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MAYER'S TRANSFER OPERATOR APPROACH TO SELBERG'S ZETA FUNCTION

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These notes are based on three lectures given by the second author at Copenhagen University (October 2009) and at Aarhus University, Denmark (December 2009). Mostly, a survey of the results of Dieter Mayer on relationships between Selberg and Smale–Ruelle dynamical zeta functions is presented. In a special situation, the dynamical zeta function is defined for a geodesic flow on a hyperbolic plane quotient by an arithmetic cofinite discrete group. More precisely, the flow is defined for the corresponding unit tangent bundle. It turns out that the Selberg zeta function for this group can be expressed in terms of a Fredholm determinant of a classical transfer operator of the flow. The transfer operator is defined in a certain space of holomorphic functions, and its matrix representation in a natural basis is given in terms of the Riemann zeta function and the Euler gamma function.

Contents

Key words: Mayer's transfer operator, Selberg's zeta function.

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§1. General theory

We quote Ruelle [14, 15] to introduce his general notion of a transfer operator and a dynamical zeta function for a given dynamical system.

First, we give the definition of a weighted dynamical system. Let Λ be a set weighted by a function $g : \Lambda \longrightarrow \mathbb{C}$. Assume that Λ describes a system, then the dynamics of the system is given by a map $F : \Lambda \longrightarrow \Lambda$. The triplet $\mathcal{D} := (\Lambda, F, g)$ is called a weighted dynamical system or simply a dynamical system.

The transfer operator method is applicable if the map F is not invertible, that is, for example, when its inverse is not unique. More precisely, the set of inverse branches of F must be finite or at least countable and discrete in a natural topology.

For such a dynamical system, the action of the Ruelle transfer operator $\mathcal{L}$ on a function $f : \Lambda \longrightarrow \mathbb{C}$ is defined by

$$(\mathcal{L}f)(x) = \sum_{y \in F^{-1}\{x\}} g(y) f(y). \tag{1.1}$$

Let the set of transfer operators for all dynamical systems of the set Λ with respect to the product $\circ$ given by $(\mathcal{L}_1 \circ \mathcal{L}_2) f = \mathcal{L}_1(\mathcal{L}_2 f)$ be an algebra denoted by $\mathcal{S}$. A trace on this algebra is a linear functional $Tr : S \longrightarrow \mathbb{C}$ such that $Tr(\mathcal{L}_1\mathcal{L}_2) = Tr(\mathcal{L}_2\mathcal{L}_1)$ for every $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{S}$. For a given trace Tr, a determinant Det for the operators of the algebra can formally be defined by

$$Det(I - z\mathcal{L}) = \exp\left(- \sum_{m=1}^{\infty} \frac{z^m}{m} Tr \mathcal{L}^m \right). \tag{1.2}$$

On the other hand, a weighted dynamical system $\mathcal{D} = (\Lambda, F, g)$ is equipped with the so-called Ruelle dynamical zeta function defined by

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x \in FixF^m} \prod_{k=0}^{m-1} g(F^k x) \right), \tag{1.3}$$

where $FixF^m$ denotes the set of all fixed points of F^m. The set $FixF^m$ is finite or countably infinite for all $m > 0$. Like other zeta functions, the Ruelle dynamical zeta function has some sort of Euler product

$$\zeta(z) = \prod_{\{P\}} \left(1 - z^{|P|} \prod_{k=0}^{|P|-1} g(F^k x_P) \right)^{-1}, \tag{1.4}$$

where $\{P\}$ denotes the set of periodic orbits of F with length $|P|$ and x_P is an arbitrary element of P. We shall assume that (1.2), (1.3), and (1.4) are absolutely convergent at least for z in a certain domain in $\mathbb{C}$.

In general, analytic properties of zeta functions give an important information about the corresponding systems in question. For example, a Tauberian theorem yields the classical prime number theorem from the positions of poles and zeros of the Riemann zeta function in the critical strip.

In the same way, we are interested in the analytic properties of the dynamical zeta function to get more information about the corresponding dynamical system.

An important method to study the analytic properties of dynamical zeta functions is the transfer operator method. In this method, the analytic properties of the zeta function are related to the spectral properties of a transfer operator through a relationship between the Fredholm determinant of the transfer operator and the dynamical zeta function.

An interesting realization of the general program described above is the Mayer transfer operator acting on some Banach space of holomorphic functions on a disk [9]. This operator is assigned to the dynamical system related to the geodesic flow on the hyperbolic plane modulo an arithmetical cofinite discrete group Γ. In this case the Fredholm determinant of the transfer operator is equal to the Selberg zeta function for the corresponding discrete group, which is one of the most important aspects of Mayer's transfer operator theory. Indeed this identity provides us a new insight to the theory of quantum chaos. It turns out that the Mayer transfer operator, which is a purely classical object, surprisingly contains all information we can obtain from the corresponding Schrödinger operator.

§2. Mayer's transfer operator for $PSL(2,\mathbb{Z})$

We start with introducing some notation and definitions. The hyperbolic plane $\mathbb{H}$ is the upper half-plane $\{x+iy \in \mathbb{C} \mid y>0\}$ equipped with the Poincaré metric $ds^2 = y^{-2}(dx^2+dy^2)$ and the measure $d\mu(z) = y^{-2}dxdy$. Thus, geodesics on $\mathbb{H}$ are the semicircles with centra and the end points on the real axis.

The group of all orientation preserving isometries of the hyperbolic plane $\mathbb{H}$ is identified with the group

$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\} \tag{2.1}$$

acting on $\mathbb{H}$ by linear fractional transformations defined by

$$z \to gz = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z \in \mathbb{H}.$$

The modular group $PSL(2,\mathbb{Z})$ is a discrete subgroup of $PSL(2,\mathbb{R})$ defined by

$$PSL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1,\ a,b,c,d \in \mathbb{Z} \right\} / \{\pm I\}\,. \tag{2.2}$$

This is a noncocompact Fuchsian group of the first kind.

Let $M = PSL(2,\mathbb{Z})\backslash\mathbb{H}$ be the quotient space of the hyperbolic plane $\mathbb{H}$ mod $PSL(2,\mathbb{Z})$. This is a surface with one cusp and two conical singularities. Consider the continuous dynamics given by the geodesic flow φ_t on T_1M, the unit tangent bundle of M. From a physics point of view, the tangent bundle said to be unit if the geodesic flow describes the motion of a free particle on M with unit magnitude of velocity.

As has already been mentioned, a transfer operator can be defined if the corresponding dynamical map has a finite or countable set of inverse branches, while the geodesic flow is continuous and determines an invertible map on T_1M. Thus, we first discretize the geodesic flow by constructing a Poincaré map of φ_t. It is known that, by a suitable choice of the Poincaré section in T_1M, the dynamics of φ_t reduces to the Poincaré map given by (see [3])

$$\begin{aligned} &P : [0,1] \times [0,1] \times \mathbb{Z}_2 \to [0,1] \times [0,1] \times \mathbb{Z}_2, \\ &P(x_1, x_2, \epsilon) = \Big(T_G x_1, \frac{1}{[\frac{1}{x_1}] + x_2}, -\epsilon\Big), \end{aligned} \tag{2.3}$$

where $[x]$ denotes the integral part of x and

$$T_G x = \begin{cases} \frac{1}{x} \mod 1 & \text{if} \quad x \in (0,1], \\ 0 & \text{if} \quad x = 0 \end{cases} \tag{2.4}$$

is the Gauss map.

Remark 2.1. The group $PSL(2,\mathbb{Z})$ does not contain the reflection relative to the y-axis. Consequently, for every geodesic on M there exists a unique geodesic on M such that their representatives on the upper half-plane are located symmetrially relative to the y-axis. The same is true for the orbits on T_1M. This fact is reflected by two possible values of the parameter ϵ.

We are interested in the expanding part of the map P, which reflects the ergodic aspects of the geodesic flow φ_t,

$$\begin{aligned} &P_{ex} : [0,1] \times \mathbb{Z}_2 \to [0,1] \times \mathbb{Z}_2, \\ &P_{ex}(x, \epsilon) = (T_G x, -\epsilon). \end{aligned} \tag{2.5}$$

It remains to select a suitable weight function. Mayer chose the following weight function:

$$g(x,\epsilon) = (T_G')^s(x) = x^{2s}, \tag{2.6}$$

where s is a complex parameter. In fact, in accordance with Sinaĭ's paper [18], the ergodic properties of φ_t are related to this weight function.

The Mayer transfer operator is a transfer operator for the weighted dynamical system

$$\mathcal{D}_1 = ([0,1] \times \mathbb{Z}_2, P_{ex}, g(x,\epsilon) = x^{2s}), \tag{2.7}$$

i.e.,

$$\mathcal{L}_s f(x,\epsilon) = \sum_{y=P_{ex}^{-1}(x,\epsilon)} g(y) f(y). \tag{2.8}$$

We notice that for $s=1$ the Mayer transfer operator reduces to the Perron–Frobenious operator for the Gauss map. The map P_{ex} has an infinite number of discrete inverse branches given by

$$P_{ex}^{-1}(x,\epsilon) = \Big(\frac{1}{x+n}, -\epsilon\Big), \quad n \in \mathbb{N}; \tag{2.9}$$

thus, the Mayer transfer operator is formally expressed as

$$\widetilde{\mathcal{L}}_s f(x,\epsilon) = \sum_{n=1}^{\infty} \Big(\frac{1}{x+n}\Big)^{2s} f\Big(\frac{1}{x+n}, -\epsilon\Big), \quad \epsilon = \pm 1. \tag{2.10}$$

Since the weighted dynamical system (2.7) and the group $PSL(2,\mathbb{Z})$ are closely related, sometimes this operator is called the Mayer transfer operator for $PSL(2,\mathbb{Z})$. Equivalently, we can write the Mayer operator (2.10) in a vector form:

$$\widetilde{\mathcal{L}}_s \overrightarrow{f}(x) = \sum_{n=1}^{\infty} \Big(\frac{1}{x+n}\Big)^{2s} M.\overrightarrow{f}\Big(\frac{1}{x+n}\Big) \tag{2.11}$$

with

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \overrightarrow{f}(x) = \begin{pmatrix} f(x,1) \\ f(x,-1) \end{pmatrix}. \tag{2.12}$$

If we take the group

$$PGL(2,\mathbb{Z}) = GL(2,\mathbb{Z})/\{\pm 1\} \tag{2.13}$$

instead of the group $PSL(2,\mathbb{Z})$, where

$$GL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = \pm 1,\ a,b,c,d \in \mathbb{Z} \right\}, \tag{2.14}$$

then the orbits corresponding to the two possible values of ϵ will be identified, because $PGL(2,\mathbb{Z})$ contains the reflection relative to the y-axis. Note that this reflection acts on $\mathbb{H}$ not by a linear fractional transformation but as the map $z \to -\overline{z}$. Therefore, the dynamical system $\mathcal{D}_1$ in (2.7) reduced to the following dynamical system:

$$\mathcal{D}_2 = ([0,1], T_G, g(x) = x^{2s}). \tag{2.15}$$

The Mayer transfer operator for $PGL(2,\mathbb{Z})$ is a transfer operator corresponding to the dynamical system $\mathcal{D}_2$,

$$\mathcal{L}_s f(x) = \sum_{n=1}^{\infty} \Big(\frac{1}{x+n}\Big)^{2s} f\Big(\frac{1}{x+n}\Big), \tag{2.16}$$

which is sometimes called the transfer operator for the billiard flow or for the Gauss map, because the dynamics of $\mathcal{D}_2$ is defined by the Gauss map T_G. The analytic properties of this operator extend easily to the original Mayer operator. Thus, from now on we consider this simplified version of the Mayer operator.

