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ON THE TOPOLOGY OF AN INTEGRABLE VARIANT OF A NONHOLONOMIC SUSLOV PROBLEM

§1. Introduction

In [1–5] A. T. Fomenko suggested a new theory of topological classification of integrable in Liouville sense Hamiltonian systems and constructed a complete topological invariant which distinguishes integrable Hamiltonian systems up to topological eguivalence. The theory was first constructed for Hamiltonian systems with two degrees of freedom which are integrable in Liouville sense on a single compact isoenergetic 3-dimensional surface $Q^3_h = \{H = h\}$, and later was generalized to multidimensional Hamiltinian systems integrable the whole phase space M^n (but not only on a single surface Q^{n-1}).

The *integrability in Liouville sense* of a Hamiltonian system $v = \operatorname{sgrad} H$ with a smooth Hamiltonian H on a symplectic manifold M^4 means that there exists an additional smooth integral F of the system which is independent of H and in involution with H, i. e., $\{F, H\} = 0$. Then the well-known Liouville theorem states that a nondegenerate connected component of a common level surface $I_{f,h} = \{F = f, H = h\}$ is diffeomorphic to a 2-dimensional torus T^2 *Liouville torus*, and the motion of the system is a quasi-periodic winding in this torus. A. T. Fomenko considers the integrability of a system on a single isoenergetic compact surface Q^3_h, when $\{F, H\} = 0$ and $d\{F, H\} = 0$ only on Q^3_h. Obviously, if the system $v = \operatorname{sgrad} H$ is integrable over the whole M^4, then it is integrable on each $Q^3_h = \{H = h\} \subset M^4$. But the converse is not true. G. G. Okuneva in [6] constructed a class of Hamiltonian systems which are integrable on each Q^3_h but not integrable on M^4. The theory also consideres Bott integrals, when the restriction of F is a Bott function, i.e., all critical submanifolds of the function F on Q^3_h are nondegenerate. Moreover, only nonresonance Hamiltonian systems (where integral trajectories form irrational windings on almost all Liouville tori in M^4) are considered. In this case the invariant, called *marked molecule*, does not depend on the choice of integral F. The marked molecule consists of letters-atoms which correspond to singularities of the Liouville foliation on Q_h^3 and are denoted by Roman letters A, B, C ets. The links between the atoms are one-parameter families of Liouville tori. The numerical marks r_s, ε_s and n_s prescribe the rules for gluing the Loiuville tori. According to the main theorem of A. T. Fomenko, there exists one-to-one correspondence between the classes of topologically equivalent integrable Hamiltonian systems (on Q^3) and the classes of isomorphic marked molecules. Two nonresonance integrable Hamiltonian systems v_1 on Q_1^3 and v_2 on Q_2^3 are *topologically equivalent* if there exist the diffeomorphism $\tau : Q_1^3 \to Q_2^3$ that takes Liouville tori of the system v_1 into the Liouville tori of the system v_2 and preserves the orientation of isoenergy surfaces and critical circles of the integrals.

Marked molecules were constructed for a large number of integrable Hamiltonian systems arising in analytic dynamics and mathematical physics [7].

The notion of a marked molecule and atom provides us with a powerful instrument for coding of complex shapes of natural objects. Recently a new method for CAD system for coding the surface of a natural three-dimensional object such as a human organ was suggested [8]. This method is based on Morse theory. An expanded method that enables us to code non-orientable surfaces such as two-dimensional projective space and Klein bottle has also been proposed [9].

In the present paper we investigate the topology of a newly obtained integrable variant of Suslov's problem of motion of a rigid body around a fixed point under a non-holonomic connection.

§2. Notation and statement of Suslov problem

Let $\overrightarrow{\omega} = (p, q, r)$ be the vector of instantaneous angular velocity of the body where p, q, r are the projections of $\overrightarrow{\omega}$ on the moving coordinate axes; $\overrightarrow{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ be the unit vector for the vertical line; $\overrightarrow{r} = (r_1, r_2, r_3)$ be mass center vector of the body and I=diag(A, B, C) the inertia tensor of the body.

