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FUNCTIONAL-DIFFERENTIAL EQUATIONS IN HARDY-TYPE CLASSES

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1. Formulation of the problem. Consider the unit disc $\mathcal{U}$ in the complex plane $\mathbb{C}$ and discs $D_k = \{z \in \mathbb{C} : \ |z - a_k| < r_k\} \subset \mathcal{U}$, $(k = 1, 2, \ldots, n)$. Denote by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the extended complex plane. Let $D = (\mathcal{U} - \bigcup_{k=1}^{n} D_k \cup \Gamma_k)$, $z = 0 \in D$, $D_0 = \{z \in \mathbb{C} : \ |z| > 1\}$, $\Gamma_k = \partial D_k$, $\Gamma_0 = \partial \mathcal{U}$. Let the curves Γ_k $(k = 1, 2, \ldots, n)$ be orientated in the counter clockwise sense, and Γ_0 be orientated in the clockwise sense. Let $\mathbf{n} = (n_1, n_2)$ be the outward unit normal vector to Γ_k $(k = 0, 1, \ldots, n)$. The normal derivative to Γ_k is introduced as follows

$$\frac{\partial}{\partial \mathbf{n}} = n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y}. \tag{1}$$

Let λ_k and γ_k be positive real numbers.

We consider the following problem: to find a function $u(z)$, (piece-wise) harmonic in D, D_k $(k = 0, 1, \ldots, n)$ continuously differentiable in the closures of the considered domains except $z = \infty$, satisfying the following conjugation conditions on all Γ_k $(k = 0, 1, \ldots, n)$:

$$\frac{\partial u^-}{\partial \mathbf{n}}(t) = \lambda_k \frac{\partial u^+}{\partial \mathbf{n}}(t), \tag{2}$$

$$\lambda_k \frac{\partial u^+}{\partial \mathbf{n}}(t) + \gamma_k (u^+(t) - u^-(t)) = 0, \tag{3}$$

and such that $u(z) - \operatorname{Re} p_N(z)$ is bounded at infinity, where $p_N(z) = A_0 + A_1 z + A_2 z^2 + \ldots + A_N z^N$ is a given polynomial of order N. In (2), (3) we denote by u^+ and u^- the boundary values of $u(z)$ from the left and the right of each boundary curve according to chosen orientation.

This problem has applications in the steady heat conduction of composites when the contact between materials is imperfect (see [1, 2]). Further, we identify the normal vector $\mathbf{n} = (n_1, n_2)$ with the complex number $n = n_1 + in_2$. Introduce the function

$$\varphi(z) = u(z) + iv(z) \tag{4}$$

analytic in D, D_k $(k = 0, 1, \ldots, n)$ continuously differentiable in the closures of the considered domains except $z = \infty$. Function $\varphi(z)$ has the principal part $p_N(z)$ at infinity, i.e. $\varphi(z) - p_N(z)$ is bounded $z \to \infty$.

Let us fix a curve Γ_k $(k = 1, 2, \ldots, n)$. The boundary values of the normal derivatives can be written in the form

$$\frac{\partial u^{\pm}(t)}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla u^{\pm}(t) = \frac{1}{r_k} \operatorname{Re}(t - a_k)(\varphi^{\pm})'(t), \quad |t - a_k| = r_k, \tag{5}$$

since $(\varphi^{\pm})'(z) = \dfrac{\partial u^{\pm}}{\partial x} - i\dfrac{\partial u^{\pm}}{\partial y}$. Let s be the natural parameter on the curve Γ_k and

$$\frac{\partial}{\partial s} = -n_2\frac{\partial}{\partial x} + n_1\frac{\partial}{\partial y} \tag{6}$$

be the tangent derivative along Γ_k. Applying the Cauchy–Riemann equations

$$\frac{\partial u^{\pm}}{\partial x} = \frac{\partial v^{\pm}}{\partial y}, \quad \frac{\partial u^{\pm}}{\partial y} = -\frac{\partial v^{\pm}}{\partial x}$$

and using (1), (6) we have

$$\frac{\partial u^{\pm}}{\partial \mathbf{n}} = n_1\frac{\partial u^{\pm}}{\partial x} + n_2\frac{\partial u^{\pm}}{\partial y} = n_1\frac{\partial v^{\pm}}{\partial y} - n_2\frac{\partial v^{\pm}}{\partial x} = \frac{\partial v^{\pm}}{\partial s}. \tag{7}$$

