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# SMALL ISLANDS OF STABILITY IN THE PHASE SPACE OF THE CARLESON MAP

ABSTRACT. We consider the Carleson map on the two dimensional torus and develop an asymptotic theory of islands of an arbitrary period.

## 1. INTRODUCTION. DESCRIPTION OF THE PROBLEM

We investigate the Carleson map on the two dimensional torus $\mathbf{T}^2 = \mathbf{R}^2/(2\pi\mathbf{Z})^2$ given by the formula

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} gf(x) - y \\ x \end{pmatrix} \bmod 1\,,$$

where $f(x) = \cos 2\pi x$ and $g$ is a real parameter.

The Carleson map is a typical example of an area-preserving map. We consider the problem of finding of some types of islands of stability for the Carleson map.

An asymptotic theory of islands of periods less then 4 for large values of the parameter $g$ of the map was obtained in [1]. In this work we extend this result and develop an asymptotic theory for the special chains of islands of arbitrary period (see Fig. 3).

The purpose of the present work is to find such areas in the phase space and intervals for real parameter $g$ for the Carleson map for which the islands of stability exist.

A characteristic feature of the problem is that the measure of the islands of stability is less than the measure of the surrounding chaotic area. The solution of this problem involves three model maps:

a quadratic map (the Hénon map) $QM : \mathbf{R}^2 \to \mathbf{R}^2$, given by the formula

$$\begin{cases} \xi_1 = \xi^2 - \eta + c\,, \\ \eta_1 = \xi \end{cases}$$

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and cubic maps Cub $M_{\pm} : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$, given by the formula

$$\begin{cases} \xi_1 = c\xi + \eta \pm \xi^3 \,, \\ \eta_1 = -\xi \,, \end{cases}$$

where $c$ is a real parameter.

We use the model maps to describe the behavior of the Carleson map in a neighborhood of the islands of stability. Our model maps are well investigated. For each of them one knows where the main islands of stability exist.

Therefore we can find the special areas in the phase space for the Carleson map and intervals for parameter $g$, when the islands of stability exist. For this objective we use substitutions of variables which conjugate the Carleson map to the corresponding model map.

For illustration of our method we represent two pictures.

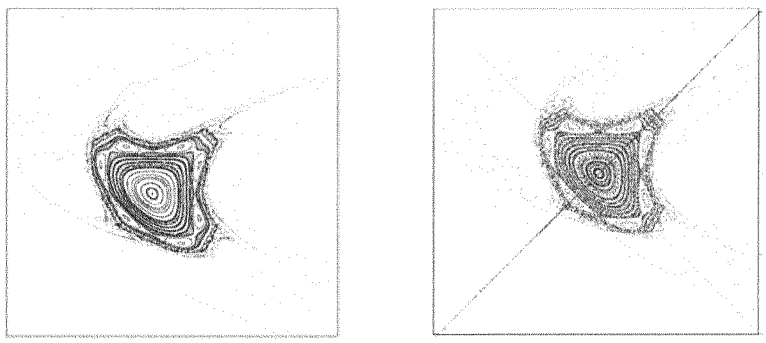

Fig. 1. Elliptic islands of the Hénon map and the Carleson map.

In the left part of Fig. 1 we can see the island of stability of the Hénon map for $c = -0.5625$ and in the right part the corresponding island of the Carleson map for $g = 77.494875929602007300$.

In the left part of Fig. 2 we can see the island of stability of the cubic map Cub $M_{+}$ for $c = 1.0$ and in the right part the corresponding island of the Carleson map for $g = 77.2484308209729$.

