Evgueniya Dyachenko, Nikolai Tarkhanov, Degeneration of Boundary Layer at Singular Points,
J. Sib. Fed. Univ. Math. Phys., 2013, Volume 6, Issue 3, 283–297

Journal of Siberian Federal University. Mathematics & Physics 2013, 6(3), 283–297

УДК 517.55

Degeneration of Boundary Layer at Singular Points

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Received 06.12.2012, received in revised form 06.02.2013, accepted 06.03.2013

We study the Dirichlet problem in a bounded plane domain for the heat equation with small parameter multiplying the derivative in t. The behaviour of solution at characteristic points of the boundary is of special interest. The behaviour is well understood if a characteristic line is tangent to the boundary with contact degree at least 2. We allow the boundary to not only have contact of degree less than 2 with a characteristic line but also a cuspidal singularity at a characteristic point. We construct an asymptotic solution of the problem near the characteristic point to describe how the boundary layer degenerates.

Keywords: Heat equation, Dirichlet problem, characteristic points, boundary layer.

Introduction

Discontinuities and quick transitions occur in various branches of physics. The mathematical questions involved are also rather classical. However, they are quite alive today and they will remain so for some time, cf. [10]. Quick transitions befall frequently in situations in which one perhaps would not speak of a discontinuity. A case in point is Prandtl's ingenious concept of the boundary layer, which he presented at the 1904 Leipzig Mathematical Congress, see [22]. This is a narrow layer along the surface of a body, traveling in a fluid, across which the flow velocity changes quickly. The paper began the study of fluid dynamical boundary layers by analysing viscous incompressible flow past an object as the Reynolds number becomes infinite. Friedrichs called asymptotic all those phenomena which show discontinuities, quick transitions, nonuniformities, or their incongruities resulting from approximate description. In the mathematical treatment of such phenomena, physicists have developed systematic mathematical procedures. In such an approach one may introduce an appropriate quantity with respect to powers of a parameter, ε. This expansion is to be set up in such a way that the quantity is continuous for $\varepsilon > 0$ but discontinuous for $\varepsilon = 0$. Naturally, a series expansion with this character must have peculiar properties. In general these series do not converge. The use of a series which does not necessarily converge is a typical instance of a "formal procedure". The idea of giving validity to these formal series goes back at least as far as Poincaré [21]. He proved that these formal series represent asymptotic expansions of actual solutions. Thus it became clear in which way formal series solutions may be regarded as "valid". Let us explain asymptotic phenomena in connection with singular perturbation problems. In a singular perturbation problem one is concerned with a differential equation of the form $A(\varepsilon)u_\varepsilon = f_\varepsilon$ with initial or boundary conditions $B(\varepsilon)u_\varepsilon = g_\varepsilon$, where ε is a small parameter. The distinguishing feature of this problem is that the orders of $A(\varepsilon)$ and $B(\varepsilon)$ for $\varepsilon \neq 0$ are higher than the orders of $A(0)$ and $B(0)$, respectively. The differential problem in question is referred to as a perturbed problem when $\varepsilon \neq 0$ and a degenerate

