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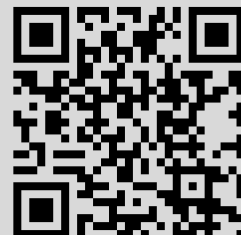
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ON DERIVATIONS ASSOCIATED WITH DIFFERENT ALGEBRAIC  
STRUCTURES IN GROUP ALGEBRAS

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**Abstract.** The paper is devoted to the comparison of derivation algebras arising in associative and Lie structures of group algebras. We shall prove that an algebra of derivations given by Lie-structure contains an algebra of associative derivations. We will give a description of Lie derivations in terms of the gruppoid associated with an inner action of the group.

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Let  $G$  be a discrete group. Denote by  $\mathbf{C}[G]$  the group algebra. The algebra  $\mathbf{C}[G]$  is associative and has the structure of a Lie algebra with the commutator  $[x, y] := xy - yx$ .

The purpose of this paper is the comparison of derivations defined by an associative structure and derivations defined by a Lie structure.

The combinatorial structure of a gruppoid  $\Gamma$  introduced in paper [1] generates derivations for an associative structure and for a Lie structure.

**Definition 1.** A linear mapping  $d : \mathbf{C}[G] \rightarrow \mathbf{C}[G]$  is called a derivation provided

$$d(uv) = d(u)v + ud(v) \quad \forall u, v \in \mathbf{C}[G]. \quad (1)$$

The derivation  $d$  is a derivation with respect to an associative structure. The set of all derivations is a Lie algebra and is denoted by  $Der(\mathbf{C}[G])$ .

**Definition 2.** A linear mapping  $\partial : \mathbf{C}[G] \rightarrow \mathbf{C}[G]$  is called a rate provided

$$\partial([u, v]) = [\partial(u), v] + [u, \partial(v)] \quad \forall u, v \in \mathbf{C}[G]. \quad (2)$$

We denote the linear space of all rates by  $Rate(\mathbf{C}[G])$ . Note that  $Rate(\mathbf{C}[G])$  is not a Lie algebra.

It is a straightforward task to ensure that a derivation is also a rate.

**Proposition 1.** A derivation  $d \in Der(\mathbf{C}[G])$  is a rate. So, the algebra  $Der(\mathbf{C}[G])$  is a subspace of the space  $Rate(\mathbf{C}[G])$ .

*Proof.* It suffices to prove that any derivation  $d \in Der(\mathbf{C}[G])$  satisfies condition (2), i.e.

$$d([u, v]) = [d(u), v] + [u, d(v)]. \quad (3)$$

From the linearity and condition (1) it follows that

$$\begin{aligned} d([u, v]) &= d(uv) - d(vu) = d(u)v + ud(v) - d(v)u - vd(u) = \\ &= d(u)v - vd(u) + ud(v) - d(v)u = [d(u), v] + [u, d(v)]. \end{aligned}$$

Hence,  $Der(\mathbf{C}[G]) \subset Rate(\mathbf{C}[G])$ . □

For an abelian group  $G$  any linear mapping  $\tau : \mathbf{C}[G] \rightarrow \mathbf{C}[G]$  is a rate. Indeed, since group  $G$  is abelian, it is easily seen that

$$0 = d([u, v]) = [d(u), v] + [u, d(v)].$$

For a nontrivial group  $G$  the algebra  $Der(\mathbf{C}[G])$  can not coincide with the space  $Rate(\mathbf{C}[G])$ . Let us illustrate this fact by the following example.

**Example 1.** Define the mapping  $\tau : \mathbf{C}[G] \rightarrow \mathbf{C}[G]$  over generators  $g \in G \subset \mathbf{C}[G]$  as follows:

$$\tau : g \rightarrow e.$$

Here  $e$  is the neutral element of the group  $G$ . Extend the mapping  $\tau$  by linearity as follows:

$$\tau \left( \sum_{g \in G} \alpha_g g \right) = \left( \sum_{g \in G} \alpha_g \right) e.$$

All the indicated sums are finite.