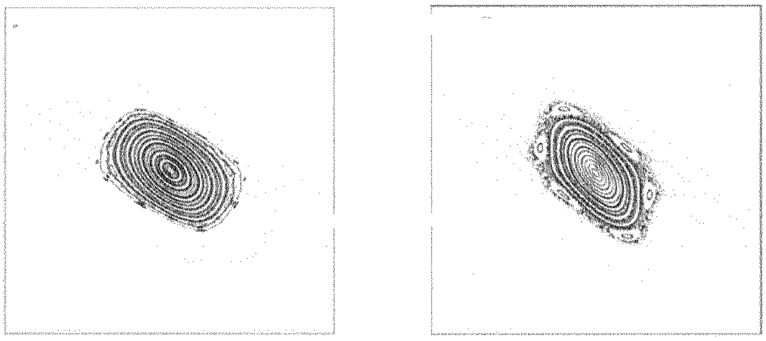

Fig. 2. Elliptic islands of the cubic map and the Carleson map.

Here we use special coordinates described in Statement 2.

The work contains the method, which allows to find the different islands of stability for fixed set of quantum numbers.

## 2. Islands of period $n = 2k + 2$. Chain $(q_k, q_{k-1}, \ldots, q_1, p, p, q_1, \ldots, q_{k-1}, q_k)$

In this paper we consider only the special chain of islands of stability (see Fig. 3).

For our purpose it is convenient to turn to the covering map defined on $\mathbf{R}^2$, the universal covering of the torus. We keep the same notation F for it. In the rest of this part all maps will be defined on $\mathbf{R}^2$.

Moreover we will consider the Carleson map as the superposition

$$T_x^{-p} T_y^{-q} F : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} gf(x) - y - p \\ x - q \end{pmatrix},$$

of the map F and the shifts

$$T_x : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x+1 \\ y \end{pmatrix}, \qquad T_y : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y+1 \end{pmatrix},$$

where $p$ and $q$ are the integers.

We consider the next chain:

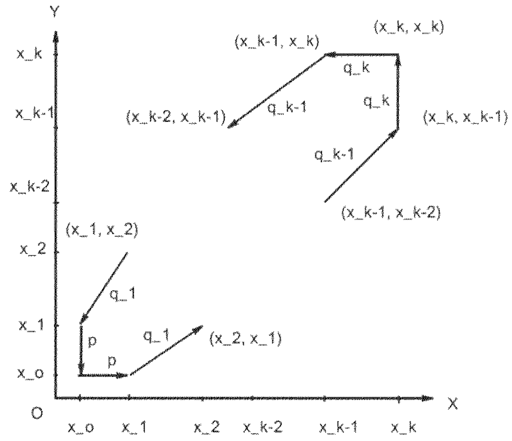


Fig. 3. Islands of period $n = 2k+2$. Chain $(q_k, q_{k-1}, \dots, q_1, p, p, q_1, \dots, q_{k-1}, q_k)$.

$$\begin{pmatrix} x_k \\ x_k \end{pmatrix} \xrightarrow{T_x^{-q_k}F} \begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \xrightarrow{T_x^{-q_{k-1}}F} \dots$$
$$\xrightarrow{T_x^{-q_1}F} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \xrightarrow{T_x^{-p}F} \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} \xrightarrow{T_x^{-p}F} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \xrightarrow{T_x^{-q_1}F} \dots$$
$$\xrightarrow{T_x^{-q_{k-1}}F} \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix} \xrightarrow{T_x^{-q_k}F} \begin{pmatrix} x_k \\ x_k \end{pmatrix}.$$

And we consider the $n$th iteration of the Carleson map:

$$\mathcal{H} \overset{\text{def}}{=} T_x^{-q_k}F \circ T_x^{-q_{k-1}}F \circ \dots \circ T_x^{-q_{-1}}F \circ T_x^{-p}F \circ$$
$$\circ T_x^{-p}F \circ T_x^{-q_1}F \circ \dots \circ T_x^{-q_{k-1}}F \circ T_x^{-q_k}F\,,$$

here $p$, $q_l$, $1 \leqslant l \leqslant k$, are the quantum numbers of our problem.

We assume that $p$ is a large positive number (asymptotic parameter) and $q_l$, $1 \leqslant l \leqslant k$, are the integers such that $|q_l| < p$.

We consider the $n$th iteration of the Carleson map $\mathcal{H}$ in the neighborhood of the elliptic fixed point $\begin{pmatrix} x_k \\ x_k \end{pmatrix}$.