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problem when $\varepsilon = 0$. We are interested not in solutions of this problem for each fixed value of the parameter ε, but in the dependence of such solutions on this parameter, in particular, in a neighbourhood of $\varepsilon = 0$. A discussion of the role of singular perturbation phenomena in mathematical physics can be found in [16]. Some difficulties are inherent in singular perturbation problems. Solutions of the degenerate problem will not in general be as smooth as solutions of the perturbed problem. Moreover, solutions of the degenerate problem usually will not satisfy as many initial or boundary conditions as do solutions of the perturbed problem. Hence, if solutions of the perturbed problem are to converge to solutions of the degenerate problem, the notion of convergence will probably have to be rather weak. Due to the "loss" of initial or boundary data it may also happen that solutions of the perturbed problem converge in a stronger sense in the interior of the underlying domain, than in the vicinity of the boundary. This is precisely the boundary layer phenomenon observed by Prandtl. There is by now a vast amount of literature on singular perturbation problems for ordinary differential equations, both linear and nonlinear. An extensive bibliography of this literature is contained in [27]. There is also a considerable amount of literature on singular perturbation problems for partial differential equations. A comprehensive theory of such problems was initiated by the remarkable paper of Vishik and Lyusternik [25]. They obtained asymptotic expressions for solutions of the perturbed problem for linear equations using boundary layer techniques. In this paper the main condition on the dependence of $A(\varepsilon)$ on a small parameter was formulated and the asymptotics as $\varepsilon \to 0$ of the solution of the Dirichlet problem was constructed. The paper [25] also contains a sizable bibliography. In [13], Huet published several theorems on convergence in singular perturbation problems for linear elliptic and parabolic partial differential equations. One particular feature distinguishes this paper from those previously mentioned. This is that convergence theorems are first proven in a Hilbert space setting and then applied to the differential problems as opposed to starting directly with the differential equations. In the elliptic case, theorems on local convergence and convergence of tangential derivatives at the boundary are also proven. The work [13] is fundamental to the considerations in [12] aimed at obtaining rate of convergence estimates for solutions of singular perturbations of linear elliptic boundary value problems. The problem can be described as follows. Let $\mathcal{X}$ be a compact smooth manifold and let ε be a positive real parameter. Consider two elliptic boundary value problems on $\mathcal{X}$, $(\varepsilon \mathcal{A}_1) + \mathcal{A}_0)u_\varepsilon = f$ and $\mathcal{A}_0 u = f$, where the order of $\mathcal{A}_1$ is greater than the order of $\mathcal{A}_0$. The problem is to determine in what sense u_ε converges to u on $\mathcal{X}$ as $\varepsilon \to 0$ and to estimate the rate of convergence. In the 1970s pseudodifferential problems with small parameter were studied in [6] and [7]. For boundary value problems of general type the theory of singular perturbations was developed in the 1980s by Frank, see [9]. In [19] the Vishik-Lyusternik method is developed for general elliptic boundary value problems in domains with conical points. However, this paper falls short of providing explicit Shapiro-Lopatinskii type condition of ellipticity with small parameter, this latter is replaced by a priori estimates for corresponding problems for ordinary differential equations on the half-axis. In [26], Volevich completed the theory of differential boundary value problems with small parameter by formulating the Shapiro-Lopatinskii type ellipticity condition and proving that it is equivalent to a priori estimates uniform in the parameter. It should be noted that paper [26] restricts itself to operators with constant coefficients in the half-space. Asymptotic analysis includes two basic steps. The first is the actual construction of asymptotics. One has to choose the form in which the formal asymptotic expansion of a solution is to be sought, and specify the way of constructing this expansion. The second step includes the justification of asymptotics, i.e., a proof that the formal asymptotic expansion is an asymptotic solution indeed. This is achieved by estimating the discrepancy. Matching of asymptotic expansions of solutions of boundary value problems is presented in the book [14]. The purpose of our paper is to describe the boundary layer near a characteristic point of the boundary. We restrict the discussion to the Dirichlet problem for the heat equation in a bounded plane domain $\mathcal{G}$ which contains a small parameter multiplying the time derivative. The boundary points at which the tangent is orthogonal to the time axis are characteristic. The boundary of $\mathcal{G}$ is moreover allowed to have singularities at characteristic points. We construct an explicit asymptotic solution of the problem in a neighbourhood of a characteristic point. It has the form of a Puiseux series in fractional powers of t/ε up to an exponential factor. Our asymptotic formula demonstrates rather strikingly that the boundary layer degenerates at a characteristic point unless the contact degree of the boundary and a characteristic line is sufficiently large (at least 2).

1. Blow-up techniques

Consider the first boundary value problem for the heat equation in a domain $\mathcal{G} \subset \mathbb{R}^2$ of the type of Fig. 1. The boundary of $\mathcal{G}$ is assumed to be C^∞ except for a finite number of

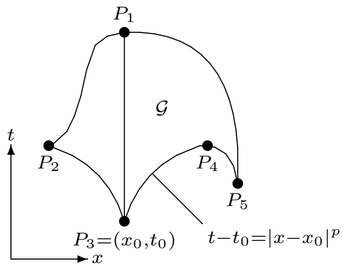

Fig. 1. Typical domain

characteristic points. At points like P_1 and P_2 the boundary curve possesses a tangent which is horizontal, hence $\partial\mathcal{G}$ is characteristic for the heat equation at such points. The characteristic touches the boundary with the degree $\geqslant 2$, which is included in the treatise [17]. At points like P_2 the boundary curve is not smooth but it touches smoothly a characteristic from below and above. Such points are therefore cuspidal singularities of the boundary, explicit treatable cases have been studied in [3].