This yields

$$\frac{\partial v^{-}}{\partial s}(t) = \lambda_k\frac{\partial v^{+}}{\partial s}(t), \quad |t-a_k| = r_k. \tag{8}$$

Integrating (8) on s we arrive at the relation

$$v^{-}(t) = \lambda_k v^{+}(t), \quad |t-a_k| = r_k. \tag{9}$$

We put the constant of integration to be zero, since the imaginary part of the function φ is determined up to an additive constant, which does not impact on the form of $u(z)$. It follows from (9):

$$\operatorname{Im}\varphi^{-}(t) = \lambda_k \operatorname{Im}\varphi^{+}(t), \quad |t-a_k| = r_k. \tag{10}$$

Using (4) and (5) we rewrite (3) in the form

$$\operatorname{Re}\left\{\varphi^{+}(t) - \varphi^{-}(t) + \frac{\lambda_k}{r_k\gamma_k}(t-a_k)(\varphi^{+})'(t)\right\} = 0, \quad |t-a_k| = r_k. \tag{11}$$

Two real relations (10) and (11) yield the following complex one

$$\varphi^{-}(t) = \phi_k(t) - \rho_k\overline{\phi_k(t)} + 2\mu_k \operatorname{Re}(t-a_k)\phi_k'(t), \quad |t-a_k| = r_k, \tag{12}$$

where

$$\phi_k(z) = \frac{\lambda_k+1}{2}\varphi^{+}(z), \quad z\in D_k, \quad \rho_k = \frac{\lambda_k-1}{\lambda_k+1}, \quad \mu_k = \frac{1+\rho_k}{2r_k\gamma_k}, \quad k=0,1,\dots,n. \tag{13}$$

Due to symmetry with respect to Γ_k one can rewrite (12) as follows

$$\varphi^{-}(t) = \phi_k(t) - \rho_k\overline{\phi_k(t)} + \mu_k(t-a_k)\phi_k'(t) + \mu_k\frac{r_k^2}{t-a_k}\overline{\phi_k'(t)}, \quad |t-a_k| = r_k. \tag{14}$$

If $k=0$, then in a similar way the boundary condition on Γ_0 takes the form

$$\varphi^{-}(t) = \phi_0(t) - \rho_0\overline{\phi_0(t)} + \mu_0 t\phi_0'(t) + \mu_0\frac{1}{t}\overline{\phi_0'(t)}, \quad |t| = 1. \tag{15}$$

The function $\phi_0(z)$ has a pole of order N at infinity, and by (13) $\phi_0(z)$ can be presented in the form

$$\phi_0(z) = \varphi_0(z) + p_N(z), \quad |z| > 1. \tag{16}$$

Without loss of generality one can suppose that $p_N(0) = 0$.

Problems similar to (2), (3) were studied in the paper [1, 2]. Our method generalizes the approach described in [3].

2. Functional-differential equation. Introduce the inversion with respect to the circle Γ_k

$$z^*_{(k)} := r_k^2/\overline{z - a_k} + a_k, \quad k = 0, 1, \ldots, n.$$

The relation $t^*_{(k)} = t$ holds for all $t \in \Gamma_k$. If $\psi(z)$ is a function, analytic in the disc $|z - a_k| < r_k$ and continuous in its closure, then the function $\overline{\psi(z^*_{(k)})}$ is analytic in $|z - a_k| > r_k$ and continuous in $|z - a_k| \geq r_k$ ($k = 0, 1, \ldots, n$).