Up to now we have seen how the ergodic aspects of the geodesic flow fix the form of the Mayer operator with no information about the function space on which $\mathcal{L}_s$ acts . At the next step we should decide about this space in such a way that the operator $\mathcal{L}_s$ become a nice operator from an analytical point of view, with well-defined trace and determinant. For this, first, we need two lemmas. Let the real x in (2.16) be extended to z in the complex domain

$$D_r = \{z \in \mathbb{C} \mid |z-1| < r\}\,. \tag{2.17}$$

Lemma 2.2. *For $n \in \mathbb{N}$, the map $\psi_n : z \to \frac{1}{z+n}$ maps holomorphically the disk D_r strictly inside itself if the radius r of the disk lies in the interval $(1, \frac{1+\sqrt{5}}{2})$. For $r = 1$ the map ψ_n touches the boundary in the limit $n \to \infty$.*

Proof. First we note that

$$\lim_{n\to\infty} \psi_n(z) = 0, \quad z \in D_r; \tag{2.18}$$

consequently, the smallest lower bound of the radius is $r = 1$. But the upper bound is at most $r < 2$ because otherwise ψ_1 is not contracting at $z = -1$. To determine the largest upper bound of r we note that the maps ψ_n are conformal, mapping circles into circles. We also note that the ψ_n's map the disk D_r to disks with centers on the real axis. Thus, to investigate the contracting properties of ψ_n, it suffices to investigate the transformations of the end-points $x_- = 1-r$ or $x_+ = 1+r$ of the diameter of D_r lying on the real axis. Moreover, since

$$\psi_{n+1}(x) < \psi_n(x), \quad n \in \mathbb{N}, \quad x > -1, \tag{2.19}$$

to find the upper bound of r it suffices to find the upper bound of r for ψ_1. But

$$\psi_1(x_+) < \psi_1(x_-), \quad 1 < r < 2, \tag{2.20}$$

so that the largest upper bound of the radius is determined by the following inequality:

$$\psi_1(x_-) = \frac{1}{2-r} < 1 + r. \tag{2.21}$$

By simple calculation we get $r < \frac{1+\sqrt{5}}{2}$, which completes the proof. □

The following lemma suggests a possible good candidate for the desired space.

Lemma 2.3. *For $r \in (1, \frac{1+\sqrt{5}}{2})$, the Banach space*

$$B(D_r) = \{f : D_r \to \mathbb{C} \mid f \text{ is holomorphic on } D_r \text{ and continuous on } \overline{D}_r\} \tag{2.22}$$

with the sup norm $||\ ||$ is invariant under the action of $\mathcal{L}_s$ for $\mathrm{Re}(s) > \dfrac{1}{2}$, that is $\mathcal{L}_s B(D_r) \subset B(D_r)$. Note that the continuity on the boundary makes the space $B(D_r)$ a subspace of the Banach space of bounded holomorphic functions.

Proof. The transfer operator for $PGL(2, \mathbb{Z})$ is given by

$$\mathcal{L}_s f(z) = \sum_{n=1}^{\infty} \Big(\frac{1}{z+n}\Big)^{2s} f\Big(\frac{1}{z+n}\Big). \tag{2.23}$$

Let f be in $B(D_r)$. The argument of f, that is the function $\psi_n(z) = \frac{1}{z+n}$, maps the disk D_r inside itself and is holomorphic in this domain for all $n \in \mathbb{N}$. Thus, the term $S_n := (\frac{1}{z+n})^{2s} f(\frac{1}{z+n})$ also belongs to $B(D_r)$. We are going to prove that $\sum_{n=1}^{\infty} S_n \in B(D_r)$. The required result comes from the general Weierstrass M-test, for which we need $||S_n|| \leqslant M_n < \infty$ for all $n \geqslant 1$ and $\sum_{n=1}^{\infty} M_n < \infty$, where M_n is a positive number. These requirements are fulfilled for $\sigma = \mathrm{Re}(s) > \frac{1}{2}$ if we take $M_n = (n - r + 1)^{-2\sigma}||f||$, where $||f||$ is bounded because $f \in B(D_r)$. This completes the proof. □

Corollary 2.4. *Let r be in the interval $(1, \frac{1+\sqrt{5}}{2})$. Then the operator $\widetilde{\mathcal{L}}_s$ for the group $PSL(2, \mathbb{Z})$ is defined on the Banach space $B(D_r) \oplus B(D_r)$ for $\mathrm{Re}(s) > \frac{1}{2}$ and leaves this space invariant.*

In Subsection 2.2 we shall see that this choice of the function space makes Mayer's operator a nuclear operator, which confirms that we are on the right track.

2.1. Matrix representation of Mayer's transfer operator, its eigenvectors and eigenvalues. In the next subsection we shall see that the transfer operator is compact and the spectrum of this operator in the space $B(D_r)$ is a discrete set of eigenvalues of finite multiplicity. Now we are going to derive a matrix representation for the transfer operator.

Let $f \in B(D_r)$ be an eigenfunction of $\mathcal{L}_s$,

$$\mathcal{L}_s f(z) = \lambda f(z); \tag{2.24}$$

the holomorphy of f on D_r allows the following expansion:

$$f(z)=\sum_{k=0}^{\infty}c_k(z-1)^k,\quad z\in D_r; \tag{2.25}$$

Inserting this in (2.24), we get

$$\sum_{n=1}^{\infty}\Big(\frac{1}{z+n}\Big)^{2s}\sum_{k=0}^{\infty}c_k\Big(\frac{1}{z+n}-1\Big)^k=\lambda\sum_{k=0}^{\infty}c_k(z-1)^k, \tag{2.26}$$

but

$$\Big(\frac{1}{z+n}-1\Big)^k=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\Big(\frac{1}{z+n}\Big)^j, \tag{2.27}$$

so that (2.26) can be written as

$$\sum_{k=0}^{\infty}c_k\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{n=1}^{\infty}\Big(\frac{1}{z+n}\Big)^{2s+j}=\lambda\sum_{k=0}^{\infty}c_k(z-1)^k. \tag{2.28}$$

On the other hand, we have the following Taylor expansion at $z=1$:

$$\Big(\frac{1}{z+n}\Big)^{2s+j}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\Gamma(2s+j+m)}{\Gamma(2s+j)}\Big(\frac{1}{1+n}\Big)^{2s+j+m}(z-1)^m, \tag{2.29}$$

which implies

$$\sum_{n=1}^{\infty}\Big(\frac{1}{z+n}\Big)^{2s+j}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\Gamma(2s+j+m)}{\Gamma(2s+j)}(\zeta(2s+j+m)-1)(z-1)^m, \tag{2.30}$$

where we have used the identity

$$\sum_{n=1}^{\infty}\Big(\frac{1}{1+n}\Big)^{\beta}=\zeta(\beta)-1. \tag{2.31}$$

Finally, inserting (2.30) in (2.28), we get

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}a_{mk}(s)c_k(z-1)^m=\lambda\sum_{k=0}^{\infty}c_k(z-1)^k, \tag{2.32}$$

where

$$a_{mk}(s)=\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^{m+k-j}}{m!}\frac{\Gamma(2s+j+m)}{\Gamma(2s+j)}(\zeta(2s+j+m)-1), \tag{2.33}$$

which leads finally to the following eigenvalue equation for the transfer operator in the matrix representation:

$$\sum_{k=0}^{\infty} a_{mk}(s)c_k = \lambda c_m. \tag{2.34}$$

Based on these calculations, we can formulate the following lemma.

Lemma 2.5. *In the natural basis* $\{(z-1)^k\}_{k=0}^{\infty}$, *in accordance with* (2.25), *the eigenfunctions* $f \in B(D_r)$ *have the representation given by the sequence* $\{c_k\}_{k=0}^{\infty}$ *satisfying* (2.34). *Moreover, the transfer operator in this basis is an infinite-dimensional matrix whose matrix entries are given by* (2.33).

Remark 2.6. Mayer [13] derived a simpler matrix representation of the transfer operator, given by

$$a_{mk}(s) = \frac{(-1)^m}{m!}\frac{\Gamma(2s+k+m)}{\Gamma(2s+k)}\zeta(2s+k+m). \tag{2.35}$$

In this representation, the basis of the space $B(D_r)$ is chosen as follows:

$$\{\zeta(2s+k, z+1)\}_{k=0}^{\infty}, \tag{2.36}$$

where $\zeta(s,w)$ denotes the Hurwitz zeta function.

2.2. Nuclear spaces, nuclear operators, and Grothendieck's theory. Let B be an arbitrary Banach space; we denote its dual by B^*, which is the space of bounded functionals on B. A linear operator $\mathcal{L}$ on B is said to be nuclear of order q if it has the representation (see [11])

$$\mathcal{L} = \sum_n \lambda_n f_n^*(.) f_n, \tag{2.37}$$

where $\{f_n\}$ and $\{f_n^*\}$ are families in B and B^* (respectively) with $||f_n|| \leqslant 1$ and $||f_n^*|| \leqslant 1$, $\{\lambda_n\}$ is a sequence of complex numbers, and

$$q = \inf\Big\{\epsilon \leqslant 1 \mid \sum_n |\lambda_n|^\epsilon < \infty\Big\}.$$

For certain classes of nuclear operators, a trace functional is available. A remarkable result of Grothendieck says that nuclear operators $\mathcal{L}$ of order $q \leqslant \frac{2}{3}$ have a trace $tr = \sum \rho_i$ as the sum of eigenvalues ρ_i counted with multiplicities. Moreover, the Fredholm determinant of $\mathcal{L}$ is defined as

$$\det(1 - z\mathcal{L}) = \exp\, tr\, \log(1 - z\mathcal{L}) = \prod_i (1 - \rho_i z), \tag{2.38}$$

or equivalently,

$$\det(1 - z\mathcal{L}) = \exp\Big(- \sum_{n=1}^{\infty} \frac{z^n}{n} tr \mathcal{L}^n \Big). \tag{2.39}$$

Consequentely, if $\mathcal{L} = \mathcal{L}(s)$ is a holomorphic function of a parameter s in a domain, then the corresponding determinant is also a holomorphic function of s in this domain (see [7, 11]).