In this paper we restrict our discussion to characteristic points like P_3 and P_5. These are cuspidal singularities of the boundary curve which touches smoothly a vertical line at P_3 and P_5. Thus, the boundary meets a characteristic at P_3 and P_5 at contact degree < 2. The study of regularity of such points for solutions of the first boundary value problem for the heat equation goes back at least as far as [11]. The classical approach of [11] rests on potential theory. A modern approach to studying boundary value problems in domains with singular points is based on the so-called blow-up techniques, cf. [23]. In [2] it was applied to the first boundary value problem for the heat equation in domains with boundary points like P_3 and P_5 to get both a regularity theorem and the Fredholm property in weighted Sobolev spaces.

The first boundary value problem for the heat equation in $\mathcal{G}$ is formulated as follows: Write Σ for the set of all characteristic points $P_1, P_2, \ldots$ on the boundary of $\mathcal{G}$. Given functions f in $\mathcal{G}$ and u_0 on $\partial\mathcal{G} \setminus \Sigma$, find a function u on $\overline{\mathcal{G}} \setminus \Sigma$ which satisfies

$$\begin{array}{rcll} \varepsilon u'_t - u''_{x,x} & = & f & \text{in} \quad \mathcal{G}, \\ u & = & u_0 & \text{at} \quad \partial\mathcal{G} \setminus \Sigma, \end{array} \tag{1.1}$$

where $\varepsilon \in (0, \varepsilon_0]$ is a small parameter. By the local principle of Simonenko [24], the Fredholm property of problem (1.1) in suitable function spaces is equivalent to the local invertibility of this problem at each point of the closure of $\mathcal{G}$. Here we focus upon the points like P_3.

Suppose the domain $\mathcal{G}$ is described in a neighbourhood of the point $P_3 = (x_0, t_0)$ by the inequality

$$t - t_0 > |x - x_0|^p, \tag{1.2}$$

where p is a positive real number. There is no loss of generality in assuming that P_3 is the origin and $|x - x_0| \leqslant 1$.

We now blow up the domain $\mathcal{G}$ at P_3 by introducing new coordinates (ω, r) with the aid of

$$\begin{array}{rcl} x & = & t^{1/p}\,\omega, \\ t & = & \varepsilon r, \end{array} \tag{1.3}$$

where $|\omega| < 1$ and $r \in (0, 1/\varepsilon)$. It is clear that the new coordinates are singular at $r = 0$, for the entire segment $[-1, 1]$ on the ω-axis is blown down into the origin by (1.3). The rectangle $(-1, 1) \times (0, 1/\varepsilon)$ transforms under the change of coordinates (1.3) into the part of the domain $\mathcal{G}$ nearby P_3 lying below the line $t = 1$. Note that for $\varepsilon \to 0$ the rectangle $(-1, 1) \times (0, 1/\varepsilon)$ stretches to the whole half-strip $(-1, 1) \times (0, \infty)$.

In the domain of coordinates (ω, r) problem (1.1) reduces to an ordinary differential equation with respect to the variable r with operator-valued coefficients. More precisely, under transformation (1.3) the derivatives in t and x change by the formulas

$$\begin{array}{rcl} \varepsilon\,\dfrac{\partial u}{\partial t} & = & \dfrac{\partial u}{\partial r} - \dfrac{1}{r}\dfrac{\omega}{p}\,\dfrac{\partial u}{\partial \omega}, \\ \dfrac{\partial u}{\partial x} & = & \dfrac{1}{(\varepsilon r)^{1/p}}\,\dfrac{\partial u}{\partial \omega}, \end{array}$$

and so (1.1) transforms into

$$\begin{array}{rcll} r^Q\,U'_r - \dfrac{1}{\varepsilon^Q}\,U''_{\omega,\omega} - r^{Q-1}\dfrac{\omega}{p}\,U'_\omega & = & r^Q F & \text{in} \quad (-1,1) \times (0, 1/\varepsilon), \\ U & = & U_0 & \text{at} \quad \{\pm 1\} \times (0, 1/\varepsilon), \end{array} \tag{1.4}$$