We consider motion of the body with the non-holonomic connection $r = 0$. This connection was first considered by G. K. Suslov in [10], where he also presented its mechanical realisation. This system belongs to so-called Chaplygin's non-holonomic systems, where the Lagrangian and equations of connections do not depend on some of the variables. In this case it is possible to choose a certain number of independent coordinates, and after the change of time $d\tau = N dt$ along the trajectories the system can be represented in a Hamiltonian form. The function N, called *Chaplygin factor* depends only on independent coordinates and can be obtained from a partial differential equation, which does not involve the potential U. This gives us a possibility to investigate Chaplygin's systems in various force fields.

The reduction of Suslov's problem to a Hamiltonian form was done by E. I. Harlamova [11]; the kinetic energy and the equation of the connection $r = 0$ do not depend on the angle ψ (ϕ, ψ, θ are the Euler angles, in terms of which the motion of rigid body is usually described), and γ_1 and γ_2 were chosen as the independent coordinates. Chaplygin's factor turned to be equal to γ_3, and after the change of time $d\tau = \gamma_3 dt$ Suslov's problem can be reduced to the Hamiltonian form in each of the domains $\gamma_3 > 0$ and $\gamma_3 < 0$, where the Hamiltonian has the following form:

$$H = \frac{1}{2}\left(\frac{p_1^2}{B} + \frac{p_2^2}{A}\right) + U(\gamma_1, \gamma_2, \gamma_3) = (Ap^2 + Bq^2) + U,$$

where $p_1 = -Bq$, $p_2 = Ap$ and U is a potential function. G. !G. Okuneva in [12–15] investigated this problem with the potential

$$U = Mg(r_1\gamma_1 + r_2\gamma_2) + \frac{\varepsilon}{2}(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2),$$

where Mg is the weight of the body.

The system proved to be integrable with the following integrals:

$$F = Ap^2 + \varepsilon(B - C)\gamma_2^2 + 2Mgr_2\gamma_2 = f,$$
$$G = Bq^2 + \varepsilon(C - A)\gamma_1^2 + 2Mgr_1\gamma_1 = g;$$

We have $F + G = H$.

The Hamiltonian H formally coincides with the Hamiltonian of the biharmonic oscillator with the frequencies

$$\varkappa_1 = \sqrt{\varepsilon(A - C)/B}, \quad \varkappa_2 = \sqrt{\varepsilon(B - C)/A},$$

where $A > B > C$. The trajectories are the Lissajous figures disposed in the rectangle Π_{fg} which is a projection of the Liouville torus on the plane (γ_1, γ_2). But the equality $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ implies that the real motion of the Suslov's problem is possible only within the common part of Π_{fg} and of the circle $D^2 = \{\gamma_1^2 + \gamma_2^2 \leq 1\}$.

The Euler-Poisson equations

$$\begin{cases} A\dfrac{dp}{dt} = -\gamma_3\dfrac{\gamma_2}{\gamma_3^4}, \\ B\dfrac{dq}{dt} = \gamma_3\dfrac{\gamma_1}{\gamma_3^4}, \\ \dfrac{d\gamma_1}{dt} = -q\gamma_3, \\ \dfrac{d\gamma_2}{dt} = p\gamma_3 \end{cases}$$

are invariant under the changes of variables $t \to -t$, $\gamma_3 \to -\gamma_3$. So, the motion achieving the circumference $\{\gamma_3 = 0\}$ in finite time repulses from it and returns along the same trajectory, and in general case reaches the circumference $\{\gamma_3 = 0\}$ again. Consequently, almost all trajectories of the Suslov's problem intersecting the circumference $\{\gamma_3 = 0\}$ are closed, independently of the relation between the frequencies $\varkappa_1$ and $\varkappa_2$ of the oscillator. The trajectories tangent to the circumference $\{\gamma_3 = 0\}$ are singular: the tangency point is reached in infinite time t.