Introduce the function

$$\Phi(z) = \begin{cases} \phi_k(z) + \mu_k(z - a_k)\phi_k'(z) + \\ + \sum\limits_{j \neq k, 0}^{n} \left(\rho_j \overline{\phi_j(z^*_{(j)})} - \mu_j \dfrac{r_j^2}{z - a_j} \overline{\phi_j'(z^*_{(j)})} \right) + \\ + \varrho_0 \overline{\phi_0(1/\overline{z})} - \mu_0 \frac{1}{z} \overline{\phi_0'(1/\overline{z})}, & z \in D_k, \quad k = 1, 2, \ldots, n; \\ \varphi(z) + \sum\limits_{j=1}^{n} \left[\rho_j \overline{\phi_j(z^*_{(j)})} - \mu_j \dfrac{r_j^2}{z - a_j} \overline{\phi_j'(z^*_{(j)})} \right] + \\ + \varrho_0 \overline{\phi_0(1/\overline{z})} - \mu_0 \frac{1}{z} \overline{\phi_0'(1/\overline{z})}, & z \in D \setminus \{0\}; \\ \phi_0(z) - \mu_0 z \phi_0'(z) + \sum\limits_{j=1}^{n} \left[\rho_j \overline{\phi_j(z^*_{(j)})} - \mu_j \frac{r_j^2}{z - a_j} \overline{\phi_j'(z^*_{(j)})} \right], & z \in D_0. \end{cases} \tag{17}$$

Lemma 1. *The function* $\Phi(z)$ *is analytic on* $\widehat{\mathbb{C}} \setminus \{0, \infty\}$.

Proof. Similarly to [3] we can see that the function Φ is analytic in D_k and has the zero jump across the curve Γ_k ($k = 1, 2, \ldots, n$). Since existenc of analytic-type singularities follows from (15), it sufficies to calculate the jump of Φ across the unit circle Γ_0:

$$\Delta_0 = \lim_{\substack{z \to t \in \Gamma_0 \\ z \in D}} \Phi(z) - \lim_{\substack{z \to t \in \Gamma_0 \\ z \in D_0}} \Phi(z).$$

By (17)

$$\Delta_0 = \lim_{\substack{z \to t \in \Gamma_0 \\ z \in D}} \left(\varphi(z) + \sum_{j=1}^{n} \left[\rho_j \overline{\phi_j(z^*_{(j)})} - \mu_j \frac{r_j^2}{z - a_j} \overline{\phi_j'(z^*_{(j)})} \right] + \rho_0 \overline{\phi_0(1/\overline{z})} - \mu_0 \frac{1}{z} \overline{\phi_0'(1/\overline{z})} \right) -$$

$$- \lim_{\substack{z \to t \in \Gamma_0 \\ z \in D_0}} \left(\phi_0(z) - \mu_0 z \phi_0'(z) + \sum_{j=1}^{n} \left[\rho_j \overline{\phi_j(z^*_{(j)})} - \mu_j \frac{r_j^2}{z - a_j} \overline{\phi_j'(z^*_{(j)})} \right] \right).$$

Hence,

$$\Delta_0 = \varphi^-(t) + \rho_0 \overline{\phi_0(t)} - \mu_0 \frac{1}{t} \overline{\phi_0'(t)} - \phi_0(t) + \mu_0 t \phi_0'(t), \quad |t| = 1.$$

Thus, by (15) we obtain $\Delta_0 = 0$. It complete the proof.

Consider the function $\Phi(z)$ in the domain D_0. Since $\phi_0(z)$ is of the form (16), hence $\Phi(z)$ has a pole of order N at infinity. It follows from (17) that $\Phi(z)$ is analytic in D except $z = 0$, and the principal parts at $z = 0$ of $\Phi(z)$ and of the function

$$\rho_0 \overline{\phi_0(1/\overline{z})} - \mu_0 \frac{1}{z} \overline{\phi_0'(1/\overline{z})}, \tag{18}$$

are the same. It follows from (16), that the principal part of (18) at $z=0$ is $\rho_0\overline{p_N(1/\overline{z})}-\mu_0\overline{p'_N(1/\overline{z})}$. Hence, $\Phi(z)$ has a pole of order N at $z=0$ and