where $U(\omega, r)$ and $F(\omega, r)$ are pullbacks of $u(x, t)$ and $f(x, t)$ under transformation (1.3), respectively, and

$$Q = \frac{2}{p}.$$

We are interested in the local solvability of problem (1.4) near the edge $r = 0$ in the rectangle $(-1, 1) \times (0, 1/\varepsilon)$. Note that the ordinary differential equation degenerates at $r = 0$, since the coefficient $r^{2/p}$ of the higher order derivative in r vanishes at $r = 0$. For the parameter values $\varepsilon > 0$, the exponent Q is of crucial importance for specifying the ordinary differential equation. If $p = 2$ then it is a Fuchs-type equation, these are also called regular singular equations. The Fuchs-type equations fit well into an algebra of pseudodifferential operators based on the Mellin transform. If $p > 2$, then the singularity of the equation at $r = 0$ is weak and so regular theory of finite smoothness applies. In the case $p < 2$ the degeneracy at $r = 0$ is strong and the equation can not be treated except by the theory of slowly varying coefficients [23].

2. Formal asymptotic solution

To determine appropriate function spaces in which a solution of problem (1.4) is sought, one constructs formal asymptotic solutions of the corresponding homogeneous problem. That is

$$\begin{array}{rcll} r^Q\,U'_r - \dfrac{1}{\varepsilon^Q}\,U''_{\omega,\omega} - r^{Q-1}\dfrac{\omega}{p}\,U'_\omega & = & 0 & \text{in} \quad (-1,1) \times (0, \infty), \\ U(\pm 1, r) & = & 0 & \text{on} \quad (0, \infty). \end{array} \tag{2.1}$$

We first consider the case $p \neq 2$. We look for a formal solution to (2.1) of the form

$$U(\omega, r) = e^{S(r)}\, V(\omega, r), \tag{2.2}$$

where S is a differentiable function of $r > 0$ and V expands as a formal Puiseux series with nontrivial principal part

$$V(\omega, r) = \frac{1}{r^{\mathfrak{e}N}} \sum_{j=0}^{\infty} V_{j-N}(\omega)\, r^{\mathfrak{e}j},$$

the complex exponent N and real exponent $\mathfrak{e}$ have to be determined. Perhaps the factor $r^{-\mathfrak{e}N}$ might be included into the definition of $\exp S$ as $\exp(-\mathfrak{e}N \ln r)$, however, we prefer to highlight the key role of Puiseux series. Substituting (2.2) into (2.1) yields

$$\begin{array}{rcll} r^Q\ (S'V + V'_r) - \dfrac{1}{\varepsilon^Q}\, V''_{\omega,\omega} - r^{Q-1}\dfrac{\omega}{p}\, V'_{\omega} & = & 0 & \text{in} \quad (-1,1) \times (0,\infty), \\ V(\pm 1, r) & = & 0 & \text{on} \quad (0,\infty). \end{array}$$

In order to reduce this boundary value problem to an eigenvalue problem we require the function S to satisfy the eikonal equation $r^Q S' = \lambda$ with a complex constant λ. This implies

$$S(r) = \lambda\, \frac{r^{1-Q}}{1-Q}$$

up to an inessential constant to be included into a factor of $\exp S$. In this manner the problem reduces to

$$\begin{array}{rcll} r^Q\, V'_r - \dfrac{1}{\varepsilon^Q}\, V''_{\omega,\omega} - r^{Q-1}\dfrac{\omega}{p}\, V'_{\omega} & = & -\lambda V & \text{in} \quad (-1,1) \times (0,\infty), \\ V(\pm 1, r) & = & 0 & \text{on} \quad (0,\infty). \end{array} \tag{2.3}$$