Let us show that the mapping  $\tau$  is a rate but not a derivation. Let elements  $g_1, g_2 \in G \subset \mathbf{C}[G]$  be generators of the group  $G$ . Then we get

$$\tau([g_1, g_2]) = 0 = [\tau(g_1), g_2] + [g_1, \tau(g_2)].$$

So, the mapping  $\tau$  is a rate. But  $\tau$  is not a derivation since

$$\tau(g_1 g_2) = e,$$

whereas

$$\tau(g_1)g_2 + g_1\tau(g_2) = g_2 + g_1 \neq e.$$

**Proposition 2.** *The space  $Rate(\mathbf{C}[G])$  is a Lie algebra.*

*Proof.* By  $\partial_1, \partial_2$  denote the following two rates:

$$\begin{aligned} \partial_1([u, v]) &= [\partial_1(u), v] + [u, \partial_1(v)] \quad \forall u, v \in C[G], \\ \partial_2([u, v]) &= [\partial_2(u), v] + [u, \partial_2(v)] \quad \forall u, v \in C[G], \end{aligned}$$

We claim that the commutator  $[\partial_1, \partial_2] := \partial_1\partial_2 - \partial_2\partial_1$  is a rate.

$$\begin{aligned} \partial_2\partial_1([u, v]) &= \partial_2([\partial_1(u), v] + [u, \partial_1(v)]) = \\ &= [\partial_2\partial_1(u), v] + [\partial_1(u), \partial_2(v)] + [\partial_2(u), \partial_1(v)] + [u, \partial_2\partial_1(v)], \end{aligned}$$

$$\begin{aligned} \partial_1\partial_2([u, v]) &= \partial_1([\partial_2(u), v] + [u, \partial_2(v)]) = \\ &= [\partial_1\partial_2(u), v] + [\partial_2(u), \partial_1(v)] + [\partial_1(u), \partial_2(v)] + [u, \partial_1\partial_2(v)]. \end{aligned}$$

Combining  $\partial_2\partial_1([u, v])$  and  $\partial_1\partial_2([u, v])$  we get

$$\begin{aligned} [\partial_2, \partial_1]([u, v]) &= \partial_2\partial_1([u, v]) - \partial_1\partial_2([u, v]) = \\ &= [\partial_2\partial_1(u), v] + [\partial_1(u), \partial_2(v)] + [\partial_2(u), \partial_1(v)] + [u, \partial_2\partial_1(v)] - \\ &- ([\partial_1\partial_2(u), v] + [\partial_2(u), \partial_1(v)] + [\partial_1(u), \partial_2(v)] + [u, \partial_1\partial_2(v)]) = \\ &= [[\partial_2, \partial_1](u), v] + [u, [\partial_2, \partial_1](v)]. \end{aligned}$$

□

Propositions 1, 2 and Example 1 imply the following assertion.

**Theorem 1.** *If  $G$  is a discrete group then the algebra of all derivations  $\text{Der}(\mathbf{C}[G])$  is a Lie subalgebra of the Lie algebra  $\text{Rate}(\mathbf{C}[G])$ . If  $G$  is a nontrivial group then the algebra of all derivations  $\text{Der}(\mathbf{C}[G])$  cannot coincide with the Lie algebra  $\text{Rate}(\mathbf{C}[G])$ .*

This theorem is presumably known in literature but it is important for the following discussion.

Let  $\partial$  be a rate. Then the action of the mapping  $\partial$  on an element  $u := \sum_{h \in G} \lambda_u^h h \in \mathbf{C}[G]$  can be written in the following way

$$\partial(u) = \sum_{g \in G} \left( \sum_{h \in G} \partial_h^g \lambda_u^h \right) g, \quad (4)$$

where all coefficients  $\partial_h^g \in \mathbf{C}$  are defined by  $\partial$ .