It turned out that the problem is also integrable in the force field with the potential

$$U_\alpha = \frac{1}{2}\frac{1}{\alpha + \gamma_3^2}, \quad \text{where } \alpha \geqslant 0.$$

The additional integral

$$F_\alpha = \frac{1}{2}\{(\alpha+1)(p_1^2+p_2^2) + \frac{B\gamma_1^2 + A\gamma_2^2}{2(\alpha+\gamma_3^2)} + \frac{1}{AB}(Ap_1\gamma_2 - Bp_2\gamma_1)^2\}$$

is independent of

$$H_\alpha = \frac{1}{2}\left(\frac{p_1^2}{B} + \frac{p_2^2}{A}\right) + U_\alpha$$

and is in involution with H_α. Everywhere below we will omit the factor $1/2$ before H_α and F_α. The index α means the dependence of integrals on the parameter α.

First, the potential function $U = 1/\gamma_3^2$ was introduced by D. N. Goryachev in [16] who considered the motion of a free rigid body (without connections) under the conditions $A = B = 4C$, $r_2 = r_3 = 0$ and $U = \alpha/\gamma_3^2 - \beta\gamma_1$. The additional integral F exists only when the constant of the momentum integral $G = 4(p\gamma_1 + q\gamma_2) + r\gamma_3$ is equal to zero. The

Hamiltonian and the integral are the following:

$$
\begin{aligned}
H =& 2\left(p^2+q^2+\frac{\alpha}{2\gamma_3^2}\right)+\frac{\alpha}{2}r^2-\beta\gamma_1,\\
F =& 2r\left(p^2+q^2+\frac{\alpha}{2\gamma_3^2}\right)+2\beta\gamma_3.
\end{aligned}
$$

The system is defined on the symplectic manifold M^4 which is diffeomorphic to $D^2 \times \mathbb{R}^2$. The motions of the system exist only in the half-sphere $\gamma_3 \lesssim 0$. This case was investigated by E. V. Anoshkina in [17–18] where the bifurcation diagrams of the integrals H and F and the marked moleciules were constructed for each Q_h^3, which turned out to be homeomorphic to S^3.

§3. Construction of the bifurcation diagram for the case $\alpha = 0$

The case where $\alpha = 0$ is essentially different from the case $\alpha > 0$. The trajectories of the system $v = sgrad\, H_0$ never reach the circumference $\{\gamma_3 = 0\}$ and are situated either in the half-sphere $\{\gamma_3 > 0\}$ or in the half-sphere $\{\gamma_3 < 0\}$. The change of time $d\tau = \gamma_3\, dt$ is nondegenerate, and the system $v = sgrad\, H_0$ is "really" Hamiltonian.

First, let us construct the bifurcation diagram, i.e., the set $\Sigma = G(N)$ where N is the set of all critical points for the momentum mapping $G : M^4 \to \mathbb{R}^2$ $(G(x) = (H_0(x), F_0(x)) \in \mathbb{R}^2)$, and the topological invariant of this system. The symlpectic manifold M^4 is diffeomorphic to $\mathbb{R}^2(p_1, p_2) \times D^2(\gamma_1, \gamma_2)$. The vector $grad\, H$ is equal to zero at the point $p_1 = p_2 = \gamma_1 = \gamma_2 = 0$ where the Hamiltonian H has an isolated minimum.

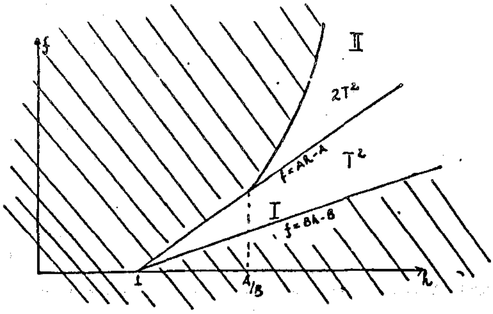

Fig. 1.