If $\mathfrak{e} = \dfrac{Q-1}{k}$ for some natural number k, then

$$\begin{array}{rcl} r^Q\, V'_r & = & \displaystyle\sum_{j=k}^{\infty} \mathfrak{e}(j-N-k)\, V_{j-N-k} r^{\mathfrak{e}(j-N)}, \\ V''_{\omega,\omega} & = & \displaystyle\sum_{j=0}^{\infty} V''_{j-N} r^{\mathfrak{e}(j-N)}, \\ r^{Q-1}\, V'_{\omega} & = & \displaystyle\sum_{j=k}^{\infty} V'_{j-N-k} r^{\mathfrak{e}(j-N)}, \end{array}$$

as is easy to check. On substituting these equalities into (2.3) and equating the coefficients of the same powers of r we get two collections of Sturm-Liouville problems

$$\begin{array}{rcll} -\dfrac{1}{\varepsilon^Q}\, V''_{j-N} + \lambda\, V_{j-N} & = & 0 & \text{in} \quad (-1,1), \\ V_{j-N} & = & 0 & \text{at} \quad \mp 1, \end{array} \tag{2.4}$$

for $j = 0, 1, \ldots, k-1$, and

$$\begin{array}{rcll} -\dfrac{1}{\varepsilon^Q}\, V''_{j-N} + \lambda\, V_{j-N} & = & \dfrac{\omega}{p}\, V'_{j-N-k} - \mathfrak{e}(j-N-k)\, V_{j-N-k} & \text{in} \quad (-1,1), \\ V_{j-N} & = & 0 & \text{at} \quad \mp 1, \end{array} \tag{2.5}$$

for $j = mk, mk+1, \ldots, mk+(k-1)$, where m takes on all natural values.

Given any $j = 0, 1, \ldots, k-1$, the Sturm-Liouville problem (2.4) has obviously simple eigenvalues

$$\lambda_n = -\frac{1}{\varepsilon^Q}\Big(\frac{\pi}{2}n\Big)^2$$

for $n = 1, 2, \ldots$, a nonzero eigenfunction corresponding to λ_n being $\sin\frac{\pi}{2}n(\omega+1)$. It follows that

$$V_{j-N}(\omega) = c_{j-N}\,\sin\frac{\pi}{2}n(\omega+1), \tag{2.6}$$

for $j = 0, 1, \ldots, k-1$, where c_{j-N} are constant. Without restriction of generality we can assume that the first coefficient V_{-N} in the Puiseux expansion of V is different from zero. Hence, $V_{j-N} = c_{j-N}V_{-N}$ for $j = 1, \ldots, k-1$. For simplicity of notation, we drop the index n.

On having determined the functions $V_{-N}, \ldots, V_{k-1-N}$, we turn our attention to problems (2.5) with $j = k, \ldots, 2k-1$. Set

$$f_{j-N} = \frac{\omega}{p}\,V'_{j-N-k} - \mathfrak{e}(j-N-k)\,V_{j-N-k},$$

then for the inhomogeneous problem (2.5) to possess a nonzero solution V_{j-N} it is necessary and sufficient that the right-hand side f_{j-N} be orthogonal to all solutions of the corresponding homogeneous problem, to wit V_{-N}. The orthogonality refers to the scalar product in $L^2(-1,1)$. Let us evaluate the scalar product (f_{j-N}, V_{-N}). We get

$$(f_{j-N}, V_{-N}) = c_{j-N-k}\Big(\frac{1}{p}\,(\omega V'_{-N}, V_{-N}) - \mathfrak{e}(j-N-k)\,(V_{-N}, V_{-N})\Big)$$

and

$$\begin{aligned}(\omega V'_{-N}, V_{-N}) &= \omega\,|V_{-N}|^2\Big|_{-1}^{1} - (V_{-N}, V_{-N}) - (V_{-N}, \omega V'_{-N}) = \\ &= -(V_{-N}, V_{-N}) - (\omega V'_{-N}, V_{-N}),\end{aligned}$$

the latter equality being due to the fact that V_{-N} is real-valued and vanishes at ± 1. Hence,

$$(\omega V'_{-N}, V_{-N}) = -\frac{1}{2}\,(V_{-N}, V_{-N})$$

and

$$(f_{j-N}, V_{-N}) = -c_{j-N-k}\,\Big(\frac{1}{2p} + \mathfrak{e}(j-N-k)\Big)(V_{-N}, V_{-N}) \tag{2.7}$$

for $j = k, \ldots, 2k-1$.