**Proposition 3.** *For any rate  $\partial$  the coefficients  $\partial_h^g$  satisfy the condition*

$$\partial_{uv}^g - \partial_{vu}^g = \partial_u^{gv^{-1}} - \partial_u^{v^{-1}g} + \partial_v^{u^{-1}g} - \partial_v^{gu^{-1}} \forall g, u, v \in G. \quad (5)$$

*Proof.* It suffices to show that Conditions (5) and (2) are equivalent. Let elements  $u, v \in G \subset \mathbf{C}[G]$  be generators. Using (4) we get by (2) that

$$\partial([u, v]) = \partial(uv) - \partial(vu) = \sum_{g \in G} (\partial_{uv}^g - \partial_{vu}^g) g. \quad (6)$$

Therefore, we obtain the following formula for the commutator  $[\partial(u), v]$

$$[\partial(u), v] = \left[ \sum_{g \in G} \partial_u^g g, v \right] = \sum_{g \in G} \partial_u^g g v - \sum_{g \in G} \partial_u^g v g. \quad (7)$$

In the same way we get that

$$[u, \partial(v)] = \left[ u, \sum_{g \in G} \partial_v^g g \right] = \sum_{g \in G} \partial_v^g u g - \sum_{g \in G} \partial_v^g g u. \quad (8)$$

Combining (7) and (8) we obtain

$$\sum_{g \in G} (\partial_{uv}^g - \partial_{vu}^g) g = \sum_{g \in G} (\partial_u^g g v - \partial_u^g v g + \partial_v^g u g - \partial_v^g g u). \quad (9)$$

Since the series in the left-hand side is equal to the series in the right-hand side, we obtain

$$\partial_{uv}^g - \partial_{vu}^g = \partial_u^{gv^{-1}} - \partial_u^{v^{-1}g} + \partial_v^{u^{-1}g} - \partial_v^{gu^{-1}}, \quad (10)$$

for all elements  $u, v, g \in G$ . In the left-hand side and in the right-hand side we have elements of a group algebra, so they are equal if the coefficients at the appropriate group elements are equal.  $\square$

Let us note that Proposition 3 is a necessary condition. To turn it into a sufficient condition we need an assumption on the finiteness of sums in formula (4).

**Definition 3** (A local finiteness condition). A mapping  $\partial : \mathbf{C}[G] \rightarrow \mathbf{C}[G]$  is called locally finite if for each  $y \in G$  the coefficient  $\partial_y^x$  is equal to zero, for almost all (except of finite number) group elements  $x \in G$ .

This means that we can get a theorem with necessary and sufficient conditions. Related questions were discussed in [1].

**Theorem 2.** *A mapping  $\partial : \mathbf{C}[G] \rightarrow \mathbf{C}[G]$  is a rate if and only if it is locally finite and satisfies condition (5).*

*Proof.* The local finiteness condition is equivalent to the fact that the map  $\partial$  is well-defined on the group algebra  $\mathbf{C}[G]$  (see [1]). Condition (5) for the mapping  $\partial$  is equivalent to condition (2) according to Proposition 3.  $\square$

Now we can discuss a combinatorial description of rates in terms of paper [1]. Recall the definition of a gruppoid  $\Gamma$  from paper [1].

Objects of a gruppoid  $\Gamma$  are group elements  $Obj(\Gamma) = \{G\}$ . Morphisms are pairs of group elements  $Mor(\Gamma) = \{G\} \times \{G\}$ . Define the composition of morphisms  $\xi = (a, b), \zeta = (c, d)$  satisfying the condition  $cd^{-1} = b^{-1}a$  as follows

$$\zeta \circ \xi := (da, db).$$

The defined structure  $\Gamma$  is a gruppoid (see [2]).

