D. Guzu, T. Hoffmann-Ostenhof, A. Laptev, On a class of sharp multiplicative Hardy inequalities, *Алгебра и анализ*, 2020, том 32, выпуск 3, 180–190

Алгебра и анализ
Том 32 (2020), №3

Dedicated to Nina Nikolaevna Ural′tseva on the occasion of her 85th birthday

ON A CLASS OF SHARP MULTIPLICATIVE HARDY INEQUALITIES

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A class of weighted Hardy inequalities is treated. The sharp constants depend on the lowest eigenvalues of auxiliary Schrödinger operators on a sphere. In particular, for some block radial weights such sharp constants are given in terms of the lowest eigenvalue of a Legendre type equation.

§1. Introduction

In this paper we consider Hardy inequalities with weights that are homogeneous functions with respect to the radial variable. Namely, let $x \in \mathbb{R}^d$, $d \geqslant 3$, and $x = (r, \vartheta)$ be polar coordinates in $\mathbb{R}^d$. Let $-\Delta_\vartheta$ be the Laplace–Beltrami operator on $\mathbb{S}^{d-1}$. Assume that $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$, $\Phi \geqslant 0$, and that the Schrödinger operator H in $L^2(\mathbb{S}^{d-1})$,

$$H = -(\nabla_\vartheta)^2 - \tau\Phi = -\Delta_\vartheta - \tau\Phi, \quad \tau \in \mathbb{R},$$

is bounded from below. Denote

$$\lambda(\tau) = \inf_{\psi \in \mathcal{H}^1(\mathbb{S}^{d-1})\setminus\{0\}} \frac{\int\limits_{\mathbb{S}^{d-1}} \left(|\nabla_\vartheta \psi|^2 - \tau\Phi\,|\psi|^2\right) d\vartheta}{\int\limits_{\mathbb{S}^{d-1}} |\psi|^2\, d\vartheta}, \tag{1.1}$$

where $\mathcal{H}^1(\mathbb{S}^{d-1})$ is the Sobolev space.

If $d \geqslant 3$, then the classical Hardy inequality in $\mathbb{R}^d$ says

$$\int\limits_{\mathbb{R}^d} |\nabla_x u(x)|^2\, dx \geqslant \Big(\frac{d-2}{2}\Big)^2 \int\limits_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\, dx, \quad u \in \mathcal{H}^1(\mathbb{R}^d).$$

Ключевые слова: Schrödinger operators, Hardy inequalities.
A. L. was partially supported by the RSF grant 19-71-30002.

Our main result states that for $d \geqslant 3$ and under some conditions on Φ, we have

$$\int\limits_{\mathbb{R}^d} |\nabla_x u(x)|^2\, dx \geqslant \tau \int\limits_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2} |u(x)|^2\, dx, \quad u \in \mathcal{H}^1(\mathbb{R}^d),$$

with sharp $\tau = \tau(\Phi)$. If $d \geqslant 3$ and $0 \leqslant V \in L^{d/2,\infty}$, then in [Mz] and [MzSh] the authors obtained the inequality

$$\begin{aligned} &\int\limits_{\mathbb{R}^d} V(x)\, |u(x)|^2\, dx \\ &\leqslant \Big(\frac{2}{d-2}\Big)^2 d^{2/d} |\mathbb{S}^{d-1}|\, \|V\|_{d/2,\infty} \int\limits_{\mathbb{R}^d} |\nabla u(x)|^2\, dx, \quad u \in \mathcal{H}^1(\mathbb{R}^d). \end{aligned} \tag{1.2}$$

where $\|\cdot\|_{d/2,\infty}$ is the Lorentz norm.