Lemma 2.7. *The Mayer transfer operator $\mathcal{L}_s$ for the group $PGL(2,\mathbb{Z})$, given by*

$$\mathcal{L}_s f(z) = \sum_{n=1}^{\infty} \Big(\frac{1}{z+n}\Big)^{2s} f\Big(\frac{1}{z+n}\Big), \tag{2.40}$$

acting on the Banach space $B(D_r)$ defined as in Lemma 2.3, *in the domain $\sigma = \mathrm{Re}(s) > \frac{1}{2}$, is a nuclear operator of order $q = \frac{1}{2\sigma}$.*

Proof. We are going to find a representation of Mayer's transfer operator in the form of (2.37). To avoid overloading, we give the proof for $r = \frac{3}{2}$. First, we insert the Taylor expansion of f at $z = 1$ in (2.40),

$$\mathcal{L}_s f(z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{f^{(m)}(1)}{m!} \Big(\frac{1}{z+n} - 1\Big)^m \Big(\frac{1}{z+n}\Big)^{2s}. \tag{2.41}$$

We introduce the family of functions

$$f_{n,m}(z) := \Big(\frac{1}{a_n}\Big)^{2\sigma} \Big(\frac{1}{z+n} - 1\Big)^m \Big(\frac{1}{z+n}\Big)^{2s}, \tag{2.42}$$

where

$$a_n := \sup_{z \in D_r} \Big| \frac{1}{z+n} \Big| = \frac{1}{n - \frac{1}{2}}. \tag{2.43}$$

Since

$$\sup_{z \in D_r} \Big| \frac{1}{z+n} - 1 \Big| \leqslant 1, \tag{2.44}$$

we have

$$\|f_{n,m}\| \leqslant 1. \tag{2.45}$$

Next we define

$$f^*_{n,m}(f) := r^m \frac{f^{(m)}(1)}{m!}. \tag{2.46}$$

Obviously, the Cauchy estimates yield

$$\|f^*_{n,m}\| \leqslant 1. \tag{2.47}$$

The desired representation is given by

$$\mathcal{L}_s f(z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \lambda_{n,m} f^*_{n,m}(f) f_{n,m}(z) \tag{2.48}$$

with

$$\lambda_{n,m} := r^{-m}(a_n)^{2\sigma}. \tag{2.49}$$

We note that

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} |\lambda_{n,m}|^{\epsilon} = \sum_{m=0}^{\infty} r^{-\epsilon m} \sum_{n=1}^{\infty} \Big(\frac{1}{n-\frac{1}{2}}\Big)^{2\epsilon\sigma}. \tag{2.50}$$

The first sum is a geometrical series absolutely convergent for any $\epsilon > 0$, and the second sum for any $\epsilon > \frac{1}{2\sigma}$ is absolutely convergent to $(2^{2\sigma\epsilon} - 1)\zeta(2\sigma\epsilon)$. Therefore,

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} |\lambda_{n,m}|^{\epsilon} = \frac{1}{1-(r^{-1})^{\epsilon}}(2^{2\sigma\epsilon} - 1)\zeta(2\sigma\epsilon), \quad \epsilon > \frac{1}{2\sigma}. \tag{2.51}$$

This shows that for $\sigma > \dfrac{1}{2}$,

$$\inf\Big\{\epsilon \mid \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} |\lambda_{n,m}|^{\epsilon} < \infty\Big\} = \frac{1}{2\sigma}, \tag{2.52}$$

which completes the proof. □

Mayer proved a stronger assertion about the nuclearity of the transfer operator in a more elegant way (see [10]). In fact he proved that the transfer operator is nuclear of order zero. This proof is based on some properties of the so-called nuclear spaces. It was Alexander Grothendieck who first introduced the class of nuclear spaces (see [7]). Roughly speaking, nuclear spaces are the maximal class of linear topological spaces with nice properties from an analytic point of view. For example, they admit a generalized kernel theorem of L. Schwartz.

Definition 2.8. A locally convex topological vector space E is a *nuclear space* if and only if every continuous linear map of E into any Banach space is nuclear (see [16, p. 100]).

The space $H(D)$ of holomorphic functions on the open disk D with the open compact topology is an example of a nuclear space. The space $H(D)$ satisfies stronger conditions than those of Deffinition 1. Indeed, every bounded linear map of the nuclear space $H(D)$ to any Banach space is not only nuclear but nuclear of order zero [7]. For more details about the nuclear spaces, we refer the reader to [7, 5, 16].

Now, to prove the nuclearity of the transfer operator, first we extend it to the nuclear space $H(D_r) \supset B(D_r)$ with $r \in (1, \frac{1+\sqrt{5}}{2})$, that is,

$$\begin{aligned}&\widehat{\mathcal{L}}_s : H(D_r) \to B(D_r),\\ &\widehat{\mathcal{L}}_s f(z) = \sum_{n=1}^{\infty} \Big(\frac{1}{z+n}\Big)^{2s} f\Big(\frac{1}{z+n}\Big).\end{aligned} \tag{2.53}$$

Lemma 2.9. *The operator* $\widehat{\mathcal{L}}_s : H(D_r) \to B(D_r)$ *for* $\mathrm{Re}(s) > \frac{1}{2}$ *is nuclear of order zero, where* $r \in (1, \frac{1+\sqrt{5}}{2})$.

Proof. As has been mentioned before, every bounded linear map of $H(D_r)$ to a Banach space is nuclear of order zero. Thus, it suffices to prove the boundedness of $\widehat{\mathcal{L}}_s$. For this, we should show that there exists a neighborhood of zero $V(0) \subset H(D_r)$ that is mapped to a bounded subset of $B(D_r)$. To begin with, we introduce the sequence of open disks

$$\mathbb{K}_n = \{w = \psi_n(z) \mid z \in D_r\} \tag{2.54}$$

whose radii r_n and centers c_n are given by

$$r_n = \frac{r}{(n+1)^2 - r^2} \tag{2.55}$$

and

$$c_n = \Big(\frac{n+1}{(n+1)^2 - r^2}, 0\Big). \tag{2.56}$$

We choose the following neighborhood of zero $V(0) \subset H(D_r)$:

$$V(0) = \Big\{f \in H(D_r) \mid \sup_{z \in \mathbb{K}_1} |f(z)| < M\Big\}, \tag{2.57}$$

where $M > 0$ is a constant. Also,

$$\mathbb{K}_n \subset \overline{\mathbb{K}}_1, \quad n \in \mathbb{N}. \tag{2.58}$$

For any $f \in H(D_r)$ we have

$$\sup_{z \in D_r} |\widehat{\mathcal{L}}_s f(z)| \leqslant \sum_{n=1}^{\infty} \Big(\frac{1}{1-r+n}\Big)^{2\sigma} \sup_{z \in D_r} f(\psi_n(z)). \tag{2.59}$$

But (2.58) implies

$$\sup_{z \in D_r} f(\psi_n(z)) \leqslant \sup_{z \in \mathbb{K}_1} f(z) = M. \tag{2.60}$$

Inserting this in (2.59) completes the proof. □

On the other hand, we have

$$\mathcal{L}_s = \widehat{\mathcal{L}}_s \circ \imath$$

where $\imath$ is the bounded injection given by

$$\imath : B(D_r) \to H(D_r), \quad \imath(f) = f.$$

Then, the nuclearity of $\widehat{\mathcal{L}}_s$ leads to that of $\mathcal{L}_s$, because the product of a bounded operator with a nuclear operator is also nuclear with the same order. Thus, we arrive at the following corrolary.

Corollary 2.10. *The Mayer transfer operator for the groups $PGL(2,\mathbb{Z})$ and $PSL(2,\mathbb{Z})$ in the domain $\mathrm{Re}(s) > \frac{1}{2}$ is nuclear of order zero.*

Note that in this corrolary we have deduced the nuclearity of the transfer operator for $PSL(2,\mathbb{Z})$ from the nuclearity of the transfer operator for $PGL(2,\mathbb{Z})$ and (2.11).

Any nuclear operator is compact (see [16, p. 99]). Thus, we obtain the following corollary.

Corollary 2.11. *The Mayer transfer operator for each of the groups $PSL(2,\mathbb{Z})$ and $PGL(2,\mathbb{Z})$ in the domain $\mathrm{Re}(s) > \frac{1}{2}$ is compact.*

§3. Integral representation of Mayer's transfer operator

In this section we discuss a new model for the Mayer transfer operator in a Hilbert space. This is important for the investigation of the transfer operator, because in this model the Grothendieck theory of nuclear operators in Banach spaces reduces to the simpler theory of linear operators in Hilbert spaces.

We follow Mayer [11] to derive an integral representation for the transfer operator in a Hilbert space for $\mathrm{Re}(s) > \frac{1}{2}$. Let $\mathcal{J}_\nu(u)$ denote the Bessel function (see [6]),

$$\mathcal{J}_\nu(u) = \sum_{k=0}^{\infty} \left(\frac{u}{2}\right)^{2k+\nu} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}. \tag{3.1}$$

Then it is not difficult to check the following inequality:

$$\int_0^\infty \int_0^\infty |\mathcal{J}_{2s-1}(2\sqrt{tt'})|^2 \frac{dt}{e^t-1} \frac{dt'}{e^{t'}-1} < \infty, \quad \mathrm{Re}(s) > \frac{1}{2}. \tag{3.2}$$

Thus, in the domain $\mathrm{Re}(s) > \frac{1}{2}$, the Bessel function $\mathcal{J}_{2s-1}(2\sqrt{tt'})$ can be viewed as a Hilbert–Schmidt kernel with respect to the measure

$$dm(t) = \frac{dt}{e^t-1}. \tag{3.3}$$

Therefore, we can define a Hilbert–Schmidt integral operator $\mathcal{K}_s$ given by

$$\mathcal{K}_s\varphi(t) = \int_0^\infty \mathcal{J}_{2s-1}(2\sqrt{tt'})\varphi(t')dm(t'), \quad \varphi \in L_2(\mathbb{R}^+, dm), \tag{3.4}$$

where $L_2(\mathbb{R}^+, dm)$ denotes the Hilbert space of square integrable functions on the positive real axis with respect to the measure dm given in (3.3). Obviously, the operator $\mathcal{K}_s$ is bounded in this space for $\mathrm{Re}(s) > \frac{1}{2}$.