Define the mapping  $\chi_\partial$  for a rate  $\partial$  as follows

$$\chi_\partial((g, h)) := \partial_h^g.$$

Given elements  $u, v \in G$ , consider the morphisms

$$\phi_1^g = (gv^{-1}, u), \psi_1^g = (v^{-1}g, u), \phi_2^g = (u^{-1}g, v), \psi_2^g = (gu^{-1}, v). \quad (11)$$

In this notation, the following equalities are valid

$$\phi_1^g \circ \phi_2^g = (g, uv), \quad \psi_2^g \circ \psi_1^g = (g, vu).$$

Hence, formula (5) takes the following form

$$\chi_\partial(\phi_1^g \circ \phi_2^g) - \chi_\partial(\psi_2^g \circ \psi_1^g) = \chi_\partial(\phi_1^g) + \chi_\partial(\phi_2^g) - \chi_\partial(\psi_1^g) - \chi_\partial(\psi_2^g). \quad (12)$$

Finally, we obtain

$$\chi_\partial(\phi_1^g \circ \phi_2^g) - \chi_\partial(\phi_1^g) - \chi_\partial(\phi_2^g) = \chi_\partial(\psi_2^g \circ \psi_1^g) - \chi_\partial(\psi_2^g) - \chi_\partial(\psi_1^g). \quad (13)$$

Note that in the special case in which  $\partial$  is a derivation (not just a rate), by paper [1] we get the following stronger condition

$$\chi_\partial(\phi_1^g \circ \phi_2^g) - \chi_\partial(\phi_1^g) - \chi_\partial(\phi_2^g) = 0.$$

In this case the mapping  $\chi_\partial$  is called a character (see [1]).

Assume that elements  $u, v, g$  do not commute, then we have the following commutative diagrams

$$\begin{array}{ccc} u^{-1}v^{-1}g & \xrightarrow{\phi_1^g \circ \phi_2^g} & gv^{-1}u^{-1} \\ & \searrow \phi_2^g & \nearrow \phi_1^g \\ & u^{-1}gv^{-1} & \end{array} \quad (*)$$

$$\begin{array}{ccc}
v^{-1}u^{-1}g & \xrightarrow{\psi_2^g \circ \psi_1^g} & gu^{-1}v^{-1} \\
\psi_1^g \searrow & & \nearrow \psi_2^g \\
& v^{-1}gu^{-1} &
\end{array} \tag{**}$$

Let us note that if some elements  $u, v, g$  commute with each other, then some of the objects of the diagrams coincide.

We can reformulate Proposition 3 in terms of morphism and diagrams (\*), (\*\*). Condition (3) is equivalent to the condition

$$\chi_{\partial}(\phi_1^g \circ \phi_2^g) - \chi_{\partial}(\phi_2^g) - \chi_{\partial}(\phi_1^g) = \chi_{\partial}(\psi_2^g \circ \psi_1^g) - \chi_{\partial}(\psi_2^g) - \chi_{\partial}(\psi_1^g). \tag{14}$$

An arbitrary map  $\chi : Mor(\Gamma) \rightarrow \mathbf{C}$  is called locally finite, if for each element  $h \in G$  we have  $\chi(g, h) = 0$  for all  $g \in G$  for almost all elements  $x \in G$ . A locally finite mapping generates a linear mapping  $\partial_{\chi} : \mathbf{C}[G] \rightarrow \mathbf{C}[G]$  defined as follows

$$\partial_{\chi}(u) = \sum_{g \in G} \left( \sum_{h \in G} \partial_h^g \lambda_u^h \right) g, \tag{15}$$

Here  $\partial_h^g := \chi(g, h)$ .

Summarizing the above we get a theorem in gruppoid terms which follows from Theorem 2. This theorem is important for description of Lie rates.

**Theorem 3.** *A locally finite mapping  $\partial_{\chi}$  defined by formula (15) defines a rate, if and only if for each elements  $g, u, v \in G$  for morphisms  $\phi_1^g, \phi_2^g, \psi_1^g, \psi_2^g$  defined by formula (11) the following condition is satisfied*

$$\chi(\phi_1^g \circ \phi_2^g) - \chi(\phi_2^g) - \chi(\phi_1^g) = \chi(\psi_2^g \circ \psi_1^g) - \chi(\psi_2^g) - \chi(\psi_1^g).$$

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