From this condition

$$\mathcal{H}\begin{pmatrix} x_k \\ x_k \end{pmatrix} = \begin{pmatrix} x_k \\ x_k \end{pmatrix}$$

we have $k+1$ equations for unknowns g, $x_0$, $x_l$, $1 \leqslant l \leqslant k$ :

$$x_0 + x_1 = gf(x_0) - p\,, \tag{0}$$

$$x_{l-1} + x_{l+1} = gf(x_l) - q_l\,, \qquad 1 \leqslant l \leqslant k-1\,, \tag{l}$$

$$x_{k-1} + x_k = gf(x_k) - q_k\,. \tag{k}$$

These equations must be supplemented with the equations which arise from the condition of the existence of an elliptic periodic trajectory.

Since

$$\begin{pmatrix} x_k \\ x_k \end{pmatrix} \qquad \text{is an elliptic point}$$

and $\mathcal{H}$ is an area-preserving map,

$$\det \mathcal{H}' \begin{pmatrix} x_k \\ x_k \end{pmatrix} = 1\,,$$

where

$$\mathcal{H}' \begin{pmatrix} x_k \\ x_k \end{pmatrix} = A_k \cdot A_{k-1} \cdot \ldots \cdot A_1 \cdot A_0 \cdot A_0 \cdot A_1 \cdot \ldots \cdot A_{k-1} \cdot A_{k-1} \cdot A_k$$

and

$$A_l = \begin{pmatrix} gf'(x_l) & -1 \\ 1 & 0 \end{pmatrix}, \qquad 0 \leqslant l \leqslant k\,,$$

we have the following condition for the trace of matrix of tangent map $\mathcal{H}'$ at the point $\begin{pmatrix} x_k \\ x_k \end{pmatrix}$. The trace does not exceed 2 in modulas:

$$\left| \mathrm{Spur} \mathcal{H}' \begin{pmatrix} x_k \\ x_k \end{pmatrix} \right| < 2\,. \tag{k+1}$$

Consequently we have to solve the problem $(0) - (k+1)$.

We seek the unknowns $g$, $x_0$, $x_l$, $1 \leqslant l \leqslant k$, in the form of the series in the inverse powers of parameter $p$ .

$$g = p + \sum_{m=0}^{\infty} \gamma_m p^{-m}\,.$$

We assume that $x_0 = \mathcal{O}(p^{-1})$ and $x_l = \mathcal{O}(1)$, $1 \leqslant l \leqslant k$ and, consequently,

$$x_0 = \sum_{m=1}^{\infty} x_{0m} p^{-m}\,, \qquad x_l = \sum_{m=0}^{\infty} x_{lm} p^{-m}\,, \qquad 1 \leqslant l \leqslant k\,.$$

Now we rewrite the condition of the existence of the elliptic trajectory in the following way.

Let us introduce the following notation

$$\mathcal{H}'\begin{pmatrix} x_k \\ x_k \end{pmatrix} = \begin{pmatrix} A_{11}^k & A_{12}^k \\ -A_{12}^k & A_{22}^k \end{pmatrix}$$

and require that all elements have the following order

$$A_{ij}^k = \mathcal{O}(1)\,, \qquad i,j = 1,2.$$

In this case the trace has the following expansion

$$Sp \stackrel{\text{def}}{=} \mathrm{Spur}\mathcal{H}'\begin{pmatrix} x_k \\ x_k \end{pmatrix} = A_{11}^k + A_{22}^k = Sp_0 + \sum_{m=1}^{\infty} Sp_m p^{-m}$$

and we require in addition that the following equation holds

$$\mathrm{Spur}\mathcal{H}'\begin{pmatrix} x_k \\ x_k \end{pmatrix} = Sp_0\,.$$

Thus we obtain that the coefficients of the expansions for $g$ and $x_l$ have to obey the following conditions

$$\begin{cases} A_{ij}^k = \mathcal{O}(1)\,, & i,j = 1,2\,, \\ |Sp_0| < 2\,, & \\ Sp_m = 0 & \text{for all } m \geqslant 1 \end{cases} \qquad ((k+1)^*)$$

instead of $(k+1)$.