In the case of $V(x) = \frac{\Phi(x/|x|)}{|x|^2}$, $\Phi \in L^{d/2}(\mathbb{S}^{d-1})$, $d \geqslant 3$, (1.2) becomes

$$\int\limits_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2} |u(x)|^2\, dx \leqslant \Big(\frac{2}{d-2}\Big)^2 |\mathbb{S}^{d-1}|\, \|\Phi\|_{d/2} \int\limits_{\mathbb{R}^d} |\nabla u(x)|^2\, dx.$$

In [HL] a stronger inequality was proved:

$$\int\limits_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2} |u(x)|^2\, dx \leqslant \Big(\frac{2}{d-2}\Big)^2 |\mathbb{S}^{d-1}|\, \|\Phi\|_{p} \int\limits_{\mathbb{R}^d} |\nabla u(x)|^2\, dx,$$

with $p \geqslant \frac{d-1}{2} + \frac{1}{2(d-1)}$, $d \geqslant 3$.

Note that a sufficient condition for the operator H to be bounded from below is $\Phi \in L^p(\mathbb{S}^{d-1})$, $\Phi \geqslant 0$, with $p > 1$ for $d = 3$ and $p = \frac{d-1}{2}$ for $d \geqslant 4$. In this case by using Sobolev's inequality we have

$$\int\limits_{\mathbb{S}^{d-1}} \Phi(\vartheta) |\psi(\vartheta)|^2\, d\vartheta \leqslant \varepsilon \int\limits_{\mathbb{S}^{d-1}} |\nabla_\vartheta \psi(\vartheta)|^2\, d\vartheta + C_\varepsilon \int\limits_{\mathbb{S}^{d-1}} |\psi(\vartheta)|^2\, d\vartheta, \tag{1.3}$$

with some $\varepsilon > 0$.

The last inequality is equivalent to compact embedding of the form

$$\int\limits_{\mathbb{S}^{d-1}} \Phi(\vartheta) |\psi(\vartheta)|^2\, d\vartheta$$

into the Sobolev space $\mathcal{H}^1(\mathbb{S}^{d-1})$ and therefore the spectrum of H is discrete.

An application of our main result is the establishing of a sharp constant τ in the case of the so-called block radial potential considered by L. Skrzypczak and C. Tintarev [ST]. In particular, in the case of $d = 4$ we have

$$\int_{\mathbb{R}^4} |\nabla_x u(x)|^2\, dx \geqslant \tau \int_{\mathbb{R}^4} \frac{|u(x)|^2}{r_1(x) r_2(x)}\, dx, \tag{1.4}$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $r_1(x) = \sqrt{x_1^2 + x_2^2}$, $r_2(x) = \sqrt{x_3^2 + x_4^2}$. We would like to emphasise that in [ST] the authors obtained inequality (1.4) only for the class of functions $u = u(r_1, r_2)$. Namely, in [ST] it is required that $u = u(r_1, r_2)$ in (1.4). Our results will hold for general $u \in \mathcal{H}^1(\mathbb{R}^4)$.

After introducing Hopf coordinates in S^3, we reduce the problem of finding the sharp constant τ in (1.4) to solving a certain minimization problem related to a Legendre type equation on the interval $[0, \pi/2]$.

The proofs of the main results are based on a method developed recently in [HL] and used a Keller–Lieb–Thirring inequality for estimating the first eigenvalue of a magnetic Schrödinger operator on $\mathbb{S}^{d-1}$ in [DELL1]. A similar inequality was also used for the study of two-dimensional Hardy's inequality related to Schrödinger operators with Bohm–Aharonov magnetic fields [DELL2]. For various aspects of Hardy's inequalities see [BEL, D, KPP, Mz, LS].

§2. Main results

Our first main result is the following statement.

Theorem 1. *Let $d \geqslant 3$, $\Phi \geqslant 0$ satisfy inequality* (1.3), *and let $\lambda(\tau)$, defined by the quotient* (1.1), *satisfy the equation*

$$\lambda(\tau_0) = -\Big(\frac{d-2}{2}\Big)^2 \text{ for some } \tau_0. \tag{2.1}$$

Then

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx \geqslant \tau_0 \int_{\mathbb{R}^d} \frac{\Phi(x/|x|)}{|x|^2}\, |u(x)|^2\, dx, \quad u \in \mathcal{H}^1(\mathbb{R}^d). \tag{2.2}$$