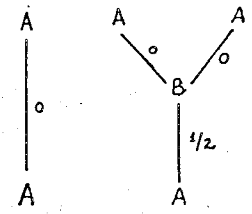

Fig. 2.

The bifurcation diagram (see Fig. 1) consists of two half-lines $f = A(h-1)$ and $f = B(h-1)$ intersecting at the point $h = 1$ and a part of the parabola $(f-h(A+B))^2-4ABh = 0$ which is tangent to $f = A(h-1)$ at the point $h = \frac{A}{B}$. The motion of the system occurs only in the non-hatched part of the diagram. The critical submanifolds of the integral F on $Q^3 = \{H = h\}$ are the following:

(1) two maximal circles given by the equations

$$\begin{cases} \dfrac{p_1^2}{Bh-\sqrt{ABh}} + \dfrac{p_2^2}{Ah-\sqrt{ABh}} = 1; \\ \dfrac{\gamma_1^2}{1-\sqrt{A/Bh}} + \dfrac{\gamma_2^2}{1-\sqrt{B/Ah}} = 1; \\ p_2^2 = \dfrac{A(\sqrt{ABh}-A)\gamma_1^2}{\gamma_1^2(A-B)+B-\sqrt{AB/h}}. \end{cases}$$

The corresponding values h and f of the integrals belong to the part $f = h(A+B) - 2\sqrt{ABh}$ of the parabola when $h > \frac{A}{B}$;

(2) saddle circle S^1:

$$\begin{cases} p_1 = 0; \\ \gamma_1 = 0; \\ \dfrac{p_2^2}{A} + \dfrac{1}{1-\gamma_2^2} = h. \end{cases}$$

In the diagram it is the half-line $f = A(h-1)$, $h > \frac{A}{B}$. The rest part of this half-line ($h < \frac{A}{B}$) corresponds to the maximum value of F on Q^3. The critical circle is given by the system above;

(3) a circle S^1, where the integral F attains its minimum value. The critical circle in this case is determined by the conditions:

$$\begin{cases} p_2 = 0; \\ \gamma_2 = 0; \\ \dfrac{p_1^2}{B} + \dfrac{1}{1-\gamma_1^2} = h. \end{cases}$$

It is the half-line $f = B(h-1)$ in the diagram.

Omitting the calculations we give the final expression for the indices of the critical circles of the momentum mapping. The indices of the critical circles that lie in the preimage of the parabola are as follows:

$$\begin{aligned} \lambda_1 =& (1-\gamma_2^2)(A\gamma_2^2 - A - B + \sqrt{AB/h}) \quad < 0; \\ \lambda_2 =& -(B + \sqrt{AB/h})/A - B\gamma_2^2/(1-\gamma_2^2) \quad < 0. \end{aligned}$$

Similar calculations for the half-lines give

$$\lambda_1\lambda_2 = -(A-B)(h - A/B), \text{ for the points in the upper line;}$$
$$\lambda_1\lambda_2 = (A-B)(h - B/A), \text{ for the points in the lower line;}$$

Since critical circles are nondegenerate, except those that lie in the preimage of the tangent point of the parabola and the half-line.

For each point (f, h) between the half-lines, the integral manifold $I_{f,h}$ is diffeomorphic to a torus T^2, and $I_{f,h}$ is diffeomorphic to two tori in the domain between the half-line $f = A(h-1)$ and the curve $f = h(A+B) - 2\sqrt{ABh}$.