$$\Phi(z)-\overline{\rho_0 p_N(1/\overline{z})}-\mu_0\frac{1}{z}\overline{\rho_0 p'_N(1/\overline{z})} \tag{19}$$

is bounded in $\mathbb{C}$. It follows from the definition (17) of the function $\Phi(z)$ in $|z|>1$ and from the relation (16) that the principal part of $\Phi(z)$ at $z=\infty$ is equal to

$$p_N(z)-\mu_0 z p'_N(z). \tag{20}$$

Therefore, the function

$$\Phi(z)-p_N(z)-\mu_0 z p'_N(z)-\overline{\rho_0 p_N(1/\overline{z})}-\mu_0\frac{1}{z}\overline{p'_N(1/\overline{z})} \tag{21}$$

is bounded in $\widehat{\mathbb{C}}$ and analytic in $\widehat{\mathbb{C}}\setminus\{0,\infty\}$. Then Liouville's theorem yields

$$\Phi(z)=p_N(z)-\mu_0 z p'_N(z)+\rho_0\overline{p_N(1/\overline{z})}+\mu_0\frac{1}{z}\rho_0\overline{p'_N(1/\overline{z})}+c, \tag{22}$$

when c is an arbitrary complex constant. Introduce

$$g(z)=p_N(z)-\mu_0 z p'_N(z)+\rho_0\overline{p_N(1/\overline{z})}+\mu_0\frac{1}{z}\rho_0\overline{p'_N(1/\overline{z})}+c.$$

Writing (17) in D_k we obtained the system of functional-differential equations with respect to $\phi_k(z)$ ($k=0,1,\dots,n$):

$$\phi_k(z)+\mu_k(z-a_k)\phi'_k(z)+\rho_0\overline{\phi_0(1/\overline{z})}-\mu_0\frac{1}{z}\overline{\phi'_0(1/\overline{z})}=$$

$$=\sum_{j\neq k,0}^{n}\left(-\rho_j\overline{\phi_j(z^*_{(j)})}+\mu_j\frac{r_j^2}{z-a_j}\overline{\phi'_j(z^*_{(j)})}\right)+g(z),\quad |z-a_k|\le r_k,\quad k=1,2,\dots,n, \tag{23}$$

$$\phi_0(z)-\mu_0 z\phi'_0(z)=\sum_{j=1}^{n}\left[-\rho_j\overline{\phi_j(z^*_{(j)})}+\mu_j\frac{r_j^2}{z-a_j}\overline{\phi'_j(z^*_{(j)})}\right]+g(z),\quad |z|\ge 1. \tag{24}$$

3. Spaces and operators. In this section we describe spaces in which the system (23), (24) has to be solved. Instead of $n+1$ function $\phi_k(z)$ $(k=0,1,\dots,n)$ we introduce formally one function analytic in $\bigcup_{k=0}^n D_k$ whose restriction on D_k coincides with ϕ_k, i.e.

$$\phi(z)=\phi_k(z),\quad z\in D_k,\quad k=0,1,\dots,n. \tag{25}$$

We introduce the space C_A of functions analytic in $\bigcup_{k=0}^n D_k$ and continuous in the closure of $\bigcup_{k=0}^n D_k$, endowed with the norm $\|f\|_{C_A}=\sup\limits_{k=0,1,\dots,n}\sup\limits_{0\le\theta\le 2\pi}|f(r_k e^{i\theta}+a_k)|$. Let C_A^1 be the subspace of C_A consisting of functions continuously differentiable in the closure of $\bigcup_{k=0}^n D_k$, endowed with the norm $\|f\|_{C_A^1}=\|f\|_{C_A}+\|f'\|_{C_A}$.