Proof. Introducing the polar coordinates $x = (r, \vartheta)$, we find

$$\int_{\mathbb{R}^d} \Big(|\nabla u(x)|^2 - \tau_0 \frac{\Phi(x/|x|)}{|x|^2}\, |u(x)|^2 \Big)\, dx$$

$$= \int_0^\infty \int_{\mathbb{S}^{d-1}} \Big(|u_r'|^2 + \frac{1}{r^2} \big(|\nabla_\vartheta u|^2 - \tau_0 \Phi(\vartheta) |u|^2 \big) \Big)\, r^{d-1}\, d\vartheta\, dr$$

$$\geqslant \int_0^\infty \int_{\mathbb{S}^{d-1}} \Big(\frac{(d-2)^2}{4} + \lambda(\tau_0) \Big) \frac{|u|^2}{r^2}\, r^{d-1}\, d\vartheta\, dr = 0.$$

Here we have used the Hardy inequality

$$\int_0^\infty |u_r'|^2\, r^{d-1}\, dr \geqslant \frac{(d-2)^2}{4} \int_0^\infty \frac{|u|^2}{r^2}\, r^{d-1}\, dr$$

and (1.1), (2.1). □

Remark 1. Finding the value of the constant τ_0 is not an easy task. Let us consider the equation

$$-\Delta_\vartheta \psi(\vartheta) - \tau\, \Phi(\vartheta)\psi(\vartheta) = \lambda\psi(\vartheta). \tag{2.3}$$

It was proved in [DEL] that for any nontrivial $\Phi \geqslant 0$, $-\lambda = -\lambda(\tau)$ is a monotone increasing convex function of τ. Clearly $\lambda(0) = 0$ and $-\lambda(\tau) \to +\infty$ as $\tau \to \infty$. Therefore there is a unique $\tau = \tau(\lambda)$ satisfying (2.3). In particular, there is a unique τ_0 and a nontrivial solution ψ of (2.3) such that $\lambda(\tau_0) = -(d-2)^2/4$.

The next result states the sharpness of inequality (2.2).

Theorem 2. *If $\Phi \geqslant 0$ satisfies inequality (1.3) and for some τ_0 the value $\lambda(\tau_0)$ is equal to $-\frac{(d-2)^2}{2}$, then the Hardy inequality (2.2) is sharp.*

Proof. From the sharpness of the classical Hardy inequality, we find that for any $\delta > 0$ there is a nontrivial $\varphi_\delta \in C_0^\infty(0,\infty)$ such that

$$\int_0^\infty \left|\frac{d}{dr}\varphi_\delta(r)\right|^2 r^{d-1}\, dr \leqslant \left(\frac{(d-2)^2}{4} + \delta\right) \int_0^\infty \frac{|\varphi_\delta(r)|^2}{r^2}\, r^{d-1}\, dr.$$

Let us consider $u_\delta(r,\vartheta) = \varphi_\delta(r)\psi_\delta(\vartheta)$, where ψ_δ is defined by the equation

$$-\Delta_\vartheta \psi_\delta(\vartheta) - \tau_\delta\, \Phi(\vartheta)\psi_\delta(\vartheta) = -\Big(\frac{(d-2)^2}{4} + \delta\Big)\, \psi_\delta(\vartheta).$$

Therefore

$$\int_{\mathbb{S}^{d-1}} \Big(|\nabla_\vartheta \psi_\delta(\vartheta)|^2 + \Big(\frac{(d-2)^2}{4} + \delta\Big)|\psi_\delta(\vartheta)|^2\Big)\, d\vartheta = \tau_\delta \int_{\mathbb{S}^{d-1}} \Phi(\vartheta)|\psi_\delta(\vartheta)|^2\, d\vartheta$$

and thus we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla_x u_\delta|^2 dx &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \Big(\Big|\frac{d}{dr}\varphi_\delta(r)\Big|^2 |\psi_\delta(\vartheta)|^2 + |\varphi_\delta(r)|^2\, \frac{|\nabla_\vartheta \psi_\delta(\vartheta)|^2}{r^2}\Big)\, r^{d-1} d\vartheta dr \\ &\leqslant \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{|\varphi_\delta(r)|^2}{r^2}\Big(|\nabla_\vartheta \psi_\delta(\vartheta)|^2 + \left(\frac{(d-2)^2}{4} + \delta\right)\, |\psi_\delta(\vartheta)|^2\Big)\, d\vartheta dr\end{aligned}$$