A. T. Fomenko proved that in general case there are five possible types of bifurcation of Liouville tori in Hamiltonian system. In our problem we have only two bifurcations, the birth (death) of a torus and deformation of the torus from the domain I into two tori from the domain II. The corresponding molecules are shown in Fig. 2. The letter A denotes the neighbourhood of minimax circle S^1 in Q^3 and the letter B denotes the neighbourhood of the saddle circle S^1 in Q^3. The topological type of Q_h^3 may change only when h crosses its critical value. Here we have only one type of Q^3, which is diffeomorphic to S^3. The numerical marks are shown on the links between the letters A and B.

Now we describe the projections P_{fh} of the Liouville tori on the plane (γ_1, γ_2) which are always situated inside the circle $D_h^2 = \{\gamma_1^2 + \gamma_2^2 \leqslant 1 - 1/h\}$. This follows directly from the inequality $p_1^2/B + p_2^2/A = h - 1/(1-\gamma_1^2-\gamma_2^2) \geqslant 0$.

After the change of variables $\gamma_1^2 = x$, $\gamma_2^2 = y$ the circle D_h^2 transforms to the quadrant $x > 0$, $y > 0$ in the plane (x, y). The boundary of the domains P_{fh} is the curve

$$\begin{aligned} D = & -B^2hx^2 + B^2h^2x^2 - Bhfx^2 - Bhfxy - 2ABhxy - Afhxy + \\ & + A^2h^2xy + B^2h^2xy - Afhy^2 - A^2hy^2 + A^2h^2y^2 + f^2x - Afhx + B^2hx - \\ & - B^2h^2x + Bfh - ABhx + ABh^2x - A^2h^2y + A^2hy + ABh^2y + Afy = 0 \end{aligned}$$

which is exactly two straight lines s_1 and s_2 intersecting at a point (x_0, y_0);

$$\begin{aligned} s_1: &\quad y = k_1x + b_1, \\ s_2: &\quad y = k_2x + b_2 \end{aligned}$$

where

$$\begin{aligned} k_{1,2} = \frac{1}{2}\frac{1}{Ah(A - Ah - f)}\{ & fh(A + B) \\ & - (A - B)^2h \pm h(B - A)\sqrt{(f - (A + B)h)^2 - 4ABh}\}, \end{aligned}$$

$$\begin{aligned} b_{1,2} = \pm(Ah - A - f)\sqrt{(f - (A + B)h)^2 - 4ABh} \\ + (Ah - A - f)(Ah - Bh - f). \end{aligned}$$

The intersection point

$$\begin{aligned} x_0 &= \frac{-A + Ah - f}{(A - B)h}, \\ y_0 &= \frac{-B + Bh - f}{(-A + B)h} \end{aligned}$$

always belongs to the straight line

$$x + y = 1 - 1/h.$$

The line s_i in general position on the plane (γ_1, γ_2) is an ellips or a hyperbola. If $k_i < 0$, we obtain an ellips, and if $k_i > 0$ we obtain a hyperbola.

The domain P_{fh} on the plane (γ_1, γ_2) corresponds to the condition $D > 0$. It is easy to see that the point $\gamma_1 = 0$, $\gamma_2 = 0$ belongs to P_{fh}

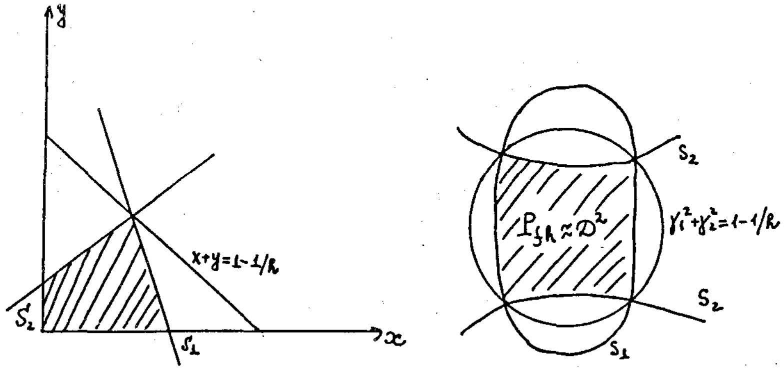

Fig. 3.