Since $V_{-N} \neq 0$, the condition $(f_{j-N}, V_{-N}) = 0$ fulfills for $j = k$ if and only if

$$\mathfrak{e}N = \frac{1}{2p}. \tag{2.8}$$

Under this condition, problem (2.5) with $j = k$ is solvable and its general solution has the form

$$V_{k-N} = V_{k-N,0} + c_{k-N}V_{-N},$$

where $V_{k-N,0}$ is a particular solution of (2.5) and c_{k-N} an arbitrary constant. Moreover, for $(f_{j-N}, V_{-N}) = 0$ to fulfill for $j = k+1, \ldots, 2k-1$ it is necessary and sufficient that $c_{1-N} = \ldots = c_{k-1-N} = 0$, i.e., all of $V_{1-N}, \ldots, V_{k-1-N}$ vanish. This in turn implies that $f_{k+1-N} = \ldots = f_{2k-1-N} = 0$, whence $V_{j-N} = c_{j-N}V_{-N}$ for all $j = k+1, \ldots, 2k-1$, where c_{j-N} are arbitrary constants. We choose the constants $c_{k-N}, \dots, c_{2k-1}$ in such a way that the solvability conditions of the next k problems are fulfilled.

More precisely, we consider the problem (2.5) for $j = 2k$, the right-hand side being

$$\begin{aligned} f_{2k-N} &= \Big(\frac{\omega}{p}\, V'_{k-N,0} - \mathfrak{e}(k-N)\, V_{k-N,0}\Big) + c_{k-N}\Big(\frac{\omega}{p}\, V'_{-N} - \mathfrak{e}(k-N)\, V_{-N}\Big) = \\ &= \Big(\frac{\omega}{p}\, V'_{k-N,0} - \mathfrak{e}(k-N)\, V_{k-N,0}\Big) + c_{k-N}\Big(f_{k-N} - \mathfrak{e}k\, V_{-N}\Big). \end{aligned}$$

Combining (2.7) and (2.8) we conclude that

$$\begin{aligned} (f_{k-N} - \mathfrak{e}k\, V_{-N}, V_{-N}) &= -\mathfrak{e}k\, (V_{-N}, V_{-N}) = \\ &= (1-Q)\, (V_{-N}, V_{-N}) \end{aligned}$$

is different from zero. Hence, the constant c_{k-N} can be uniquely defined in such a way that $(f_{2k-N}, V_{-N}) = 0$. Moreover, the functions $f_{2k+1-N}, \dots, f_{3k-1-N}$ are orthogonal to V_{-N} if and only if $c_{k+1-N} = \ldots = c_{2k-1-N} = 0$. It follows that V_{j-N} vanishes for each $j = k+1, \dots, 2k-1$.

Continuing in this fashion we construct a sequence of functions $V_{j-N}(\omega, \varepsilon)$, for $j = 0, 1, \dots$, satisfying equations (2.4) and (2.5). The functions $V_{j-N}(\omega, \varepsilon)$ are defined uniquely up to a common constant factor c_{-N}. They depend smoothly on the parameter ε^p. Moreover, V_{j-N} vanishes identically unless $j = mk$ with $m = 0, 1, \dots$. Therefore,

$$\begin{aligned} V(\omega, r, \varepsilon) &= \frac{1}{r^{\mathfrak{e}N}} \sum_{m=0}^{\infty} V_{mk-N}(\omega, \varepsilon)\, r^{\mathfrak{e}mk} = \\ &= \frac{1}{r^{Q/4}} \sum_{m=0}^{\infty} \tilde{V}_m(\omega, \varepsilon)\, r^{(Q-1)m} \end{aligned}$$

is a unique (up to a constant factor) formal asymptotic solution of problem (2.3) corresponding to $\lambda = \lambda_n$.

Theorem 2.1. *Let $p \neq 2$. Then an arbitrary formal asymptotic solution of homogeneous problem (2.1) has the form*

$$U(\omega, r, \varepsilon) = \frac{c}{r^{Q/4}}\, \exp\Big(\lambda\, \frac{r^{1-Q}}{1-Q}\Big) \sum_{m=0}^{\infty} \frac{\tilde{V}_m(\omega, \varepsilon)}{r^{(1-Q)m}},$$

where λ is one of eigenvalues $\lambda_n = -\dfrac{1}{\varepsilon^Q}\Big(\dfrac{\pi}{2}n\Big)^2$.