$$= \int_0^\infty \int_{\mathbb{S}^{d-1}} \tau_\delta \Phi(\vartheta) \frac{|\varphi_\delta(r)\psi_\delta(\vartheta)|^2}{r^2} r^{d-1}\, d\vartheta dr$$

$$= \tau_\delta \int_{\mathbb{R}^d} \tau_0 \Phi(x/|x|) \frac{|u_\delta(x)|^2}{|x|^2}\, dx.$$

Note that since τ_δ is a continuous function of δ and $\tau_\delta \to \tau_0$ as $\delta \to 0$, we complete the proof. □

Remark 2. Note that there is no minimiser for inequality (2.2).

The next statement was already proved in [HL].

Theorem 3. *Let* $d \geqslant 3$ *and* $0 \leqslant \Phi \in L^p(\mathbb{S}^{d-1})$ *with* $p \geqslant \frac{d-1}{2} + \frac{1}{2(d-1)}$. *Then*

$$\int_{\mathbb{R}^d} |\nabla u|^2\, dx \geqslant \tau \int_{\mathbb{R}^d} \frac{\Phi(x/|x|)|u|^2}{|x|^2} dx, \quad u \in \mathcal{H}^1(\mathbb{R}^d). \tag{2.4}$$

Theorem 3 provides an estimate for τ_0.

Corollary 1. *Under the conditions given in Theorem* 3, *we have*

$$\tau_0 \geqslant \frac{(d-2)^2}{4} |\mathbb{S}^{d-1}|^{1/p} \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}.$$

Remark 3. If for example $d = 3$, then the lowest possible value of p that is allowed in Theorem 3 equals $5/4$.

The proof of Theorem 3 was based on the estimate of the lowest eigenvalue λ of the operator $\Delta_\vartheta - \Phi$ obtained in [DEL].

§3. Block radial Hardy inequalities, some examples

One of the results proved in [ST] is a Hardy inequality for block radial functions. Let $x \in \mathbb{R}^d$ and let $r = |x|$, $r_1 = (x_1^2 + \cdots + x_k^2)^{1/2}$ and $r_2 = (x_{k+1}^2 + \cdots + x_d^2)^{1/2}$, $1 \leqslant k \leqslant d-1$. In particular, Theorem 1 in [ST] states that if u is block radial such that $u(x) = u(r_1, r_2)$, then

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx \geqslant C \int_{\mathbb{R}^d} \frac{|u(x)|^2}{r_1^\alpha r_2^{2-\alpha}}\, dx, \quad u \in \mathcal{H}^1(\mathbb{R}^d), \tag{3.1}$$

with $0 < \alpha < 2$ with some constant $C = C(k)$.

In this section we give some examples of inequality (3.1) for not necessarily block radial functions u and describe the sharp value of the constant C in this inequality. The potential functions Φ considered below satisfy all the conditions from Theorems 1 and 2.

3.1. $\mathbb{R}^4$**-case.** To demonstrate our idea, we start with the case of $d = 4$, $\alpha = 1$, and $k = 2$. Note that in this case $(d-2)^2/4 = 1$ and $r_1 = \sqrt{x_1^2 + x_2^2}$, $r_2 = \sqrt{x_3^2 + x_4^2}$, $r^2 = r_1^2 + r_2^2$. We introduce

$$\Phi(r_1, r_2) = \frac{r^2}{r_1\, r_2} = \frac{r_1^2 + r_2^2}{r_1\, r_2} \geqslant 0.$$