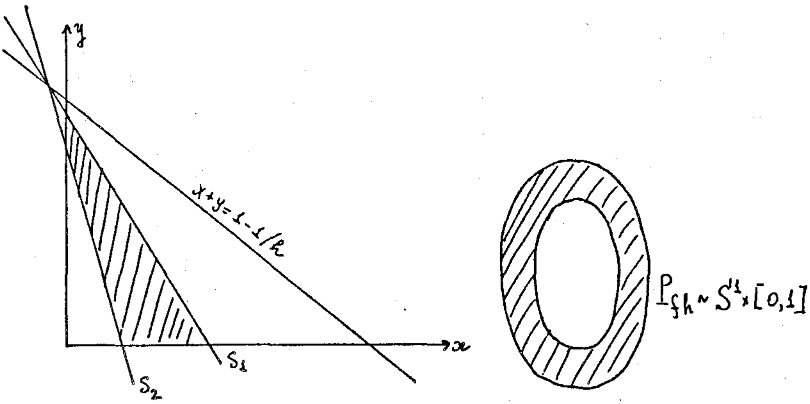

Fig. 4.

only in the domain I on the bifurcation diagram (Fig. 1), and does not belong to P_{fh} in the domain II.

Finally, we obtain the following structure of P_{fh}:

a) In the domain I we have one Liouville torus which is projected on the domain P_{fh} diffeomorphic to a two-dimensional disk (Fig. 3). Four points of the Liouville torus are projected to one interior point of P_{fh};

boundary points of P_{fh} correspond to two points of T^2, and each of four boundary points of intersection of the ellips and hyperbola correspond to one point of T^2.

b) In the domain II two Liouville tori are projected into a ring, each interior point of which corresponds to two points in each torus, and each boundary point of the ring corresponds to one point in each torus (Fig. 4).

§4. Construction of the bifurcation diagram and integral manifolds in the case $\alpha > 0$

When $\alpha > 0$, the trajectories intersect the circumference $\gamma_3 = 0$ as in the case of the linear-quadratic potential, and the system is not Hamiltonian on the whole sphere $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$.

Note that after a canonical change of variables

$$p_1' = \sqrt{\alpha+1}\,p_1, \quad p_2' = \sqrt{\alpha+1}\,p_2,$$
$$\gamma_1' = \frac{\gamma_1}{\sqrt{\alpha+1}}, \quad \gamma_2' = \frac{\gamma_2}{\sqrt{\alpha+1}}.$$

the Hamiltonian H_α and the integral F_α have the following form:

$$H_\alpha' = \frac{H_0'}{\alpha+1}, \quad F_\alpha' = F_0'.$$

The factor $1/(\alpha+1)$ together with the prime indices can be omitted without loss of generality.

After this change of variables the circle $D^2 = \{\gamma_1^2+\gamma_2^2 = 1\}$ transforms to the circle $D_\alpha^2 = \{{\gamma_1'}^2 + {\gamma_2'}^2 \leqslant 1/(\alpha+1)\}$. So, the real motion of the system only takes place within the common part of the projections P_{fh} of the Liouville tori of the system $v = sgrad\, H_0$ and the circle D_α^2.

The procedure of construction of integral manifolds is the same as in Section 1. We must intersect the domains P_{fh} with the circle D_α^2, delete parts corresponding to $P_{fh} \setminus (P_{fh} \cap D_\alpha^2)$ from the Liouville tori, and duplicate two-dimensional manifolds thus obtained.