We are going to explain how this operator is related to the Mayer transfer operator. First, we consider the following integral transform:

$$(T_s\varphi)(z) = \int_0^\infty e^{-zt}t^{s-\frac{1}{2}}\varphi(t)dm(t), \quad \varphi \in L_2(\mathbb{R}^+, dm), \quad z \in \mathbb{C}/(-\infty, -1]. \tag{3.5}$$

This transform can be regarded as the composition of multiplication by the exponential function and the Mellin transform; since both of them are formally invertible, so is T_s. For a complex s with $\mathrm{Re}(s) > \frac{1}{2}$, consider the space $\mathcal{H}_1$ of holomorphic functions in the domain $z \in \mathbb{C}/(-\infty, -1]$ that is the image of $L_2(\mathbb{R}^+, dm)$ under T_s,

$$\mathcal{H}_1 = \left\{f(z) = (T_s\varphi)(z) \mid \varphi \in L_2(\mathbb{R}^+, dm),\ z \in \mathbb{C}/(-\infty, -1]\right\}. \tag{3.6}$$

The operator

$$\mathcal{L}'_s : \mathcal{H}_1 \to \mathcal{H}_1 \tag{3.7}$$

given by

$$\mathcal{L}'_s = T_s\mathcal{K}_sT_s^{-1} \tag{3.8}$$

is isomorphic to $\mathcal{K}_s$ and has the same spectrum. Shortly we shall see that $\mathcal{L}'_s$ has the same form as Mayer's operator.

Lemma 3.1. *For* $\mathrm{Re}(s) > \frac{1}{2}$*, on the space* $\mathcal{H}_1$ *we have*

$$\mathcal{L}'_s f(z) = \sum_{n=1}^\infty \Big(\frac{1}{z+n}\Big)^{2s} f\Big(\frac{1}{z+n}\Big). \tag{3.9}$$

Proof. Let $f(z) = (T_s\varphi)(z)$. Then

$$\mathcal{L}'_s f(z) = T_s(z)\mathcal{K}_s\varphi = \int_0^\infty dm(t)e^{-zt}t^{s-\frac{1}{2}}(\mathcal{K}_s\varphi)(t), \tag{3.10}$$

or

$$\mathcal{L}'_s f(z) = \int_0^\infty dt\frac{t^{s-\frac{1}{2}}}{e^t-1}e^{-zt}\int_0^\infty \mathcal{J}_{2s-1}(2\sqrt{tt'})\varphi(t')dm(t'). \tag{3.11}$$

Formula (3.1) implies

$$\mathcal{J}_{2s-1}(2\sqrt{tt'}) = (tt')^{s-\frac{1}{2}}\sum_{k=0}^\infty \frac{(-tt')^k}{k!\Gamma(k+2s)}. \tag{3.12}$$

Inserting (3.12) in (3.11) and rearranging the terms, we get

$$\mathcal{L}'_s f(z) = \int_0^\infty t'^{s-\frac{1}{2}} \varphi(t') dm(t') \int_0^\infty dt \frac{t^{2s-1}}{e^t - 1} e^{-zt} \sum_{k=0}^\infty \frac{(-tt')^k}{k!\Gamma(k+2s)}, \tag{3.13}$$

or

$$\mathcal{L}'_s f(z) = \int_0^\infty t'^{s-\frac{1}{2}} \varphi(t') dm(t') \sum_{k=0}^\infty \frac{(-t')^k}{k!\Gamma(k+2s)} \int_0^\infty dt \frac{t^{2s+k-1} e^{-zt}}{e^t - 1}. \tag{3.14}$$

But the Hurwitz zeta function has the integral presentation [6],

$$\zeta(w;q) = \frac{1}{\Gamma(w)} \int_0^\infty \frac{t^{w-1} e^{-qt}}{1 - e^{-t}} dt, \quad \mathrm{Re}(w) > 1. \tag{3.15}$$

Thus, the integral over t in (3.14) can be replaced by the Hurwitz zeta function,

$$\mathcal{L}'_s f(z) = \int_0^\infty t'^{s-\frac{1}{2}} \varphi(t') dm(t') \sum_{k=0}^\infty \frac{(-t')^k}{k!} \zeta(k+2s; z+1). \tag{3.16}$$

On the other hand, for $\mathrm{Re}(w) > 1$ the Hurwitz zeta function is defined by

$$\zeta(w;q) = \sum_{n=0}^\infty \Big(\frac{1}{q+n}\Big)^w, \tag{3.17}$$

which leads to the formula

$$\mathcal{L}'_s f(z) = \int_0^\infty t'^{s-\frac{1}{2}} \varphi(t') dm(t') \sum_{k=0}^\infty \frac{(-t')^k}{k!} \sum_{n=1}^\infty \Big(\frac{1}{z+n}\Big)^{2s+k}, \tag{3.18}$$

or

$$\mathcal{L}'_s f(z) = \sum_{n=1}^\infty \Big(\frac{1}{z+n}\Big)^{2s} \int_0^\infty t'^{s-\frac{1}{2}} \varphi(t') dm(t') \sum_{k=0}^\infty \frac{(-t')^k}{k!} \Big(\frac{1}{z+n}\Big)^k. \tag{3.19}$$

Since the sum over k is the Taylor expansion of the function $e^{-t'(\frac{1}{z+n})}$, we get

$$\mathcal{L}'_s f(z) = \sum_{n=1}^\infty \Big(\frac{1}{z+n}\Big)^{2s} \int_0^\infty t'^{s-\frac{1}{2}} e^{-t'(\frac{1}{z+n})} \varphi(t') dm(t'), \tag{3.20}$$

or, by (3.5),

$$\mathcal{L}'_s f(z) = \sum_{n=1}^\infty \Big(\frac{1}{z+n}\Big)^{2s} (T_s\varphi)\Big(\frac{1}{z+n}\Big). \tag{3.21}$$

This means that

$$\mathcal{L}'_s f(z) = \sum_{n=1}^\infty \Big(\frac{1}{z+n}\Big)^{2s} f\Big(\frac{1}{z+n}\Big). \tag{3.22}$$

□

As we saw in this lemma, the Mayer transfer operator $\mathcal{L}_s$ and the operator $\mathcal{L}'_s$ are of the same form. Using this fact and considering the spaces on which these operators act, we can see that every eigenfunction of $\mathcal{L}'_s$ is an eigenfunction of $\mathcal{L}_s$. In [13], Mayer proved the converse; thus, we arrive at the following lemma.

Lemma 3.2. *For* $\mathrm{Re}(s) > \frac{1}{2}$, *the operators* $\mathcal{L}'_s$ *and* $\mathcal{L}_s$ *have the same spectrum.*

Corollary 3.3. *In the domain* $\mathrm{Re}(s) > \frac{1}{2}$, *the integral operator* $\mathcal{K}_s$ *and the Mayer transfer operator* $\mathcal{L}_s$ *for* $PGL(2,\mathbb{Z})$ *have the same spectrum counted with multiplicities.*

§4. Calculation of the trace

As was explained in §2, the transfer operator is of trace class. In this part we are going to calculate the trace of the transfer operator and its powers. We illustrate two different approaches for the calculation of trace; the first is based on the contracting property of the map

$$\psi_n(z) = \frac{1}{z+n}, \tag{4.1}$$

which allows us to apply the method of geometric trace, and the second employs the integral representation of the transfer operator.

By using (4.1), the Mayer transfer operator for $PGL(2,\mathbb{Z})$ can be written in the form

$$\mathcal{L}_s f(z) = \sum_{n=1}^{\infty} \psi_n(z)^{2s} f(\psi_n(z)). \tag{4.2}$$

To calculate the trace of $\mathcal{L}_s$, first we need to calculate the trace of the terms

$$\mathcal{L}_{s,n} f(z) = \psi_n(z)^{2s} f(\psi_n(z)), \quad n \in \mathbb{N}. \tag{4.3}$$

With the same arguments as in Subsection 2.1, we can show that the operator $\mathcal{L}_{s,n}$ is nuclear of order zero for all $n \in \mathbb{N}$. For $r \in [1, \frac{1+\sqrt{5}}{2})$, the map $\psi_n(z)$ on D_r has a unique fixed point z_n^* given by

$$z_n^* = -\frac{n}{2} + \frac{\sqrt{n^2+4}}{2}, \tag{4.4}$$

which is obtained by solving the equation

$$\frac{1}{z+n} = z. \tag{4.5}$$

The existence of a unique solution for (4.5) in D_r is crucial for the calculation of the trace of $\mathcal{L}_{s,n}$. Before proceeding further we quote a lemma from [11] concerning the eigenvalues of a general composition operator.

Lemma 4.1. *Let $D \subset \mathbb{C}$ be an arbitrary domain, ψ a holomorphic map on D with a unique fixed point $z^* \in D$, and φ an arbitrary function in the Banach space $B(D)$. Then the spectrum of the composition operator $\mathcal{C}f = \varphi\psi \circ f$ on $B(D)$ consists of simple eigenvalues $\lambda_n = \varphi(z^*)(\psi'(z^*))^n$, $n = 0, 1, \ldots$, which converge to zero as $n \to \infty$.*

Remark 4.2. A contracting map ψ on a domain D is said to be a map of the domain D strictly inside itself if

$$\inf_{z\in D, z'\in\mathbb{C}\setminus D} \|\psi(z) - z'\| \geqslant \delta > 0. \tag{4.6}$$

Such maps allways have a unique fixed point in D.

In accordance with this lemma, the eigenvalues of $\mathcal{L}_{s,n}$ are given by

$$\lambda_m(n) = (\psi_n(z_n^*))^{2s}(\psi_n'(z_n^*))^m = (-1)^m(z_n^*)^{2s}(z_n^*)^{2m}, \quad m \in \mathbb{N}\cup\{0\}\,, \tag{4.7}$$

which are all simple. Therefore, the trace of $\mathcal{L}_{s,n}$ is simply the sum of them,

$$tr\mathcal{L}_{s,n} = \sum_{m=0}^{\infty} \lambda_m(n) = \frac{(z_n^*)^{2s}}{1+(z_n^*)^2}. \tag{4.8}$$

Then we obtain the trace of $\mathcal{L}_s$ for $\mathrm{Re}(s) > \frac{1}{2}$ by summing the traces of all $\mathcal{L}_{s,n}$'s,

$$tr\mathcal{L}_s = \sum_{n=1}^{\infty} \frac{(z_n^*)^{2s}}{1+(z_n^*)^2}, \tag{4.9}$$

where $\mathrm{Re}(s) > \frac{1}{2}$ ensures absolute convergence.