For each fixed $k=0,1,\dots,n$, we define the Hilbert spaces $\mathcal{L}^2(\Gamma_k)$ of functions $f(t)$ which satisfy the condition $f(r_k e^{i\theta}+a_k)\in\mathcal{L}^2(0,2\pi)$ with respect to the variable $\theta\in(0,2\pi)$ endowed with the norm $\|f\|^2_{\mathcal{L}^2(\Gamma_k)}=\int_0^{2\pi}|f(r_k e^{i\theta}+a_k)|^2\,d\theta$. For fixed $k=1,2,\dots,n$ the space $\mathcal{H}^2(D_k)$ is introduced as the space of analytic functions on D_k satisfying the condition

$$\sup_{0<r<r_k}\int_0^{2\pi}|f(re^{i\theta}+a_k)|^2\,d\theta<\infty$$

and endowed with the norm $\|f\|^2_{\mathcal{H}^2(D_k)} = \sup\limits_{0<r<r_k} \int_0^{2\pi} |f(re^{i\theta}+a_k)|^2\, d\theta$. Note that for $r_k = 1$ and $a_k = 0$ we obtain the classical Hardy space. For $k = 0$ the space $\mathcal{H}^2(D_0)$ is introduced as the space of analytic functions on D_0 satisfying the condition $\sup\limits_{1<r<\infty} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta < \infty$.

Introduce $\mathcal{H}^2$ as the space of all function ϕ analytic in $\bigcup_{k=0}^n D_k$ for which $\phi_k \in \mathcal{H}^2(D_k)$ endowed with the norm $\|\phi\|^2_{\mathcal{H}^2} := \sum_{k=0}^n \|\phi_1\|^2_{\mathcal{H}^2(D_k)}$, where $\phi_k(z) = \phi(z)$ in D_k $(k = 0, 1, \ldots, n)$. The introduced spaces are related as follows

$$C_A^1 \subset C_A \subset \mathcal{H}^2.$$

Introduce the operator $\mathrm{L} : C_A^1 \to \mathcal{H}^2$ define as follows

$$\mathrm{L}f(z) = \mathrm{L}_k f(z), \quad z \in D_k, \quad k = 0, 1, \ldots, n,$$

where

$$\mathrm{L}_0 f(z) = f(z) - \mu_0 z f'(z), \quad z \in D_0,$$

$$\mathrm{L}_k f(z) = f(z) + \mu_k (z - a_k) f'(z), \quad |z - a_k| \le r_k, \quad k = 1, 2, \ldots, n.$$

Theorem 1. *Let* $\mu_k > 0$, $k = 0, 1, \ldots, n$. *Then the operator* L^{-1} *is bounded in* $\mathcal{H}^2$ *and* $\|\mathrm{L}^{-1}\| \le 1$.

Proof. We consider equation $\mathrm{L}f = \psi$ with a given function $\psi \in \mathcal{H}^2$ and unknown function $f \in \mathcal{H}^2$. This equation is equivalent to $n+1$ equations $\mathrm{L}_k f_k = \psi_k$, in $\mathcal{H}^2(D_k)$ $(k = 0, 1, \ldots, n)$. We us examine the case $k = 0$. For all other curves we can proceed in the same way. Represent the functions $f_0(z)$ and $\psi_0(z)$ in D_0 in form of their Taylor series

$$f_0(z) = \sum_{m=0}^{\infty} \frac{\alpha_{0m}}{z^m}, \quad \psi_0(z) = \sum_{m=0}^{\infty} \frac{\beta_{0m}}{z^m}.$$

Calculate

$$z f_0'(z) = -\sum_{m=0}^{\infty} \frac{m\alpha_{0m}}{z^m}.$$

Then equation $\mathrm{L}_0 f_0 = \psi_0$ in the space $\mathcal{H}^2(D_0)$ is equivalent to the equation in l^2

$$\alpha_{0m} + \mu_0 m \alpha_{0m} = \beta_{0m}, \quad m = 0, 1, \ldots, \tag{26}$$

where l^2 is the space of the sequences $\alpha_k = \{\alpha_{km}\}_{m=0}^{\infty}$ endowed with the norm