The bifurcation diagram for the case where $\alpha > 0$ contains three new lines in comparisson with the diagram for the case $\alpha = 0$:

a) The vertical line $h = 1/(1-\beta)$, where $\beta = 1/(\alpha+1)$, $0 < \beta < 1$, which corresponds to the moment when the circles $D_h^2 = \{\gamma_1^2 + \gamma_2^2 \leqslant 1 - 1/h\}$ and D_α^2 coincide.

b) The straight line $f = Bh(\beta + A/B) + A/(\beta - 1)$, which corresponds to the moment when the line s_1 passes through the point $(\beta, 0)$.

c) The straight line $f = Ah(\beta + B/A) + B/(\beta - 1)$, which corresponds to the moment when the line s_2 passes throug the point $(0, \beta)$.

To the left of the line $h = 1/(1-\beta)$ we obtain the topological structure of the phase space as in the case $\alpha = 0$ where the inclusion $D_h^2 \subset D^2 = \{\gamma_1^2 + \gamma_2^2 \leqslant 1\}$ always holds. The lines b) and c) take place only to the left of $h = 1/(1-\beta)$, where $D_h^2 \supset P_{fh} \cap D_\alpha^2 \neq \emptyset$.

The bifurcation diafgam in the case $1/(1-\beta) > A/B$ (Fig. 5) differs from the diagram for the case $1/(1-\beta) < A/B$ (Fig. 6), when the line $h = 1/(1-\beta)$ devides the diagram into two parts: to the left we obtain the "Hamiltonian" surgeries of the Liouville tori, and in the rest part of the diagram we observe the surgeries of closed 2-dimensional manifolds obtained after the procedure of duplication.

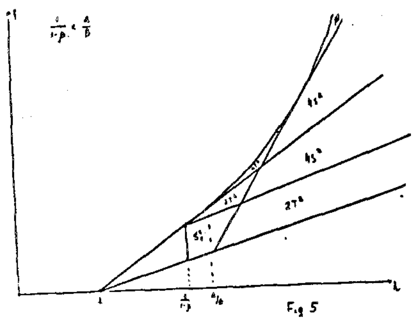

Fig. 5.

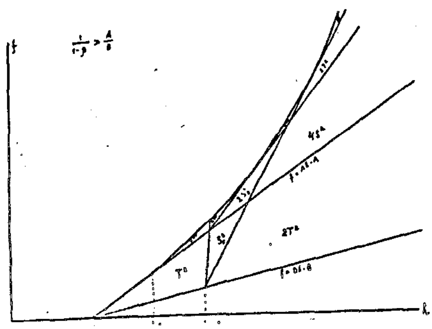

Fig. 6.

Example 1. Let us describe how two spheres $2S_3^2$ with 3 handles appear in the case $1/(1-\beta) > A/B$. The respective domain corrsponds to the following conditions: $h > 1/(1-\beta)$, $f > A(h-1)$, $f > Bh(\beta + A/B) + A/(\beta - 1)$, $f < Ah(\beta + B/A) + B/(\beta - 1)$. The position of the lines s_1, s_2 and $x + y = \beta$ is shown in Fig. 7. The hatched domain corresponds to two tori, each with two disks cut off (Fig. 7). After the duplication we obtain S_3^2.

Example 2. The sphere S_5^2 with 5 handles appears in both diagrams when $h < 1/(1-\beta)$, $f > B(h-1)$, $f < A(h-1)$, $f < Bh(\beta + A/B) + A/(\beta - 1)$. Here P_{fh} is homeomorphic to a disk with four boundary disks cut off (Fig. 8). Four disks are also deleted from the torus in phase space, and after the duplication we obtain S_5^2.

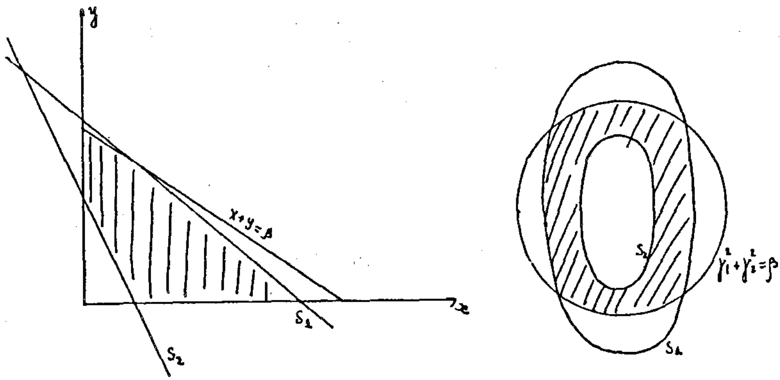

Fig. 7.