Next we calculate the trace of the powers of the transfer operator. First, we note that

$$\mathcal{L}_s^n = \sum_{i_1\geqslant 1}\sum_{i_2\geqslant 1}\cdots\sum_{i_n\geqslant 1} \mathcal{L}_{s,i_1}\mathcal{L}_{s,i_2}\ldots\mathcal{L}_{s,i_n}, \tag{4.10}$$

where as before

$$\mathcal{L}_{s,i_k}f(z) = \psi_{i_k}(z)^{2s} f(\psi_{i_k}(z)), \tag{4.11}$$

and the composition operator $\mathcal{L}_{s,i_1}\mathcal{L}_{s,i_2}\ldots\mathcal{L}_{s,i_n}$ has the form

$$\begin{aligned} &\mathcal{L}_{s,i_1}\mathcal{L}_{s,i_2}\ldots\mathcal{L}_{s,i_n} f(z) \\ &\quad = \left\{\psi_{i_1}(z)[\psi_{i_2}\psi_{i_1}(z)]\ldots[\psi_{i_n}\psi_{i_{n-1}}\ldots\psi_{i_1}(z)]\right\}^{2s} f(\psi_{i_n}\ldots\psi_{i_1}(z)). \end{aligned} \tag{4.12}$$

For convenience, without fear of confusion we introduce the following notation:

$$\mathcal{L}_{s,\underline{n}} := \mathcal{L}_{s,i_1}\mathcal{L}_{s,i_2}\ldots\mathcal{L}_{s,i_n}, \quad \psi_{\underline{n}} := \psi_{i_n}\ldots\psi_{i_1}(z); \tag{4.13}$$

then formula (4.12) can be written as follows:

$$\mathcal{L}_{s,\underline{n}} f(z) = \left\{ \prod_{k=1}^{n} \psi_{\underline{k}}(z) \right\}^{2s} f(\psi_{\underline{n}}(z)). \tag{4.14}$$

On the other hand, the composition map $\psi_{\underline{n}}$ has a unique fixed point $z_{\underline{n}}$ on D_r, given by a periodic continuous fraction,

$$z_{\underline{n}} = [0, \overline{i_n, i_{n-1}, \dots, i_1}]. \tag{4.15}$$

The uniqueness of the fixed point $z_{\underline{n}}$ enables us to apply Lemma 4.1 to the composition operator $\mathcal{L}_{s,\underline{n}}$ in (4.14). Thus, we get the eigenvalues of $\mathcal{L}_{s,\underline{n}}$:

$$\lambda_m(z_{\underline{n}}) = \left\{ \prod_{k=1}^{n} \psi_{\underline{k}}(z_{\underline{n}}) \right\}^{2s} \Big(\frac{d\psi_{\underline{n}}}{dz}|_{z=z_{\underline{n}}} \Big)^m, \quad m \in \mathbb{N}. \tag{4.16}$$

Using the chain rule and observing that

$$\frac{d\psi_{i_k}}{dz} = (-1)(\psi_{i_k})^2, \quad i_k \in \mathbb{N}, \tag{4.17}$$

we can write the derivative of the composition function $\psi_{\underline{n}}(z)$ as

$$\frac{d}{dz}\psi_{\underline{n}}(z)|_{z=z_{\underline{n}}} = (-1)^n \bigg(\prod_{k=1}^{n} \psi_{i_k} \psi_{\underline{k-1}}(z_{\underline{n}}) \bigg)^2 = (-1)^n \bigg(\prod_{k=1}^{n} \psi_{\underline{k}}(z_{\underline{n}}) \bigg)^2, \tag{4.18}$$

where $\psi_{\underline{0}} := id$ and the last identity simply comes from the definition of $\psi_{\underline{k}}$ in (4.13). By inserting (4.18) in (4.16), the eigenvalues $\lambda_m(z_{\underline{n}})$ can be written as

$$\lambda_m(z_{\underline{n}}) = (-1)^{nm} \left\{ \prod_{k=1}^{n} \psi_{\underline{k}}(z_{\underline{n}}) \right\}^{2s+2m}, \quad m \in \mathbb{N}. \tag{4.19}$$

Moreover, we have

$$\psi_{\underline{k}}(z_{\underline{n}}) = T^{n-k} z_{\underline{n}}, \tag{4.20}$$

where T is the Gauss map given in (2.4). Thus, the eigenvalues of $\mathcal{L}_{s,\underline{n}}$ in (4.19) can be written as

$$\lambda_m(z_{\underline{n}}) = (-1)^{nm} \left\{ \prod_{k=0}^{n-1} T^k z_{\underline{n}} \right\}^{2s+2m}, \quad m \in \mathbb{N}. \tag{4.21}$$

Because of the simplicity of the eigenvalues, the trace of $\mathcal{L}_{\underline{n}}$ is the sum of all $\lambda_m(z_{\underline{n}})$'s,

$$tr\mathcal{L}_{s,\underline{n}} = \sum_{m=0}^{\infty} \lambda_m(z_{\underline{n}}) = \frac{\left\{ \prod_{k=0}^{n-1} T^k z_{\underline{n}} \right\}^{2s}}{1 - (-1)^n \left(\prod_{k=0}^{n-1} T^k z_{\underline{n}} \right)^2}, \tag{4.22}$$

or with a more explicit notation for $\mathcal{L}_{s,\underline{n}}$ and $z_{\underline{n}}$ given in (4.13) and (4.15) respectively,

$$tr\mathcal{L}_{s,i_1}\mathcal{L}_{s,i_2}\dots\mathcal{L}_{s,i_n} = \frac{\left\{\prod_{k=0}^{n-1} T^k[0,\overline{i_n,i_{n-1},\dots,i_1}]\right\}^{2s}}{1-(-1)^n\left(\prod_{k=0}^{n-1} T^k[0,\overline{i_n,i_{n-1},\dots,i_1}]\right)^2}. \tag{4.23}$$

Finally, by (4.10), we obtain the trace of $\mathcal{L}_s^n$ by summing the contribution of all $\mathcal{L}_{i_1}\mathcal{L}_{i_2}\dots\mathcal{L}_{i_n}$'s:

$$tr\mathcal{L}_s^n = \sum_{i_1\geqslant 1}\sum_{i_2\geqslant 1}\cdots\sum_{i_n\geqslant 1} \frac{\left\{\prod_{k=0}^{n-1} T^k[0,\overline{i_n,i_{n-1},\dots,i_1}]\right\}^{2s}}{1-(-1)^n\left(\prod_{k=0}^{n-1} T^k[0,\overline{i_n,i_{n-1},\dots,i_1}]\right)^2}. \tag{4.24}$$

We note that, in formula (4.24), the sum is over the set of all fixed points $[0,\overline{i_n,i_{n-1},\dots,i_1}]$ of the map $\psi_{\underline{n}}$, as mentioned in (4.15). But this set coincides with the set of fixed points of the Gauss map $FixT^n$. Thus, (4.24) can be written in a more compact form

$$tr\mathcal{L}_s^n = \sum_{x\in FixT^n} \frac{\left\{\prod_{k=0}^{n-1} T^k(x)\right\}^{2s}}{1-(-1)^n\left(\prod_{k=0}^{n-1} T^k(x)\right)^2}. \tag{4.25}$$

4.1. Calculation of the trace via the integral representation. In this subsection we calculate the trace of the integral operator

$$\mathcal{K}_s\varphi(t) = \int_0^\infty \mathcal{J}_{2s-1}(2\sqrt{tt'})\varphi(t')dm(t'), \quad \varphi\in L_2(\mathbb{R}^+,dm), \tag{4.26}$$

for $\mathrm{Re}(s) > \frac{1}{2}$. By the standard results of the theory of linear operators in Hilbert spaces, the trace of $\mathcal{K}_s$ is given by the integral

$$tr\mathcal{K}_s = \int_0^\infty \mathcal{J}_{2s-1}(2t)dm(t), \tag{4.27}$$

or by inserting the measure (3.3),

$$tr\mathcal{K}_s = \int_0^\infty \frac{\mathcal{J}_{2s-1}(2t)}{e^t-1}dt. \tag{4.28}$$

To calculate this integral, we use the identity

$$\frac{1}{e^t-1} = \sum_{n=1}^\infty e^{-nt}, \tag{4.29}$$

obtaining

$$tr\mathcal{K}_s = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt} \mathcal{J}_{2s-1}(2t)dt. \tag{4.30}$$

The integral above can be calculated,

$$\int_0^{\infty} e^{-nt} \mathcal{J}_{2s-1}(2t)dt = \frac{x_n^{2s}}{1+x_n^2}, \quad x_n = -\frac{n}{2} + \frac{\sqrt{n^2+4}}{2}. \tag{4.31}$$

Therefore,

$$tr\mathcal{K}_s = \sum_{n=1}^{\infty} \frac{x_n^{2s}}{1+x_n^2} \tag{4.32}$$

which coincides with (4.9), i.e.,

$$tr\mathcal{K}_s = tr\mathcal{L}_s, \tag{4.33}$$

as was expected in view of Corollary 3.3. To calculate the trace of powers of $\mathcal{K}_s$, it is not difficult to show that

$$tr\mathcal{K}_s^n = \int_0^{\infty} dm(t_n) \\ \dots \int_0^{\infty} dm(t_1)\mathcal{J}_{2s-1}(2\sqrt{t_1 t_2}) \dots \mathcal{J}_{2s-1}(2\sqrt{t_{n-1}t_n})\mathcal{J}_{2s-1}(2\sqrt{t_n t_1}). \tag{4.34}$$

Calculating this integral, we arrive at the expected result:

$$tr\mathcal{K}_s^n = tr\mathcal{L}_s^n. \tag{4.35}$$

We do not know of any simple direct proof of (4.35) (see [11]; a similar calculation was done in [20]).

§5. Ruelle's zeta function and transfer operator

In this section we shall denote various zeta functions by other letters, not necessarily ζ.

As was mentioned in the §1, the Ruelle zeta function for a given weighted dynamical system (Λ, F, g) is defined by

$$\zeta_R(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in Fix F^n} \prod_{k=0}^{n-1} g(F^k x) \right). \tag{5.1}$$

The Ruelle zeta function for the dynamical system $\mathcal{D}_2$ defined in (2.15) at the point $z = 1$ reduces to the dynamical function $\xi(s)$ given by

$$\xi(s) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{x\in FixT^n}\prod_{k=0}^{n-1}(T^k x)^{2s}\right). \tag{5.2}$$

In the following lemma we see the close relationship between the Mayer transfer operator and the dynamical function $\xi(s)$.