Applying Theorem 1, we find

$$\int\limits_{\mathbb{R}^4} |\nabla u(x)|^2\, dx \geqslant \tau_0 \int\limits_{\mathbb{R}^4} \frac{\Phi(x/|x|)}{|x|^2} |u(x)|^2\, dx = \tau_0 \int\limits_{\mathbb{R}^4} \frac{1}{r_1 r_2} |u(x)|^2\, dx,$$

where τ_0 is defined by (2.1). Thus (2.1) becomes

$$\lambda(\tau_0) = \inf_{\psi \in \mathcal{H}^1(\mathbb{S}^3)\setminus\{0\}} \frac{\int\limits_{\mathbb{S}^3} (|\nabla_\vartheta \psi|^2 - \tau_0 \Phi\, |\psi|^2)\, d\vartheta}{\int\limits_{\mathbb{S}^3} |\psi|^2\, d\vartheta} = -1. \tag{3.2}$$

On $\mathbb{S}^3$ we introduce the classical Hopf coordinates that are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2.$$

If $x \in \mathbb{R}^4$, then

$$x_1 = r \cos \xi_1 \sin \eta, \qquad x_2 = r \sin \xi_1 \sin \eta,$$
$$x_3 = r \cos \xi_2 \cos \eta, \qquad x_4 = r \sin \xi_2 \cos \eta.$$

Here η runs over the range 0 to $\pi/2$ and ξ_1 and ξ_2 can take any values between 0 and 2π. For any fixed value of $\eta \in [0, \pi/2]$ the coordinates (ξ_1, ξ_2) parameterise a 2-dimensional torus. In the degenerate cases, when η equals 0 or $\pi/2$, these coordinates describe a circle.

The Jacobian of the mapping $x \mapsto (r, \vartheta)$, $\vartheta = (\xi_1, \xi_2, , \eta)$, is defined by

$$dx = r^3 \sin \eta \cos \eta\, dr d\xi_1 d\xi_2 d\eta$$

and $r_1 = r \sin \eta$, $r_2 = r \cos \eta$. Then the function Φ in these coordinates equals

$$\Phi(r_1, r_2) = \frac{1}{\sin \eta \cos \eta} \geqslant 0, \quad \eta \in (0, \pi/2).$$

Simple computations imply that for a function ψ on $\mathbb{S}^3$ in Hopf coordinates we have

$$\int\limits_{\mathbb{S}^3} |\nabla_\vartheta \psi|^2\, dx = \int\limits_0^{\frac{\pi}{2}} \int\limits_0^{2\pi} \int\limits_0^{2\pi} \left(\frac{|\psi'_{\xi_1}|^2}{\sin^2 \eta} + \frac{|\psi'_{\xi_2}|^2}{\cos^2 \eta} + |\psi'_\eta|^2 \right) \sin \eta\, \cos \eta\, d\xi_1 d\xi_2 d\eta. \tag{3.3}$$

The class of functions in (3.3) satisfies periodic boundary conditions with respect to the variables ξ_1 and ξ_2 on the interval $(0, 2\pi)$.

Let $\psi_{nm}(\eta)$ be the Fourier coefficients of ψ with respect to ξ_1 and ξ_2:

$$\psi_{nm}(\eta) = \int_0^{2\pi}\int_0^{2\pi} e^{-i(n\,\xi_1 + m\,\xi_2)}\psi(\xi_1, \xi_2, \eta)\,d\xi_1 d\xi_2\,d\eta.$$

Then using the Parseval identity we can rewrite the expression (3.3):

$$\int_{\mathbb{S}^3} |\nabla_\vartheta \psi|^2\,dx = (2\pi)^2 \sum_{nm} \int_0^{\frac{\pi}{2}} \left(\frac{n^2 |\psi_{nm}(\eta)|^2}{\sin^2(\eta)} + \frac{m^2|\psi_{nm}(\eta)|^2}{\cos^2(\eta)} + |\psi_{nm}(\eta)'|^2 \right) d\eta.$$