$$\|f\|_{l^2} = \sum_{m=0}^{\infty} |\alpha_m|^2 r_k^{2m}.$$

The following estimate

$$\|\mathrm{L}^{-1}\psi\|^2_{\mathcal{H}^2} = \sup_{\|\psi\|_{\mathcal{H}^2}=1} \left(\sum_{j=0}^{n} \|\mathrm{L}_j^{-1}\psi_j\|^2 \right) \le \|\beta\|^2_{h^2} = \|\psi\|_{\mathcal{H}^2}, \tag{27}$$

is valid, where h^2 is the space of the sequences $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$, $\beta_j \in l^2$ $(j = 0, 1, \ldots, n)$ endowed with the norm $\|\beta\|_{h^2} = (\sum_{j=0}^n \|\beta_j\|^2_{l^2})^{1/2}$. Then (27) yields the inequality $\|\mathrm{L}^{-1}\| \le 1$ in $\mathcal{H}^2$.

4. Solution to functionals equation. Define the operator $\mathcal{A}$ in the space $\mathcal{H}^2$

$$\mathcal{A}\phi(z) = -\rho_0\overline{\phi_0(1/\overline{z})} + \mu_0\frac{1}{z}\overline{\phi_0'(1/\overline{z})} - \sum_{j\neq k,0}\left(-\rho_j\overline{\phi_j(z^*_{(j)})} - \mu_j\frac{r_j^2}{z-a_j}\overline{\phi_j'(z^*_{(j)})}\right)$$

for $|z-a_k|\leq r_k$ ($k=1,2,\ldots,n$) and

$$\mathcal{A}\phi(z) = \sum_{j=1}^{n}\left[-\rho_j\overline{\phi_j(z^*_{(j)})} + \mu_j\frac{r_j^2}{z-a_j}\overline{\phi_j'(z^*_{(j)})}\right], \quad |z|\geq 1.$$

The system of equations (23), (24) can be written as the operator equation in $\mathcal{H}^2$

$$\mathrm{L}\phi = \mathcal{A}\phi + g.$$

Then similarly to [3] we deduce the following theorems.

Theorem 2. *Operator $\mathcal{A}$ is compact in $\mathcal{H}^2$.*

Theorem 3. *For sufficiently small coefficients $|\rho_k|<1$ and $\mu_k\geq 0$ ($k=0,1,\ldots,n$) boundary value problem (14), (15) has a unique solution in $\mathcal{H}^2$ which can be found by the method of successive approximations applied to equation*

$$\phi = \mathrm{L}^{-1}\mathcal{A}\phi + \mathrm{L}^{-1}g. \tag{28}$$

Proof follows from the structure of the operator $\mathcal{A}$ and Theorem 1.

Remark 1. If $g\in C_A^1$, then $\phi\in C_A^1$. Theorem 3 yields an efficient algorithm to solve equations (23), (24) in the space $\mathcal{H}^2$.

Remark 2. The solution $u(z)$ to the original boundary problem (2), (3) has the form

$$u(z) = \begin{cases} \dfrac{2}{\lambda_k+1}\operatorname{Re}\phi_k(z), & z\in \mathrm{cl}D_k, \quad k=1,2,\ldots,n, \\ \operatorname{Re}\varphi(z), & z\in D, \end{cases} \tag{29}$$

where

$$\varphi(z) = -\sum_{j=1}^{n}\left[\rho_j\overline{\phi_j(z^*_{(j)})} - \mu_j\frac{r_j^2}{z-a_j}\overline{\phi_j'(z^*_{(j)})}\right] - \varrho_0\overline{\phi_0(1/\overline{z})} + \mu_0\frac{1}{z}\overline{\phi_0'(1/\overline{z})},$$

$\phi_k(z)$ is a solution of (23), (24) found due to Theorem 3. One can see that the solution $u(z)$ determined by (29) depends on an arbitrary additive real constant $\operatorname{Re} c$.

References

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Functional-differential equations in Hardy-type classes

Summary

We consider a conjugation problem for harmonic functions in multiply connected circular domains. The problem is rewritten in the form of the $\mathbb{R}$-linear boundary value problem which is solved in Hardy-type classes by using equivalent functional-differential equations.