Lemma 5.1. *Let $\mathcal{L}_s$ be the Mayer transfer operator for $PGL(2,\mathbb{Z})$ given in* (2.40)*, and let $\xi(s)$ be the zeta function defined by* (5.2)*. For* $\mathrm{Re}(s) > \frac{1}{2}$*, we have*

$$\frac{\det(1+\mathcal{L}_{s+1})}{\det(1-\mathcal{L}_s)} = \xi(s), \tag{5.3}$$

where the determinant of the transfer operator is defined in the sense of Grothendieck by (2.39)*.*

Proof. From Lemma 2.9 we know that for $\mathrm{Re}(s) > \frac{1}{2}$, the transfer operators $\mathcal{L}_s$ and $\mathcal{L}_{s+1}$ are both nuclear of order zero. Therefore, by (2.39) we have

$$\det(1+\mathcal{L}_{s+1}) = \exp\left(-\sum_{n=1}^{\infty}\frac{(-1)^n}{n} tr\mathcal{L}_{s+1}^n\right) \tag{5.4}$$

and

$$\det(1-\mathcal{L}_s) = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n} tr\mathcal{L}_s^n\right). \tag{5.5}$$

Dividing (5.4) by (5.5), we get

$$\frac{\det(1+\mathcal{L}_{s+1})}{\det(1-\mathcal{L}_s)} = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\left\{-(-1)^n tr\mathcal{L}_{s+1}^n + tr\mathcal{L}_s^n\right\}\right), \quad \mathrm{Re}(s) > \frac{1}{2}. \tag{5.6}$$

On the other hand, inserting the traces of $\mathcal{L}_s$ and $\mathcal{L}_{s+1}$ from (4.25), we obtain

$$\left\{-(-1)^n tr\mathcal{L}_{s+1}^n + tr\mathcal{L}_s^n\right\} = \sum_{x\in FixT^n}\prod_{k=0}^{n-1}\left(T^k(x)\right)^{2s}, \tag{5.7}$$

which completes the proof. □

Now we consider the dynamical system $\mathcal{D}_1$ defined in (2.7), which is closely related to the geodesic flow on the upper half-plane mod $PSL(2,\mathbb{Z})$. The Ruelle zeta function (5.1) for the dynamical system $\mathcal{D}_1$ at $z = 1$ reduces to the zeta function $\eta(s)$ given by

$$\eta(s) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{(x,\epsilon)\in FixP_{ex}^n}\prod_{k=0}^{n-1} g(P_{ex}^k(x,\epsilon))\right). \tag{5.8}$$

From (2.5) we have

$$P_{ex}(x,\epsilon) = (Tx, -\epsilon), \tag{5.9}$$

where T is the Gauss map. Formula (5.9) shows that, obviously, the odd powers of P_{ex} have no fixed points, and therefore, summation in (5.8) is restricted to the even integers $n \in \mathbb{N}$,

$$\eta(s) = \exp\left(\sum_{n\ even} \frac{1}{n} \sum_{(x,\epsilon)\in FixP_{ex}^n} \prod_{k=0}^{n-1} g(P_{ex}^k(x,\epsilon))\right). \tag{5.10}$$

On the other hand, by iterating (5.9) we get

$$P_{ex}^k(x,\epsilon) = (T^k x, (-1)^k \epsilon)). \tag{5.11}$$

Therefore, (2.6) and (5.11) imply

$$g(P_{ex}^k(x,\epsilon)) = (T^k x)^{2s}. \tag{5.12}$$

Inserting (5.12) in (5.10), we have

$$\eta(s) = \exp\left(\sum_{n\ even} \frac{1}{n} \sum_{(x,\epsilon)\in FixP_{ex}^n} \prod_{k=0}^{n-1} (T^k x)^{2s}\right). \tag{5.13}$$

Finally, we note that for even $n \in \mathbb{N}$ every pair of fixed points $(x, \pm 1)$ of the map P_{ex}^n corresponds to the fixed point x of the map T^n. Consequently, summation over $FixP_{ex}^n$ can be replaced by twice the sum over the set $FixT^n$ defined in (**??**), that is,

$$\eta(s) = \exp\left(2\sum_{n\ even} \frac{1}{n} \sum_{x\in FixT^n} \prod_{k=0}^{n-1} (T^k x)^{2s}\right). \tag{5.14}$$

The next lemma shows how $\eta(s)$ is related to the transfer operator.

Lemma 5.2. *For* $\mathrm{Re}(s) > \frac{1}{2}$, *we have*

$$\frac{\det(1-\mathcal{L}_{s+1}^2)}{\det(1-\mathcal{L}_s^2)} = \eta(s), \tag{5.15}$$

where $\mathcal{L}_s$ *denotes the transfer operator for* $PGL(2,\mathbb{Z})$, *and the determinants are defined in the sense of Grothendieck.*

Proof. As in the previous lemma, in accordance with Grothendieck's Fredholm determinant (2.39) for nuclear operators, we have

$$\det(1-\mathcal{L}_{s+1}^2) = \exp\left(-\sum_m \frac{1}{m} tr\mathcal{L}_{s+1}^{2m}\right), \quad \mathrm{Re}(s) > \frac{1}{2}, \tag{5.16}$$

and

$$\det(1-\mathcal{L}_s^2)=\exp\Big(-\sum_m \frac{1}{m} tr\mathcal{L}_s^{2m}\Big),\quad \mathrm{Re}(s)>\frac{1}{2}. \tag{5.17}$$

Then

$$\frac{\det(1-\mathcal{L}_{s+1}^2)}{\det(1-\mathcal{L}_s^2)}=\exp\Big(\sum_m \frac{1}{m}\left\{-tr\mathcal{L}_{s+1}^{2m}+tr\mathcal{L}_s^{2m}\right\}\Big),\quad \mathrm{Re}(s)>\frac{1}{2}, \tag{5.18}$$

or equivalently

$$\frac{\det(1-\mathcal{L}_{s+1}^2)}{\det(1-\mathcal{L}_s^2)}=\exp\Big(2\sum_{n\ even} \frac{1}{n}\left\{-tr\mathcal{L}_{s+1}^{n}+tr\mathcal{L}_s^{n}\right\}\Big),\quad \mathrm{Re}(s)>\frac{1}{2}. \tag{5.19}$$

On the other hand, by (4.25) and a simple algebraic calculation, we get

$$\left\{-tr\mathcal{L}_{s+1}^{n}+tr\mathcal{L}_s^{n}\right\}=\sum_{x\in FixT^n}\prod_{k=0}^{n-1}\Big(T^k(x)\Big)^{2s},\quad n \text{ even}. \tag{5.20}$$

Inserting this in (5.19), we arrive at the desired result. □

Corollary 5.3. *Let* $\widetilde{\mathcal{L}}_s$ *denote the transfer operator for* $PSL(2,\mathbb{Z})$*. For* $\mathrm{Re}(s)>\frac{1}{2}$*, we have*

$$\frac{\det(1-\widetilde{\mathcal{L}}_{s+1})}{\det(1-\widetilde{\mathcal{L}}_s)}=\eta(s). \tag{5.21}$$

Proof. The representation of Mayer's transfer operator for $PSL(2,\mathbb{Z})$ given in (2.11) immediately leads to

$$\det(1-\widetilde{\mathcal{L}}_s)=\det(1-\mathcal{L}_s)\det(1+\mathcal{L}_s)=\det(1-\mathcal{L}_s^2). \tag{5.22}$$

Together with the previous lemma, this gives the desired result. □

§6. Selberg's zeta function and transfer operator

In this section we illustrate the relationship between the Selberg zeta function and the Mayer transfer operator, which is a most important aspect of Mayer's theory. First, we recall the definition of the Selberg zeta function. Let Γ be a Fuchcian group of the first kind. The Selberg zeta function for Γ is defined in the domain $\mathrm{Re}(s)>1$ by an absolutely convergent infinite product (see [17])

$$Z_\Gamma(s)=\prod_{k=0}^{\infty}\prod_{\{P\}_\Gamma}(1-\mathcal{N}(P)^{-k-s}), \tag{6.1}$$

where P runs over all primitive hyperbolic conjugacy classes of Γ and $\mathcal{N}(P) > 1$ denotes the norm of P. By definition, every hyperbolic element P of the group Γ is conjugated by an element from $PSL(2,\mathbb{R})$ to a (2×2)-matrix

$$\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \tag{6.2}$$

with $\rho > 1$, which gives the norm of P as $\mathcal{N}(P) = \rho^2$. With the help of the Selberg trace formula, it is proved that

(1) $Z_\Gamma(s)$ has analytic (meromorphic) continuation to the entire complex s-plane;

(2) $Z_\Gamma(s)$ satisfies the functional equation

$$Z_\Gamma(1-s) = \Psi(s) Z_\Gamma(s)$$

with some known function Ψ;

(3) the nontrivial zeros of $Z_\Gamma(s)$ are related to eigenvalues and resonances of the automorphic Laplacian $A(\Gamma)$ for the group Γ.

It is well known that there is a one-to-one correspondence between the primitive hyperbolic conjugacy classes of Γ represented by P with norm $\mathcal{N}(P)$ and the prime closed geodesics c on the Riemann surface $H \setminus \Gamma$(with obvious singularities in some cases) with length $\ell(c)$ such that

$$\mathcal{N}(P) = e^{\ell(c)}. \tag{6.3}$$

This fact enables us to define the Selberg zeta function in the following equivalent form:

$$Z_\Gamma(s) = \prod_{k=0}^{\infty} \prod_{\{c\}_\Gamma} (1 - \exp(-(s+k)\ell(c))), \tag{6.4}$$

where $\{c\}_\Gamma$ runs over all prime closed geodesics on $H \setminus \Gamma$.

In the sequel we shall need an Euler product for the dynamical function $\eta(s)$ defined in (5.14),

$$\eta(s) = \exp\Big(\sum_{n\ even} \frac{1}{n} \sum_{x \in Fix T^n} \Big(\prod_{k=0}^{n-1} T^k x \Big)^{2s} \Big). \tag{6.5}$$

Following Ruelle [15], we rewrite $\eta(s)$ as an Euler product. To begin with, we recall that for $x \in [0,1]$, the set

$$\phi = \{x, Tx, \dots, T^k x, \dots\} \tag{6.6}$$

is an orbit of the Gauss map T given in (2.4). The orbit ϕ is periodic if there exists an integer $u \in \mathbb{N}$ such that

$$T^u x = x. \tag{6.7}$$

The integer $u \in \mathbb{N}$ is called a period of the periodic orbit ϕ. We say that a periodic orbit ϕ has primitive of minimal period m if m is the minimum of the set of all periods of ϕ. We denote the set of all primitive periodic orbits of T of minimal period m by $Per(m)$.