The infimum of the above expression is attained at $n = m = 0$. Therefore the terms with ψ'_{ξ_1} and ψ'_{ξ_2} in (3.3) can be omitted and thus the infimum in (3.2) is reduced to the class of functions ψ depending only on η:

$$\inf_{\psi\in\mathcal{H}^1((0,\pi/2),d\rho)\setminus\{0\}} \frac{\int_0^{\pi/2} \left(|\psi'_\eta|^2 - \frac{\tau_0}{\sin\eta\,\cos\eta}\,|\psi|^2\right)\,\sin\eta\,\cos\eta\,d\eta}{\int_0^{\pi/2} |\psi|^2 \sin\eta\cos\eta\,d\eta} = -1, \qquad (3.4)$$

where $d\rho(\eta) = \sin\eta\cos\eta\,d\eta$ and

$$\mathcal{H}^1((0,\pi/2), d\rho) = \big\{\psi \in L^2((0,\pi/2), d\rho) :\ \psi' \in L^2((0,\pi/2), d\rho)\big\}.$$

Note that for ψ defined on $\mathbb{S}^3$ and depending only on η we have

$$\|\psi\|_{L^2(\mathbb{S}^3)} = 4\pi^2 \int_0^{\pi/2} |\psi|^2 \sin\eta\,\cos\eta\,d\eta.$$

Due to Remark 1, τ_0 in (3.4) is uniquely achieved. The quadratic form

$$\int_0^{\pi/2} (|\psi'(\eta)|^2 \sin\eta\cos\eta - \tau|\psi(\eta)|^2)\,d\eta, \quad \tau > 0, \quad \psi \in H^1((0,\pi/2), d\rho),$$

defines a singular selfadjoint Sturm–Liouville operator

$$A_\tau\psi(\eta) = -\psi''(\eta)\sin 2\eta - 2\psi'(\eta)\cos 2\eta - 2\tau\psi(\eta),$$

acting in $\mathcal{H}^2((0,\pi/2), d\rho)$.

Therefore, finding ψ and τ_0 in (3.4) is reduced to finding the minimal value of $\tau_0 > 0$ satisfying the equation

$$A_{\tau_0}\psi(\eta) = -\psi(\eta)\sin 2\eta. \qquad (3.5)$$

Theorem 4. *Let $\tau = \tau_0$ be the minimal value for which equation* (3.5) *has a nontrivial solution ψ. Then*

$$\int_{\mathbb{R}^4} |\nabla u(x)|^2 dx \geqslant \tau_0 \int_{\mathbb{R}^4} \frac{|u(x)|^2}{r_1 r_2}\, dx, \quad u \in \mathcal{H}^1(\mathbb{R}^4). \tag{3.6}$$

The constant τ_0 in (3.6) *is sharp.*

Proof. This statement immediately follows from Theorems 1 and 2. □

Remark 4. Note that inequality (3.6) differs substantially from the standard Hardy inequality in $\mathbb{R}^4$ because

$$\frac{1}{r_1 r_2} \geqslant \frac{2}{r_1^2 + r_2^2}$$

and it has singularities not only at zero but also at $r_1 = 0$ and $r_2 = 0$.

Remark 5. In the results that will be given in the following theorems we use a similar reduction that we had when obtaining (3.4), (3.5) due to the periodicity of the test functions on intervals $(0, 2\pi)$.

The next statement is related to inequality (3.1) in dimension four with α not necessarily equal to one. Let

$$\Phi_\alpha(r_1, r_2) = \frac{r^2}{r_1^\alpha r_2^{2-\alpha}} = \frac{1}{\sin^\alpha \eta \cos^{2-\alpha} \eta}, \quad \eta \in (0, \pi/2),$$

where $0 < \alpha < 2$. In this case the quotient (3.4) becomes

$$\inf_{\psi \in \mathcal{H}^1(0,\pi/2)\setminus\{0\}} \frac{\int_0^{\pi/2} \left(|\psi'_\eta|^2 - \frac{\tau_0}{\sin^\alpha \eta \cos^{2-\alpha} \eta} |\psi|^2 \right) \sin\eta \, \cos\eta \, d\eta}{\int_0^{\pi/2} |\psi|^2 \sin\eta \cos\eta \, d\eta} = -1. \tag{3.7}$$

Therefore as in (3.5) the problem is reduced to finding τ_0 that appears in the equation

$$-\psi''(\eta) \sin 2\eta - 2\psi'(\eta) \cos 2\eta - \frac{2\tau_0}{\sin^{\alpha-1} \eta \cos^{1-\alpha} \eta} \psi(\eta) = -\psi(\eta) \sin 2\eta, \tag{3.8}$$

and we have the following statement.

