Short communications

EURASIAN MATHEMATICAL JOURNAL
ISSN 2077-9879
Volume 8, Number 2 (2017), 97 – 104

ON THE BOUNDEDNESS OF HAUSDORFF OPERATORS ON MORREY-TYPE SPACES

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Communicated by N.A. Bokayev

Dedicated to the 70th birthday of Professor Ryskul Oinarov

Key words: Hausdorff operator, Morrey space.

AMS Mathematics Subject Classification: 47B38, 46E30, 42B10.

Abstract. We give conditions ensuring the boundedness of Hausdorff operators on Morrey-type spaces. Sharpness of the obtained results is studied, and classes of the Hausdorff operators are described for which the necessary and sufficient conditions coincide.

1 Introduction

The Hausdorff means are known long ago in connection with summability of number series. It is also worth mentioning that many classical operators are particular cases of the Hausdorff ones, the Cesàro means (the Hardy operator) among them. However, modern theory of these operators has begun to develop quite recently, [14] is a recognized base point.

In the previous episodic works Hausdorff operators were considered mostly on Lebesgue spaces. The main feature of today study is a variety of spaces on which the boundedness of the Hausdorff operators is proved.

The main step in [14] as compared with preceding works was involving the real Hardy space in place of the Lebesgue spaces. Further results are surveyed in detail in [13] and [6].

In the present paper the Morrey-type spaces are the main field on which the game is played. These spaces have already come to light in line with Hausdorff operators (see, e.g., [10] or [8]) but surprisingly not in straightforward estimates of the norms of operators.

Originally, the Morrey space was introduced by Morrey in [16] while studying some quasi-linear elliptic partial differential equations. More general Morrey-type spaces are defined as follows. Let $0 < p, \theta \leq \infty$.

The local Morrey-type space $LM^{\lambda}_{p\theta}$, where

$$\lambda > 0 \quad \text{if} \quad \theta < \infty \quad \text{and} \quad \lambda \geq 0 \quad \text{if} \quad \theta = \infty, \tag{1.1}$$

is the set of all functions $L^p_{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{LM^{\lambda}_{p\theta}} = \left(\int_0^{\infty} \left(\frac{\|f\|_{L^p(B(0,r))}}{r^{\lambda}} \right)^{\theta} \frac{dr}{r} \right)^{\frac{1}{\theta}} < \infty$$

if $\theta < \infty$ and

$$\|f\|_{LM^{\lambda}_{p\infty}} = \sup_{r>0} \frac{\|f\|_{L^p(B(0,r))}}{r^{\lambda}} < \infty.$$

The global Morrey-type space $GM^{\lambda}_{p\theta}$, where

$$0 < \lambda < \frac{n}{p} \quad \text{if} \quad \theta < \infty \quad \text{and} \quad 0 \le \lambda \le \frac{n}{p} \quad \text{if} \quad \theta = \infty, \tag{1.2}$$

is the set of all functions $L^p_{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{GM^{\lambda}_{p\theta}} = \sup_{x\in\mathbb{R}^n} \left(\int_0^{\infty} \left(\frac{\|f\|_{L^p(B(x,r))}}{r^{\lambda}} \right)^{\theta} \frac{dr}{r} \right)^{\frac{1}{\theta}} < \infty$$

if $\theta < \infty$ and

$$\|f\|_{GM^{\lambda}_{p\infty}} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} \frac{\|f\|_{L^p(B(x,r))}}{r^{\lambda}} < \infty.$$

If $\theta = \infty$, then $GM^{\lambda}_{p\infty} \equiv M^{\lambda}_p$ is the classical Morrey space.

If $\lambda = 0$, then $LM^0_{p\infty} = GM^0_{p\infty} = L^p(\mathbb{R}^n)$. Also, in the limiting case $\lambda = \frac{n}{p}$, $GM^{\frac{n}{p}}_{p\infty} = L^{\infty}(\mathbb{R}^n)$. If conditions (1.1) and (1.2) are not satisfied for the local Morrey-type space $LM^{\lambda}_{p\theta}$, the global Morrey-type space $GM^{\lambda}_{p\theta}$ respectively, then these spaces are trivial (consist only of functions equivalent to 0 on $\mathbb{R}^n$).

As for the Hausdorff operators, we consider them in one of their most general forms

$$(\mathcal{H}f)(x) = (\mathcal{H}_{\Phi}f)(x) = (\mathcal{H}_{\Phi,A}f)(x) = \int_{\mathbb{R}^n} \Phi(u) f\big(xA(u)\big)\, du, \tag{1.3}$$

where Φ is an averaging function and $A = A(u) = (a_{ij})^n_{i,j=1} = \big(a_{ij}(u)\big)^n_{i,j=1}$ is an $n \times n$ matrix with the entries $a_{ij}(u)$ being measurable functions of u. This matrix may be degenerate at most on a set of measure zero; $xA(u)$ is the row n-vector obtained by multiplying the row n-vector x by the matrix A.