Now we are going to replace the sum over $FixT^n$ in (6.5) by a sum over the primitive periodic orbits $Per(m)$. For this, first we introduce the subset $MFixT^m \subset FixT^m$, containing the periodic continued fractions with minimum period m. Then

$$\sum_{x\in FixT^n}\left(\prod_{k=0}^{n-1}T^k x\right)^{2s}=\sum_{m|n}\ \sum_{x\in MFixT^m}\ \prod_{k=0}^{(\frac{n}{m})m-1}(T^k x)^{2s}, \tag{6.8}$$

or, observing that $x \in MFixT^m$ is of period m,

$$\sum_{x\in FixT^n}\left(\prod_{k=0}^{n-1}T^k x\right)^{2s}=\sum_{m|n}\ \sum_{x\in MFixT^m}\left[\prod_{k=0}^{m-1}(T^k x)^{2s}\right]^{\frac{n}{m}}. \tag{6.9}$$

Now by replacing the sum over $MFixT^m$ by a sum over $Per(m)$, we get

$$\sum_{x\in FixT^n}\left(\prod_{k=0}^{n-1}T^k x\right)^{2s}=\sum_{m|n}\ \sum_{\phi\in Per(m)} m\left[\prod_{k=0}^{m-1}(T^k x_\phi)^{2s}\right]^{\frac{n}{m}}, \tag{6.10}$$

where x_ϕ is an arbitrary point of the orbit $\phi \in Per(m)$ and the factor m comes from the fact that a periodic orbit ϕ containing a point $x \in MFixT^m$ contains also the set of points

$$\left\{T^k x \in MFixT^m \mid k=0,\ldots,m-1\right\}. \tag{6.11}$$

Inserting (6.10) in (6.5), we see that

$$\eta(s)=\exp\left(2\sum_{n\ even}\frac{1}{n}\sum_{m|n}\ \sum_{\phi\in Per(m)} m\left[\prod_{k=0}^{m-1}(T^k x_\phi)^{2s}\right]^{\frac{n}{m}}\right), \tag{6.12}$$

or, by rearranging the sum,

$$\eta(s)=\exp\left(2\sum_{r\ even}\ \sum_{\phi\in Per(r)}\ \sum_{q=0}^{\infty}\frac{1}{q}\left[\prod_{k=0}^{r-1}(T^k x_\phi)^{2s}\right]^{q}\right). \tag{6.13}$$

Since

$$-\log(1-w)=\sum_{q=1}^{\infty}\frac{1}{q}w^q, \tag{6.14}$$

formula (6.13) reduces to

$$\eta(s) = \exp\Bigg(-2 \sum_{r\ even} \sum_{\phi \in Per(r)} \log \Bigg[1 - \prod_{k=0}^{r-1} (T^k x_\phi)^{2s} \Bigg] \Bigg), \tag{6.15}$$

or

$$\eta(s) = \exp\Bigg(\sum_{r\ even} \sum_{\phi \in Per(r)} \log \Bigg[1 - \prod_{k=0}^{r-1} (T^k x_\phi)^{2s} \Bigg]^{-2} \Bigg), \tag{6.16}$$

which leads finally to the desired Euler product,

$$\eta(s) = \prod_{r\ even} \prod_{\phi \in Per(r)} \frac{1}{(1 - \prod_{k=0}^{r-1} (T^k x_\phi)^{2s})^2}. \tag{6.17}$$

The Euler product for $\eta(s)$ given above is crucial for the following lemma which is a bridge between the Mayer transfer operator and Selberg zeta function.

Lemma 6.1. *For* $\mathrm{Re}(s) > 1$*, we have*

$$Z(s)^{-1} = \prod_{l=0}^{\infty} \eta(s+l), \tag{6.18}$$

where $Z(s)$ *is the Selberg zeta function for the group* $PSL(2,\mathbb{Z})$ *and* $\eta(s)$ *is defined in* (6.5) *with the Euler product given in* (6.17)*.*

Proof. First, we need to rewrite (6.17) as a product over the primitive priodic orbits of the map P_{ex} defined in (2.5). Let $\widehat{Per}(r)$ denote the set of primitive periodic orbits of minimal period $r = 2a$ with $a \in \mathbb{N}$ for the map P_{ex}. In accordance with (5.11), an element $\widehat{\phi} \in \widehat{Per}(r)$ passing the point (x, ϵ) is defined by the following set:

$$\widehat{\phi} = \Big\{ (T^k x, (-1)^k \epsilon) \mid k = 0, \ldots, r-1,\ \epsilon = \pm 1 \Big\}, \tag{6.19}$$

where $x \in [0,1]$ and $\epsilon = \pm 1$. We note that all periodic orbits of P_{ex} have even period r. Obviously, every two elements of $\widehat{Per}(r)$ correspond to an element of $Per(r)$. Noting that the terms in the product (6.17) do not depend on ϵ, we see that the power 2 in the denominator of (6.17) disappears if we replace $Per(r)$ by $\widehat{Per}(r)$,

$$\eta(s) = \prod_{r\ even} \prod_{\widehat{\phi} \in \widehat{Per}(r)} \frac{1}{1 - \prod_{k=0}^{r-1} (T^k x_{\widehat{\phi}})^{2s}}, \tag{6.20}$$

where $x_{\widehat{\phi}}$ is an arbitrary point of $\widehat{\phi}$. On the other hand, according to Series [2] and Adler and Flatto [1], there is a one-to-one correspondence between $\widehat{Per}(r)$

and the set of primitive periodic orbits ϑ on the unit tangent bundle $T_1 M$, $M = PSL(2,\mathbb{Z})\backslash\mathbb{H}$, with the period (see [12])

$$\tau(\vartheta) = -2ln \prod_{k=0}^{r-1} T^k x_{\widehat{\phi}}. \tag{6.21}$$

These facts recover the physical situation behind the abstract number-theoretic appearance of the problem, leading to the following equivalent formula for the dynamical zeta function in (6.20):

$$\eta(s) = \prod_{\{\vartheta\}_\Gamma} \frac{1}{1 - \exp(-s\tau(\vartheta))}, \tag{6.22}$$

where $\{\vartheta\}_\Gamma$ denotes the set of all primitive periodic orbits on $T_1 M$, $M = \Gamma\backslash\mathbb{H}$ with $\Gamma = PSL(2,\mathbb{Z})$. But there is also a one-to-one correspondence between the primitive periodic orbits on $T_1 M$ and the closed geodesics c on M with length $\ell(c) = \tau(\vartheta)$. Thus, we have another equivalent formula for our zeta function

$$\eta(s) = \prod_{\{c\}_\Gamma} \frac{1}{1 - \exp(-\beta\ell(c))}. \tag{6.23}$$

Here $\{c\}_\Gamma$ denotes the set of primitive closed geodesics on $M = \Gamma\backslash\mathbb{H}$ with $\Gamma = PSL(2,\mathbb{Z})$, and $\ell(c)$ is the length of the closed geodesic c. Note that because of the unity of the tangent bundle, $\ell(c) = \tau(\vartheta)$. Finally, inserting (6.23) in (6.4) provides the desired result. □

This lemma immediately leads to the most important feature of the Mayer transfer operator theory, namely, the following is true.

Theorem 6.2. *For* $\mathrm{Re}(s) > 1$, *we have*

$$\det(1 - \widetilde{\mathcal{L}}_s) = Z(s) \tag{6.24}$$

and

$$\det(1 - \mathcal{L}_s^2) = Z(s), \tag{6.25}$$

where $Z(s)$ *denotes the Selberg zeta function for the group* $PSL(2,\mathbb{Z})$, $\widetilde{\mathcal{L}}_s$ *is the transfer operator also for* $PSL(2,\mathbb{Z})$, *and* $\mathcal{L}_\beta$ *is the transfer operator for* $PGL(2,\mathbb{Z})$.

Proof. It suffices to insert (5.21) and (5.15) in (6.18). □

Remark 6.3. The domain of validity of (6.24) and (6.25) extends immediately to all $s \in \mathbb{C}$ except possibly some small singular set, because the Selberg zeta function is a meromorphic function on the entire plane $\mathbb{C}$.

§7. Number theoretic approach to the relationship between Selberg's zeta function and Mayer's transfer operator

In the previous section, based on the one-to-one correspondence between the primitive periodic orbits of P_{ex} and primitive periodic orbits on the phase space T_1M, we passed from the number theoretic appearance of the problem to its dynamical nature. In this physical realization of the problem, we can see the relationship of the Selberg zeta function with the Mayer transfer operator.

Efrat [4] and later Lewis and Zagier [8] reproved Mayer's result in a purely number theoretic approach. In this section we are going to illustrate the alternative approach of Lewis and Zagier. First, we introduce their notation. Let $\gamma \in GL(2,\mathbb{Z})$ act on D_r via a linear fractional transformation. The right action of the semigroup

$$\Xi = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}) \mid \gamma(D_r) \subseteq D_r \right\} \tag{7.1}$$

on the space $B(D_r)$ is given by

$$\pi_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (cz+d)^{-2s} f\Big(\frac{az+b}{cz+d}\Big), \tag{7.2}$$

where D_r and $B(D_r)$ are the same as in §2. Then the Mayer transfer operator for $PGL(2,\mathbb{Z})$ in the domain $\mathrm{Re}(s) > \frac{1}{2}$ can be represented by

$$\mathcal{L}_s = \sum_{n=1}^{\infty} \pi_s \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}. \tag{7.3}$$

We note that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & n \end{smallmatrix}\right) \in \Xi$ for all $n \in \mathbb{N}$. The set of the so-called reduced elements of $SL(2,\mathbb{Z})$ is defined by

$$\mathrm{Red} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid 0 \leqslant a \leqslant b,\ c \leqslant d \right\}. \tag{7.4}$$

For a hyperbolic element γ of $SL(2,\mathbb{Z})$, a positive integer $k = k(\gamma)$ is defined to be the largest integer such that $\gamma = \gamma_1^k$ for some $\gamma_1 \in SL(2,\mathbb{Z})$. Therefore, for a primitive hyperbolic element we have $k = 1$. Now we quote the heart of the proof of Lewis and Zagier, based on a classical reduction theory for quadratic forms, as a lemma whose proof one can find in [8].

Lemma 7.1.