Theorem 5. *Let τ_0 be the minimal value for which equation* (3.8) *has a nontrivial solution. If $0 < \alpha < 2$, then*

$$\int_{\mathbb{R}^4} |\nabla u(x)|^2 dx \geqslant \tau_0 \int_{\mathbb{R}^4} \frac{|u(x)|^2}{r_1^\alpha r_2^{2-\alpha}}\, dx, \quad u \in \mathcal{H}^1(\mathbb{R}^4).$$

The constant τ_0 in the last inequality is sharp.

3.2. The case of $d = 3$. Let us introduce the standard spherical coordinates in $\mathbb{R}^3$:

$$x_1 = r\sin\eta, \quad x_2 = r\cos\xi\cos\eta, \quad x_3 = r\sin\xi\cos\eta$$

with $\xi \in [0, 2\pi]$, $\eta \in [-\pi/2, \pi/2]$, $r > 0$.

The Jacobian of the mapping $x \mapsto (r, \vartheta)$, $\vartheta = (\xi, \eta)$ is defined by

$$dx = r^2 \cos\eta \, dr d\xi d\eta.$$

Denote $r_1 = |x_1|$ and $r_2 = \sqrt{x_2^2 + x_3^2}$ and thus $|x|^2 = r^2 = r_1^2 + r_2^2$. As before, we introduce

$$\Phi(r_1, r_2) = \frac{r_1^2 + r_2^2}{r_1^\alpha r_2^{2-\alpha}}, \qquad 0 < \alpha < 1.$$

Simple computations give

$$\int\limits_{\mathbb{R}^3} |\nabla u|^2 \, dx = \int\limits_0^\infty \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} \left(|u_r'|^2 + \frac{|u_\xi'|^2}{r_2^2} + \frac{|u_\eta'|^2}{r^2} \right) r^2 \cos\eta \, d\eta d\xi dr.$$

The version of the quotient (3.4) for defining the sharp constant τ_0 becomes

$$\inf_{\psi \in \mathcal{H}^1(-\pi/2, \pi/2) \setminus \{0\}} \frac{\int\limits_{-\pi/2}^{\pi/2} \left(|\psi_\eta'|^2 - \frac{\tau_0}{\sin^\alpha \eta \cos^{2-\alpha} \eta} |\psi|^2 \right) \cos\eta \, d\eta}{\int\limits_0^{\pi/2} |\psi|^2 \cos\eta \, d\eta} = -\frac{1}{4}. \tag{3.9}$$

Theorem 6. *Let $0 < \alpha < 1$ and let τ_0 be defined by the minimising problem (3.9). Then*

$$\int\limits_{\mathbb{R}^3} |\nabla u(x)|^2 dx \geqslant \tau_0 \int\limits_{\mathbb{R}^3} \frac{|u(x)|^2}{r_1^\alpha r_2^{2-\alpha}} \, dx, \quad u \in H^1(\mathbb{R}^3).$$

The constant $\tau_0 = \tau_0(\alpha)$ in the last inequality is sharp.

3.3. Hopf coordinates in $\mathbb{R}^8$. In this subsection we give one more example that is related to the classical Hopf bundle

$$\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{S}^4.$$

If $x \in \mathbb{R}^8$, we introduce the coordinates

$$\begin{aligned}
x_1 &= r\cos\xi_1 \cos\eta_2 \sin\eta_1, & x_2 &= r\sin\xi_1 \cos\eta_2 \sin\eta_1, \\
x_3 &= r\cos\xi_2 \sin\eta_2 \sin\eta_1, & x_4 &= r\sin\xi_2 \sin\eta_2 \sin\eta_1, \\
x_5 &= r\cos\xi_3 \cos\eta_3 \cos\eta_1, & x_6 &= r\sin\xi_3 \cos\eta_3 \cos\eta_1, \\
x_7 &= r\cos\xi_4 \sin\eta_3 \cos\eta_1, & x_8 &= r\sin\xi_4 \sin\eta_3 \cos\eta_1.
\end{aligned}$$