In the last three decades there is a lot of interest in studying boundedness of various operators of real analysis (maximal operator, fractional maximal operator, Riesz potential, singular integrals, etc) in Morrey-type spaces and general Morrey-type spaces. For references, see the recent surveys [4], [5] and monograph [1].

This paper is devoted to the study of the boundedness of Hausdorff operator (1.3) in the local and global Morrey-type spaces $LM^{\lambda}_{p\theta}$, $GM^{\lambda}_{p\theta}$ respectively. We present the obtained results but omit details concerning the proofs, which will appear elsewhere.

2 Boundedness of Hausdorff operators

In fact, general Hausdorff means of a Fourier-Stieltjes transform were introduced in [7], but only in L^1. In the real Hardy space on $\mathbb{R}$, for the Hausdorff operator defined, by means of $\varphi \in L^1(\mathbb{R})$, as

$$(\mathcal{H}f)(x) = (\mathcal{H}_\varphi f)(x) = \int_{\mathbb{R}} \frac{\varphi(u)}{|u|} f\Big(\frac{x}{u}\Big)\, du \tag{2.1}$$

the boundedness of this operator taking $H^1(\mathbb{R})$ into $H^1(\mathbb{R})$ was proved in [14].

Observe that, as in [7], in [2]-[3] and [15] similar problems are considered for the Hausdorff operators defined by suitable measures, while (2.1) is a particular case of absolutely continuous measures. Here and in what follows we restrict ourselves to the latter case for the sake of brevity and convenience; we are aiming at a different generality, more transparent in a simpler setting. Extensions to arbitrary measures go through as in the cited papers.

Since our main goal is the multi-dimensional setting, we shall use a somewhat more advanced one-dimensional version of the Hausdorff operator, apparently first introduced in [10]:

$$(\mathcal{H}f)(x) = (\mathcal{H}_{\varphi,g} f)(x) = \int_{\mathbb{R}} \frac{\varphi(t)}{|t|} f(g(t)x)\, dt\,.$$

It contains (2.1) as a particular case, while (1.3) is one of the possible general multivariate extensions.

Theorem 2.1. *Let $1 \le p, \theta \le \infty$, φ be a Lebesgue measurable function on $\mathbb{R}$, g be a Lebesgue measurable function on $\mathbb{R}$ such that $g(t) \neq 0$ for almost all $t \in \mathbb{R}$, and for each set $D \subset \mathbb{R}$ of measure 0 the set $g^{-1}(D)$ is also of measure 0.*

If (1.1) is valid, then the condition

$$A(\varphi, g, \lambda, p) = \int_{\mathbb{R}} \frac{|\varphi(t)|}{|t|} |g(t)|^{\lambda - \frac{1}{p}} dt < \infty \tag{2.2}$$

is sufficient for the boundedness of $\mathcal{H}_{\varphi,g}$ on $LM^\lambda_{p\theta}$ and

$$\|\mathcal{H}_{\varphi,g}\|_{LM^\lambda_{p\theta} \to LM^\lambda_{p\theta}} \le A(\varphi, g, \lambda, p).$$

Moreover, if $\varphi(t) \ge 0$, $t \in \mathbb{R}$, and φ is not equivalent to zero on $\mathbb{R}$, then condition (2.2) is necessary and sufficient for the boundedness of $\mathcal{H}_{\varphi,g}$ on $LM^\lambda_{p\theta}$ and

$$\|\mathcal{H}_{\varphi,g}\|_{LM^\lambda_{p\theta} \to LM^\lambda_{p\theta}} = A(\varphi, g, \lambda, p).$$

If (1.2) is satisfied, then the above statement holds with the space $LM^\lambda_{p\theta}$ being replaced by the space $GM^\lambda_{p\theta}$.

In the multidimensional case the situation is, as usual, more complicated. As is mentioned, we consider the Hausdorff operators (like in, say, [11] and similar to [3] and [15]) to be of one of the most general forms (1.3).

Let $\|B\|_2 = \max_{|x|=1} |Bx^T|$, where $|\cdot|$ denotes the Euclidean norm and B is an $n \times n$ matrix.

