \Omega_1 \\ 0 & 6q_1 & -60q_1^2 - 2p_2 & 0 & 2\partial + \Omega_2 \\ -2q_2 & -p_2 & -\Omega_1 & 2 - \Omega_2 & 4(\partial q_1 + q_1 \partial) \end{pmatrix}, \tag{80}$$

where $\Omega_1 = 40q_1^3 - 20q_2^2 - 2c_2$, $\Omega_2 = -40q_1q_2 - 2p_1$ and $\partial = \frac{\partial}{\partial \xi}$, $\xi = t_2$. Once again, the importance of the forms of these Hamiltonian structures is that they immediately reduce to those of the stationary flow [4]. The details can be found in [1].

5. Conclusions

In this paper we have been interested in the relationship between an integrable nonlinear evolution equation (PDE) and its stationary flow. We were particularly interested in the reduction of the infinite dimensional Hamiltonian structures to their finite dimensional counterparts. This reduction is most transparent when the PDE is written as a flow in a larger space whose coordinates are those of the stationary manifold, together with their t−derivatives.

Starting from a zero-curvature representation (reversing the roles of U and V) we gave a systematic construction of the isospectral flows and their Hamiltonian structures. We have adopted the approach presented in [2, 3], which *simultaneously* constructs the isospectral flows, time evolutions of the wave functions, the Hamiltonians and Hamiltonian structures. This close relationship between the zero curvature representation and the Hamiltonian formulation of the equations is perhaps best seen by the formula (76).

Whilst the importance of our results is mainly in the realm of the stationary reductions, we have, in passing, answered a number of the questions raised in [4], where they believed they had found a *new* hierarchy of equations. We have seen that, in fact, this hierarchy is just the KdV hierarchy in disguise.

In this paper we presented only the simplest example of the stationary KdV equation, but the method is general and has been applied to the KdV, DWW, Ito and Boussinesq hierarchies [1].

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REFERENCES

1. M. Antonowicz, M. Blaszak , *On a non-standard Hamiltonian description of NLEE.* S. Carillo, O. Ragnisco, euditors , — Nonlinear Evolution Equations and Dynamical Systems (1990), 152–156 , Berlin , Springer .
2. M. Antonowicz, A. P. Fordy , *Factorisation of energy dependent Schrödinger operators: Miura maps and modified systems.* — Commun. Math. Phys. **124** (1989), 465–86 .
3. M. Antonowicz, A. P. Fordy , *Hamiltonian structures of nonlinear evolution equations.* Fordy. pp. 273–312 .
4. M. Antonowicz, A. P. Fordy, S. Wojciechowski , *Integrable stationary flows : Miura maps and bi-Hamiltonian structures.* — Phys. Letts. A **124** (1987), 143–50 .
5. O. I. Bogoyavlenskii, S. P. Novikov , *The relationship between Hamiltonian formalisms of stationary and nonstationary problems.*

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