Remark 1. Parts of the diagrams where the motion of the system takes place are different in the cases $\alpha = 0$ and $\alpha > 0$. In the last case the part of the diagram between the part of parabola $(f - (A+B)h)^2 - 4ABh = 0$ and the line $f = A(h-1)$ corresponds to no motion. Here P_{fh} is diffeomophic to a ring, and D_α^2 is always inside this ring (Fig. 9).

Remark 2. When $A = B$ and $\alpha = 0$ the Hamiltonian H and the integral F of Suslov's problem coincide with the Hamiltonian

$$H^* = H|_{A=B} = \frac{1}{2A}(p_1^2 + p_2^2) + \frac{1}{2}\frac{1}{(1 - \gamma_1^2 - \gamma_2^2)},$$

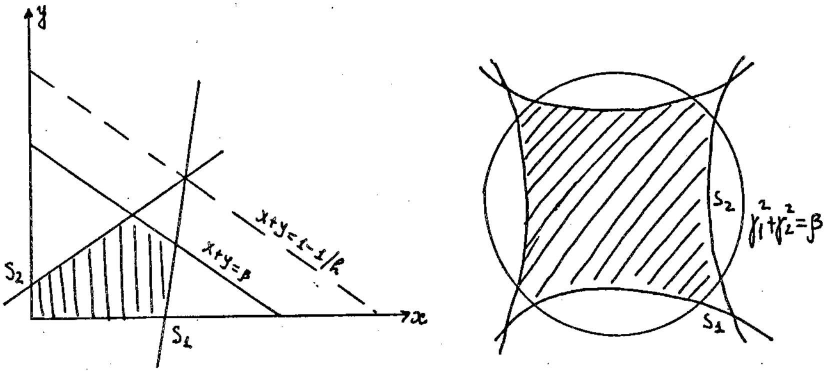

Fig. 8.

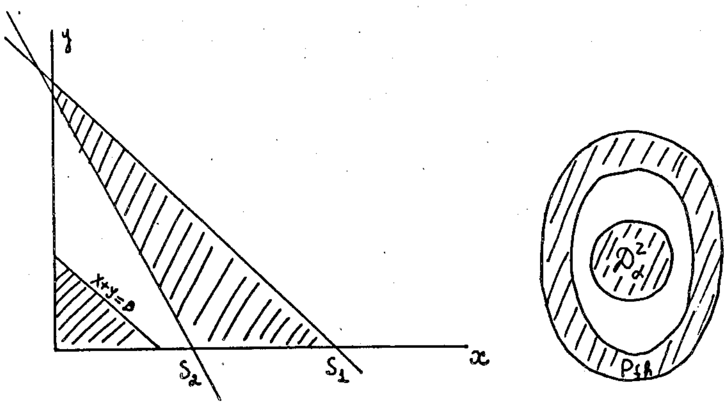

Fig. 9.

and the integral

$$F^* = F|_{A=B} = \frac{1}{2A}(p_1^2 + p_2^2) + \frac{1}{2}\frac{1}{(1-\gamma_1^2-\gamma_2^2)} + \frac{1}{2A}(p_1\gamma_2 - p_2\gamma_1)^2$$

of a plane motion of a particle under the central force field respectively. Note that the system $v = sgrad\, H^*$ has an additional angular momentum integral $K = p_1\gamma_2 - p_2\gamma_1$, and F^* is a linear combination of H^* and

K^2.

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