We substitute the series for unknowns $g$, $x_0$, $x_l$, $1 \leqslant l \leqslant k$, into equations $(0) - (k+1^*)$ and expand the left and the right hand sides of these equations in series in inverse powers of the parameter $p$. After canceling the terms of the same order in $p$, one obtains the infinite recurrent system of algebraic equations for determining the coefficients of our series. By solving this recurrent system, it is easy to write out the explicit formulae for our unknown coefficients.

It is easy to see that the following Statement is valid.

**Statement 1.** *Let $k$ be a positive integer. Let $p$ be a large positive integer, and $q_l$, $1 \leqslant l \leqslant k$, be integers satisfying $|q_l| < p$, ($p$, $q_1$, $q_2, \ldots$, $q_n$ are the quantum numbers).*

*Let $c_0$ be any real number such that $|c_0| < 2$.*

*Then for all such* $c_0$, $p$, *and* $q_1, \ldots, q_k$ *the problem* $(0)-(k+1)^*$ *has two solutions. Either of the two corresponds the elliptic periodic trajectory in the phase space of the Carleson map (* $n = 2k+2$ *is a period of trajectory).*

1) *The first solution begins with*

$$x_{00} = 0\,, \qquad x_{01} = -\frac{1}{4\pi^2}\,, \qquad \textit{and} \qquad \mathrm{Spur}\mathcal{H}'\begin{pmatrix} x_k \\ x_k \end{pmatrix} = c_0\,.$$

*The subsequent coefficients of our expansions for* $g$, $x_0\,, \ldots\, x_k$ *are uniquely defined. Moreover,* $\gamma_{2k+2}\,$, $x_{0,2k+1}\,$, *and* $x_{l,2k+2}\,$, $1 \leqslant l \leqslant k$, *are the first coefficients that depend on* $c_0$.

2) *The second solution begins with*

$$x_{00} = 0\,, \qquad x_{01} = +\frac{1}{4\pi^2}\,, \qquad \textit{and} \qquad \mathrm{Spur}\mathcal{H}'\begin{pmatrix} x_k \\ x_k \end{pmatrix} = c_0\,.$$

*The subsequent coefficients of our expansions for* $g$, $x_0\,, \ldots\, x_k$ *are uniquely defined. Moreover* $\gamma_{2k+1}\,$, $x_{0,2k+1}\,$, *and* $x_{l,2k+2}\,$, $1 \leqslant l \leqslant k$, *are the first coefficients that depend on* $c_0$.

*In both cases it is easy to write out the explicit recurrent formulae for our unknown coefficients.* □

Due to Statement 1 we know the coordinates of the center of our island and interval of the values of parameter $g$ for which the island exists. But, unfortunately, we don't know the size of our island. Therefore it is very difficult to discover this island. The connection between the Carleson map and the model maps helps us to overcome this obstacle.

## 3. Connection between the Carleson map and the model maps

We consider again the $n$th iteration of the Carleson map

$$\mathcal{H} \overset{\mathrm{def}}{=} T_x^{-q_k} F \circ T_x^{-q_{k-1}} F \circ \cdots \circ T_x^{-q_{-1}} F \circ T_x^{-p} F \circ$$
$$\circ\, T_x^{-p} F \circ T_x^{-q_1} F \circ \cdots \circ T_x^{-q_{k-1}} F \circ T_x^{-q_k} F\,,$$

that corresponds the following chain

$$\begin{pmatrix} x_k \\ x_k \end{pmatrix} \xrightarrow{T_x^{-q_k}F} \begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix} \xrightarrow{T_x^{-q_{k-1}}F} \dots$$

$$\xrightarrow{T_x^{-q_1}F} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \xrightarrow{T_x^{-p}F} \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} \xrightarrow{T_x^{-p}F} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \xrightarrow{T_x^{-q_1}F} \dots$$

$$\xrightarrow{T_x^{-q_{k-1}}F} \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix} \xrightarrow{T_x^{-q_k}F} \begin{pmatrix} x_k \\ x_k \end{pmatrix},$$

here $p$, $q_l$, $1 \leqslant l \leqslant k$, are the quantum numbers of our problem.