Theorem 2.2. *Let $1 \leq p, \theta \leq \infty$, Φ be a Lebesgue measurable function on $\mathbb{R}^n$, all the entries $a_{ij}(u)$ of the matrix $A(u)$ be Lebesgue measurable functions on $\mathbb{R}^n$, and for each set $D \subset \mathbb{R}^n$ of measure 0 the set $a_{ij}^{-1}(D)$ be also of measure 0 for all i, j, $1 \leq i, j \leq n$.*

If (1.1) is valid, then

$$\|\mathcal{H}_{\Phi,A}\|_{LM^{\lambda}_{p\theta} \to LM^{\lambda}_{p\theta}} \leq \int_{\mathbb{R}^n} |\Phi(u)| \, |\det A(u)|^{-\frac{1}{p}} \|A(u)\|_2^{\lambda} \, du. \tag{2.3}$$

Moreover, if $\Phi(u) \geq 0$ for almost all $u \in \mathbb{R}^n$, then

$$\|\mathcal{H}_{\Phi,A}\|_{LM^{\lambda}_{p\theta} \to LM^{\lambda}_{p\theta}} \geq C \int_{\mathbb{R}^n} \Phi(u) |\det A(u)^{-1}| \, \|A(u)^{-1}\|_2^{-\lambda - \frac{n}{p'}} \, du, \tag{2.4}$$

where $C > 0$ depends only on n, p, θ, λ.

If (1.2) is satisfied, then the above statement holds with the space $LM^{\lambda}_{p\theta}$ being replaced by the space $GM^{\lambda}_{p\theta}$.

Corollary 2.1. *Let $1 \leq p, \theta \leq \infty$, Φ be a Lebesgue measurable function on $\mathbb{R}^n$, all the entries $a_{ij}(u)$ of the matrix $A(u)$ be Lebesgue measurable functions on $\mathbb{R}^n$, and for each set $D \subset \mathbb{R}^n$ of measure 0 the set $a_{ij}^{-1}(D)$ be also of measure 0 for all i, j, $1 \leq i, j \leq n$. Assume that for some $c > 0$,*

$$\|A(u)^{-1}\|_2 \leq c \|A(u)\|_2^{-1} \tag{2.5}$$

for almost all $u \in \mathbb{R}^n$, and $\Phi(u) \geq 0$ for almost all $u \in \mathbb{R}^n$.

If (1.1) is valid, then the condition

$$\int_{\mathbb{R}^n} \Phi(u) \, |\det A(u)|^{-\frac{1}{p}} \|A(u)\|_2^{\lambda} \, du < \infty \tag{2.6}$$

is necessary and sufficient for the boundedness of $H_{\Phi,A}$ on $LM^{\lambda}_{p\theta}$.

If (1.2) is satisfied, then the above statement holds with the space $LM^{\lambda}_{p\theta}$ being replaced by the space $GM^{\lambda}_{p\theta}$.

Remark 1. *Let $|\lambda(u)|_{\min}$, $|\lambda(u)|_{\max}$ respectively, denote the minimal, the maximal respectively, of the absolute values of the eigenvalues of the matrix $A(u)$. Since $\|A(u)\|_2 = |\lambda(u)|_{\max}$, it follows that condition (2.5) is equivalent to the following one:*

$$|\lambda(u)|_{\max} \leq c \, |\lambda(u)|_{\min} \tag{2.7}$$

for almost all $u \in \mathbb{R}^n$.

3 Examples

Let us start with a multidimensional example where necessary and sufficient conditions coincide and we thus get a criterion for the boundedness of a Hausdorff operator on the Morrey spaces. In recent papers, see, e.g., [17], an important particular case of the Hausdorff operator has been considered:

$$(\mathcal{H}_{\frac{\Phi(\cdot)}{|u|^n},A_d} f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(u)}{|u|^n} f\big(\frac{x}{|u|}\big)\, du. \tag{3.1}$$

Here A_d denotes the diagonal matrix with all the diagonal entries $\frac{1}{|u|}$. Since in this case routine calculations give that the norms on the right-hand sides of (2.3) and (2.4) coincide, Theorem 2.2 reduces to

Corollary 3.1. *Let $1 \le p,\theta \le \infty$ and $\Phi(u) \ge 0$. If (1.1) is valid, then the condition*

$$\int_{\mathbb{R}^n} \Phi(u)\,|u|^{-\frac{n}{p'}-\lambda} du < \infty$$

is necessary and sufficient for the boundedness of $\mathcal{H}_{\frac{\Phi(\cdot)}{|u|^n},A_d}$ on $LM^\lambda_{p\theta}$ and there exist $C_1, C_2 > 0$ depending only on n, p, θ, λ such that

$$C_1 \int_{\mathbb{R}^n} \Phi(u)\,|u|^{-\frac{n}{p'}-\lambda} du \le \|\mathcal{H}_{\frac{\Phi(\cdot)}{|u|^n},A_d}\|_{LM^\lambda_{p\theta}\to LM^\lambda_{p\theta}} \le C_2 \int_{\mathbb{R}^n} \Phi(u)\,|u|^{-\frac{n}{p'}-\lambda} du.$$

The same, under assumption (1.2), is true for $GM^\lambda_{p\theta}$.