(1) *Every reduced matrix $\gamma \in Red$ has a unique decomposition of the form*

$$\begin{pmatrix} 0 & 1 \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & n_{2l} \end{pmatrix}, \quad n_1, \dots, n_{2l} \geqslant 1, \tag{7.5}$$

for a unique positive integer $l = l(\gamma)$, called the length of γ.

(2) *There are $2l(\gamma)/k(\gamma)$ reduced representatives with the same length $l(\gamma)$ in every hyperbolic conjugacy class of $SL(2,\mathbb{Z})$ containing γ.*

Next, consider the Selberg zeta function for the group $\Gamma = SL(2,\mathbb{Z})$ for $\mathrm{Re}(s) > 1$,

$$Z_\Gamma(s) = \prod_{m=0}^{\infty} \prod_{\{P\}_\Gamma} (1 - \mathcal{N}(P)^{-m-s}). \tag{7.6}$$

Taking the logarithm of both sides and invoking the Taylor expansion of $\log(1 - \mathcal{N}(P)^{-m-s})$, we get

$$-\log Z(s) = \sum_{\{P\}_\Gamma} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} \mathcal{N}(P)^{-k(m+s)}. \tag{7.7}$$

The absolute convergence of the product (7.6) implies that of the sum above. Thus, the interchange of the sums over m and k is allowed,

$$-\log Z(s) = \sum_{\{P\}_\Gamma} \sum_{k=1}^{\infty} \frac{1}{k} \mathcal{N}(P)^{-ks} \sum_{m=0}^{\infty} \mathcal{N}(P)^{-km}, \tag{7.8}$$

but the sum over m is the Taylor expansion of $(1 - \mathcal{N}(P)^{-k})^{-1}$, so that

$$-\log Z(s) = \sum_{\{P\}_\Gamma} \sum_{k=1}^{\infty} \frac{1}{k} \frac{\mathcal{N}(P)^{-ks}}{1 - \mathcal{N}(P)^{-k}}. \tag{7.9}$$

Since $\mathcal{N}(P)^k = \mathcal{N}(P^k)$, we can consider the double sum over k and $\{P\}_\Gamma$, the primitive hyperbolic conjugacy classes, as a single sum over all, not only primitive hyperbolic conjugacy classes, denoted by $\{\gamma\}_\Gamma$,

$$-\log Z(s) = \sum_{\{\gamma\}_\Gamma} \frac{1}{k(\gamma)} \frac{\mathcal{N}(\gamma)^{-s}}{1 - \mathcal{N}(\gamma)^{-1}}. \tag{7.10}$$

The second part of Lemma 7.1 enables us to replace the sum over the hyperbolic conjugacy classes by the the sum over the set Red of reduced matrices:

$$-\log Z(s) = \sum_{\gamma \in \mathrm{Red}} \frac{1}{2l(\gamma)} \frac{\mathcal{N}(\gamma)^{-s}}{1 - \mathcal{N}(\gamma)^{-1}}. \tag{7.11}$$

Now we need the following lemma.

Lemma 7.2. *For $\gamma \in \Xi$, the trace of the operator $\pi_s(\gamma)$ acting on $B(D_r)$ is given by*

$$tr(\pi_s(\gamma)) = \frac{\mathcal{N}(\gamma)^{-s}}{1 - \mathcal{N}(\gamma)^{-1}}. \tag{7.12}$$

Proof. Let ψ_γ denote the action of $\gamma \in \Xi$ on $\overline{D}_r$,

$$\psi_\gamma(z) = \gamma z := \frac{az+b}{cz+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{7.13}$$

We also put

$$j(\gamma, z) = cz + d. \tag{7.14}$$

Then the operator $\pi_s(\gamma)$ is written in the form

$$\pi_s(\gamma) f(z) = j(\gamma, z)^{-2s} f(\psi_\gamma(z)). \tag{7.15}$$

Since the definiton of Ξ in (7.1) shows that ψ_γ maps $\overline{D}_r$ strictly inside itself, we can apply Lemma 4.1 to get the eigenvalues of $\pi_s(\gamma)$, which are all simple. These eigenvalues are given by

$$\lambda_m(\gamma) = j(\gamma, x^*)^{-2s} \left[\frac{d\psi_\gamma}{dz}|_{z=x^*} \right]^m, \tag{7.16}$$

where x^* is a unique fixed point of ψ_γ in $\overline{D}_r$. The existence of a unique fixed point comes from Remark 4.2. On the other hand, we note that

$$\frac{d\psi_\gamma}{dz} = \frac{1}{j(\gamma, z)^2}, \tag{7.17}$$

whence (7.16) reduces to

$$\lambda_m(\gamma) = j(\gamma, x^*)^{-2s-2m}. \tag{7.18}$$

The sum of all $\lambda_m(\gamma)$ gives the trace of $\pi_s(\gamma)$,

$$tr(\pi_s(\gamma)) = \sum_{m=1}^{\infty} j(\gamma, x^*)^{-2s-2m} = \frac{j(\gamma, x^*)^{-2s}}{1 - j(\gamma, x^*)^{-2}}. \tag{7.19}$$

To complete the proof we must show that $\mathcal{N}(\gamma)^{-1} = j(\gamma, x^*)^{-2}$. For this, we observe that

$$j(g\gamma g^{-1}, gx^*) = j(\gamma, x^*), \quad g \in SL(2, \mathbb{R}), \tag{7.20}$$

but there exists an element $g \in SL(2, \mathbb{R})$ such that

$$gx^* = 0, \quad g\gamma g^{-1} = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix} \in \Xi, \quad \rho > 1. \tag{7.21}$$

Thus,

$$j(\gamma, x^*) = j(g\gamma g^{-1}, gx^*) = \rho = \sqrt{\mathcal{N}(\gamma)}, \tag{7.22}$$

where the last identity follows from the definition of the norm in §6. Inserting (7.22) in (7.19) completes the proof. □

We note that Red ⊂ Ξ; thus, we can insert (7.12) in (7.11),

$$-\log Z(s) = tr\left(\sum_{\gamma\in\mathrm{Red}} \frac{1}{2l(\gamma)}\pi_s(\gamma)\right) \tag{7.23}$$

but part 1) of Lemma 7.1 allows us to write

$$-\log Z(s) = tr\left(\sum_{l=1}^{\infty} \frac{1}{2l}\left(\sum_{n=1}^{\infty}\pi_s\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}\right)^{2l}\right). \tag{7.24}$$

That is, we have

$$-\log Z(s) = \sum_{l=1}^{\infty} \frac{1}{2l} tr\,(\mathcal{L}_s)^{2l}, \tag{7.25}$$

or

$$Z(s) = \exp\left(-\sum_{l=1}^{\infty} \frac{1}{2l} tr\,(\mathcal{L}_s)^{2l}\right). \tag{7.26}$$

Finally, since the right-hand side of the equation above coincides with the Fredholm determinant of $\mathcal{L}_s^2$, we get the desired result, namely,

$$Z(s) = \det(1 - \mathcal{L}_s^2). \tag{7.27}$$

References

[1] Adler R. L., Flatto L., *Cross section map for the geodesic flow on the modular surface*, Conference in Modern Analysis and Probability (New Haven, 1982), Contemp. Math., vol. 26, Amer. Math. Soc., Providence, RI, 1984, pp. 9–24.

[2] Bowen R., Series C., *Markov maps associated with Fuchsian groups*, Inst. Hautes Études Sci. Publ. Math. No. 50 (1979), 153–170.

[3] Chang C.-H., Mayer D., *Thermodynamic formalism and Selberg's zeta function for modular groups*, Regul. Chaotic Dyn. **5** (2000), 281–312.

[4] Efrat I., *Dynamics of the continued fraction map and the spectral theory of $SL(2,\mathbb{Z})$*, Invent. Math. **114** (1993), 207–218.

[5] Гельфанд И. М., Виленкин Н. Я., *Обобщенные функции.* Вып. 4. *Некоторые применения гармонического анализа. Оснащенные гильбертовы пространства*, Физматгиз, 1961; Пер. на англ. яз., Acad. Press, New York–London, 1964.

[6] Градштейн И. С., Рыжик И. М., *Таблицы интегралов, сумм, рядов и произведений*, Физматгиз, М., 1963; Пер. на англ. яз., Acad. Press, New York–London, 1965.

[7] Grothendieck A., *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16 (1955), 140 pp.

[8] Lewis J., Zagier D., *Period functions and the Selberg zeta function for the modular group*, The Mathematical Beauty of Physics (Saclay, 1996), Adv. Ser. Math. Phys., vol. 24, World Sci., Singapore, 1997, pp. 83-97.

[9] Mayer D., *The thermodynamic formalism approach to Selberg's zeta function for* $PSL(2,\mathbb{Z})$, Bull. Amer. Math. Soc. (N.S.) **25** (1991), 55–60.

[10] Mayer D., *On a zeta function related to the continued fraction transformation*, Bull. Soc. Math. France **104** (1976), 195–203.

[11] Mayer D., *Continued fractions and related transformations*, Ergodic Theory, Symbolic Dynamics, and Hyperbolic Spaces (Trieste, 1989), Oxford Univ. Press, New York, 1991, pp. 175–222.

[12] Mayer D., *Thermodynamics formalism and quantum mechanics on the modular surface*, From Phase Transitions to Chaos, World Sci. Publ., River Edge, NJ, 1992, pp. 521–529.

[13] Mayer D., *On the thermodynamic formalism for the Gauss map*, Comm. Math. Phys. **130** (1990), 311–333.

[14] Ruelle D., *Dynamical zeta functions and transfer operators*, Notices Amer. Math. Soc. **49** (2002), 887–895.

[15] Ruelle D., *Dynamical zeta functions for piecewise monotone maps of the interval*, CRM Monogr. Ser., vol. 4, Amer. Math. Soc., Providence, RI, 1994.

[16] Schaefer H. H., *Topological vector spaces*, The Macmillan Co., New York, 1966.

[17] Selberg A., *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) **20** (1956), 47–87.

[18] Синай Я. Г., *Гиббсовские меры в эргодической теории*, Успехи мат. наук **27** (1972), №4, 21–64; Пер. на англ. яз., Russian Math. Surveys **27** (1972), no. 4, 21–69.

[19] Венков А. Б., *Спектральная теория автоморфных функций*, Тр. Мат. ин-та АН СССР **153** (1981), 171 с.; Пер. на англ. яз., Kluwer Acad. Publ. Group, Dordrecht, 1990.

[20] Венков А. Б., *Об автоморфной матрице рассеяния для группы Гекке* $\Gamma[2\cos(\pi/q)]$, Тр. Мат. ин-та АН СССР **163** (1984), 32–36.

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