The following Statement holds.

**Statement 2. Part 1 (quadratic model)**. *Let* $k$ ,$c_0$ , $p$ , $q_l$ , $1 \leqslant l \leqslant k$ , $x_0$ , $x_1$ , ... , $x_k$ , *and* $g$ *be defined by Statement 1, part 1.*

*In this case*

$$x_{01} = -\frac{1}{4\pi^2} \qquad \text{and} \qquad \mathrm{Spur}\mathcal{H}' \begin{pmatrix} x_k \\ x_k \end{pmatrix} = c_0 \, ,$$

*and a period of an elliptic trajectory is equal to* $n = 2k + 2$.

*Then the substitution of variables* $(\xi, \eta) \longrightarrow (x, y)$ *given by the formulae*

$$x = x_k + \frac{1}{p^{3k+1}}(a\xi + b), \qquad y = x_k + \frac{1}{p^{3k+1}}(a\eta + b),$$

$$a = -\frac{1}{B}, \qquad b = \frac{1}{2}\frac{c_0}{B},$$

$$B \stackrel{def}{=} 4\pi^2 \prod_{l=1}^{k} (-2\pi \sin 2\pi x_{l0})^3$$

*converts the* $n$*th iteration of the Carleson map* $\mathcal{H}$ *with the precision up to the term of order* $\mathcal{O}(p^{-1})$ *to the quadratic map:*

$$QM: \qquad \begin{cases} \xi_n = \xi^2 - \eta + c \, , \\ \eta_n = \xi \, . \end{cases}$$

*The parameter* $c$ *satisfies:* $\quad c = -\frac{1}{4}(c_0)^2 + c_0 \quad$ *and* $\quad -3 < c < 1$.

**Part 2 ( cubic model)**. *Let* $k$ ,$c_0$ , $p$ , $q_l$ , $1 \leqslant l \leqslant k$ , $x_0$ , $x_1$ , ... , $x_k$ , *and* $g$ *be defined by Statement 1, part 2.*

*In this case*

$$x_{01} = +\frac{1}{4\pi^2} \qquad \text{and} \qquad \mathrm{Spur}\mathcal{H}' \begin{pmatrix} x_k \\ x_k \end{pmatrix} = c_0\,,$$

*and a period of an elliptic trajectory is equal to* $n = 2k + 2$.

*Then the substitution of variables* $(x, y) \longrightarrow (\xi, \eta)$ *given by the formulae*

$$x = x_k + \frac{1}{p^{2k+1}} a\xi, \qquad y = x_k + \frac{1}{p^{2k+1}} a\eta\,,$$

$$a = \frac{1}{\sqrt{B}}, \qquad B \stackrel{def}{=} 8\pi^4 \prod_{l=1}^{k} (-2\pi \sin 2\pi x_{l0})^4$$

*converts the* $n$*th iteration of the Carleson map* $\mathcal{H}$ *with the precision up to the term of order* $\mathcal{O}(p^{-1})$ *to the cube map:*

$$\mathrm{Cub}\, M_- : \qquad \begin{cases} \xi_3 = c\xi + \eta - \xi^3\,, \\ \eta_3 = -\xi. \end{cases}$$

*The parameter* $c$ *satisfies:* $\quad c = c_0 \quad$ *and* $\quad -2 < c < 2.$ □

The quadratic map $QM$ and the cubic map $\mathrm{Cub}\, M_-$ are well investigated. The island of stability for the quadratic map $QM$ exists for values of parameter $c$ in interval $(-3, 1)$.

We can show that an island of stability for the cubic map exists when $|c| < 2$ too.

Therefore (for sufficiently large $p$), we can find special areas in the phase space for the Carleson map and intervals for parameter $g$, when the islands of stability exist, just by specifying areas in the phase space of models maps, intervals for parameter $c$ and using the formulae above.

## References

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