Moreover, we can immediately generalize this statement by introducing a somewhat more general Hausdorff operator than (3.1) by letting the matrix to be diagonal with equal but arbitrary diagonal entries $G(u)$. We denote this matrix by A_G and define the operator by

$$(\mathcal{H}_{\frac{\Phi(\cdot)}{|u|^n},A_G} f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(u)}{|u|^n} f(G(u)x)\, du.$$

Thus, the next corollary is a direct extension of Theorem 2.1.

Corollary 3.2. *Let $1 \le p,\theta \le \infty$ and $\Phi(u), G(u) \ge 0$. If (1.1) is valid, then the condition*

$$\int_{\mathbb{R}^n} \frac{\Phi(u)}{|u|^n}\, G(u)^{\lambda-\frac{n}{p}} du < \infty$$

is necessary and sufficient for the boundedness of $\mathcal{H}_{\frac{\Phi(\cdot)}{|u|^n},A_G}$ on $LM^\lambda_{p\theta}$ and there exist $C_1, C_2 > 0$ depending only on n, p, θ, λ such that

$$C_1 \int_{\mathbb{R}^n} \frac{\Phi(u)}{|u|^n}\, G(u)^{\lambda-\frac{n}{p}} du \le \|\mathcal{H}_{\frac{\Phi(\cdot)}{|u|^n},A_G}\|_{LM^\lambda_{p\theta}\to LM^\lambda_{p\theta}}$$
$$\le C_2 \int_{\mathbb{R}^n} \frac{\Phi(u)}{|u|^n}\, G(u)^{\lambda-\frac{n}{p}} du.$$

The same, under assumption (1.2), is true for $GM^\lambda_{p\theta}$.

Acknowledgments

The research of the first author was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number 02.a03.21.008).

References

[1] D.R. Adams, *Morrey spaces*, Birkhäuser, 2015.

[2] K.F. Andersen, *Boundedness of Hausdorff operators on $L^p(\mathbb{R}^n), H^1(\mathbb{R}^n)$, and $BMO(\mathbb{R}^n)$*, Acta Sci. Math. (Szeged) 69 (2003), 409–418.

[3] G. Brown, F. Móricz, *Multivariate Hausdorff operators on the spaces $L^p(\mathcal{R}^n)$*, J. Math. Anal. Appl. 271 (2002), 443–454.

[4] V.I. Burenkov, *Recent progress in the problem of the boundedness of classical operators of real analysis in general Morrey-type spaces. I*, Eurasian Math. J. 3 (2012), no. 3, 11–32.

[5] V.I. Burenkov, *Recent progress in the problem of the boundedness of classical operators of real analysis in general Morrey-type spaces. II*, Eurasian Math. J. 4 (2013), no. 1, 21–45.

[6] J. Chen, D. Fan, S. Wang, *Hausdorff operators on Euclidean spaces*, Appl. Math. J. Chinese Univ. (Ser. B) (4) 28 (2014), 548–564.

[7] C. Georgakis, *The Hausdorff mean of a Fourier-Stieltjes transform*, Proc. Am. Math. Soc. 116 (1992), 465–471.

[8] A. Hussain, G. Gao, *Multidimensional Hausdorff operators and commutators on Herz-type spaces*, J. Inequal. Appl. 594 (2013), 1–12.

[9] Y. Kanjin, *The Hausdorff operators on the real Hardy spaces $H^p(\mathbb{R})$*, Studia Math. 148 (2001), 37–45.

[10] J.C. Kuang, *Generalized Hausdorff operators on weighted Morrey-Herz spaces*, Acta Math. Sinica (Chin. Ser.) 55 (2012), 895–902 (Chinese; Chinese, English summaries).

[11] A. Lerner, E. Liflyand, *Multidimensional Hausdorff operator on the real Hardy space*, J. Austr. Math. Soc. 83 (2007), 79–86.

[12] E. Liflyand, *Boundedness of multidimensional Hausdorff operators on $H^1(\mathbb{R}^n)$*, Acta Sci. Math. (Szeged) 74 (2008), 845–851.

[13] E. Liflyand, *Hausdorff operators on Hardy spaces*, Eurasian Math. J. 4 (2013), no. 4, 101–141.

[14] E. Liflyand, F. Móricz, *The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$*, Proc. Am. Math. Soc. 128 (2000), 1391–1396.

[15] F. Móricz, *Multivariate Hausdorff operators on the spaces $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$*, Analysis Math. 31 (2005), 31–41.

[16] C. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43 (1938), 126–166.

[17] J. Ruan, D. Fan, *Hausdorff operators on the power weighted Hardy spaces*, J. Math. Anal. Appl. 433 (2016), 31–48.

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Received: 31.08.2016