Here $\xi_j \in [0, 2\pi]$, $j = 1, \dots, 4$, and $\eta_j \in |0, \pi/2]$, $j = 1, 2, 3$. Clearly,

$$r^2 = \sum_{j=1}^{8} x_j^2.$$

Using elementary computations, we deduce that the Jacobian of the mapping $x \mapsto (r, \xi, \eta)$ is defined by

$$dx = \frac{1}{8} r^7 \, \sin^3(2\eta_1) \sin 2\eta_2 \sin 2\eta_3 \; dr \, d\xi \, d\eta,$$

where $\xi = (\xi_1, \dots \xi_4)$, and $\eta = (\eta_1, \eta_2, \eta_3)$.

Moreover, if $Q_1 = (0, 2\pi)^4$ and $Q_2 = (0, \pi/2)^3$, then

$$\int\limits_{\mathbb{R}^8} |\nabla u(x)|^2 dx = \frac{1}{8} \int\limits_0^\infty \int\limits_{Q_1} \int\limits_{Q_2} \Bigg(|u'_r|^2 + \frac{1}{r^2 \cos^2 \eta_1} \left(\frac{|u'_{\xi_1}|^2}{\cos^2 \eta_2} + \frac{|u'_{\xi_2}|^2}{\sin^2 \eta_2} + |u'_{\eta_2}|^2 \right)$$
$$+ \frac{1}{r^2 \sin^2 \eta_1} \left(\frac{|u'_{\xi_3}|^2}{\cos^2 \eta_3} + \frac{|u'_{\xi_4}|^2}{\sin^2 \eta_3} + |u'_{\eta_3}|^2 \right) + |u_{\eta_1}|^2 \Bigg)$$
$$\times r^7 \, \sin^3(2\eta_1) \sin 2\eta_2 \sin 2\eta_3 \; d\eta \, d\xi \, dr.$$

Now, we introduce

$$r_1^2 = x_1^2 + x_2^2, \quad r_2^2 = x_3^2 + x_4^2, \quad r_3^2 = x_5^2 + x_6^2, \quad r_4^2 = x_7^2 + x_8^2.$$

Then

$$\Phi(r_1, r_2, r_3, r_4) = \frac{r^2}{\left(\prod\limits_{j=1}^{4} r_j \right)^{1/2}} = \frac{4}{\sin(2\eta_1) \sqrt{\sin(2\eta_2) \sin(2\eta_3)}}.$$

Let define

$$d\mu = \sin^3 2\eta_1 \sin 2\eta_2 \sin 2\eta_3 \; d\eta.$$

Theorem 7. *We have*

$$\int\limits_{\mathbb{R}^8} |\nabla u(x)|^2 \, dx \geqslant \tau_0 \int\limits_{\mathbb{R}^8} \frac{|u(x)|^2}{\left(\prod\limits_{j=1}^{4} r_j \right)^{1/2}} \, dx,$$

where τ_0 *is defined by the equation*

$$\inf_{\psi \in \mathcal{H}^1(Q_2) \setminus \{0\}} \frac{\int\limits_{Q_2} \left(\frac{|\psi'_{\eta_2}|^2}{\cos^2 \eta_1} + \frac{|\psi'_{\eta_3}|^2}{\sin^2 \eta_1} + |\psi'_{\eta_1}|^2 - \tau_0 \frac{4 \, |\psi|^2}{\sin(2\eta_1) \sqrt{\sin(2\eta_2) \sin(2\eta_3)}} \right) d\mu}{\int\limits_{Q_2} |\psi|^2 \, d\mu} = -9.$$

Proof. The reduction given in this theorem can be obtained much as the reduction of (3.2) to (3.4). But of course now we cannot reduce the problem to an ODE. □

Acknowledgment. The authors are grateful to the referee whose remarks substantially improved the paper.

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Поступило 7 августа 2019 г.

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