Proof. The theorem follows readily from (2.2). □

In the original coordinates (x, t) close to the point P_3 in $\mathcal{G}$ the formal asymptotic solution looks like

$$u(x, t, \varepsilon) = c\Big(\frac{\varepsilon}{t}\Big)^{Q/4} \exp\Big(\frac{\lambda}{1-Q}\Big(\frac{t}{\varepsilon}\Big)^{1-Q}\Big) \sum_{m=0}^{\infty} \tilde{V}_m\Big(\frac{x}{t^{1/p}}, \varepsilon\Big)\Big(\frac{\varepsilon}{t}\Big)^{(1-Q)m} \tag{2.9}$$

for $\varepsilon > 0$. If $1 - Q > 0$, i.e., $p > 2$, expansion (2.9) behaves in much the same way as boundary layer expansion in singular perturbation problems, since the eigenvalues are all negative. The threshold value $p = 2$ is a turning contact order under which the boundary layer degenerates.

3. The exceptional case p = 2

In this section we consider the case $p = 2$ in detail. For $p = 2$, problem (2.1) takes the form

$$\begin{array}{rcll} r\,U'_r - \dfrac{1}{\varepsilon}\,U''_{\omega,\omega} - \dfrac{\omega}{2}\,U'_\omega & = & 0 & \text{in} \quad (-1,1)\times(0,\infty), \\ U(\pm 1, r) & = & 0 & \text{on} \quad (0,\infty). \end{array} \tag{3.1}$$

The problem is specified as Fuchs-type equation on the half-axis with coefficients in boundary value problems on the interval $[-1, 1]$. Such equations have been well understood, see [8] and elsewhere.

If one searches for a formal solution to (3.1) of the form $U(\omega, r) = e^{S(r)}\,V(\omega, r)$, then the eikonal equation $rS' = \lambda$ gives $S(r) = \lambda \ln r$, and so $e^{S(r)} = r^\lambda$, where λ is a complex number. It makes therefore no sense to looking for $V(\omega, r)$ being a formal Puiseux series in fractional powers of r. The choice $\mathfrak{e} = (Q-1)/k$ no longer works, and so a good substitute for a fractional power of r is the function $1/\ln r$. Thus,

$$V(\omega, r) = \sum_{j=0}^{\infty} V_{j-N}(\omega) \left(\frac{1}{\ln r}\right)^{j-N}$$

has to be a formal asymptotic solution of

$$\begin{array}{rcll} r\,V'_r - \dfrac{1}{\varepsilon}\,V''_{\omega,\omega} - \dfrac{\omega}{2}\,V'_\omega & = & -\lambda V & \text{in} \quad (-1,1)\times(0,\infty), \\ V(\pm 1, r) & = & 0 & \text{on} \quad (0,\infty), \end{array}$$

N being a nonnegative integer. Substituting the series for $V(\omega, r)$ into these equations and equating the coefficients of the same powers of $\ln r$ yields two collections of Sturm-Liouville problems

$$\begin{array}{rcll} -\dfrac{1}{\varepsilon}\,V''_{-N} - \dfrac{\omega}{2}\,V'_{-N} + \lambda\,V_{-N} & = & 0 & \text{in} \quad (-1,1), \\ V_{-N} & = & 0 & \text{at} \quad \mp 1, \end{array} \tag{3.2}$$

for $j = 0$, and

$$\begin{array}{rcll} -\dfrac{1}{\varepsilon}\,V''_{j-N} - \dfrac{\omega}{2}\,V'_{j-N} + \lambda\,V_{j-N} & = & (j-N-1)V_{j-N-1} & \text{in} \quad (-1,1), \\ V_{j-N} & = & 0 & \text{at} \quad \mp 1, \end{array} \tag{3.3}$$

for $j \geq 1$.

Problem (3.2) has a nonzero solution V_{-N} if and only if λ is an eigenvalue of the operator

$$v \mapsto \frac{1}{\varepsilon}\,v'' + \frac{\omega}{2}\,v'$$

whose domain consists of all functions $v \in H^2(-1, 1)$ vanishing at ∓ 1. Then, equalities (3.3) for $j = 1, \ldots, N$ mean that $V_{-N+1}, \ldots, V_0$ are actually root functions of the operator corresponding to the eigenvalue λ. In other words, $V_{-N}, \ldots, V_0$ is a Jordan chain of length $N+1$ corresponding to the eigenvalue λ. Note that for $j = N + 1$ the right-hand side of (3.3) vanishes, and so $V_1, V_2, \ldots$ is also a Jordan chain corresponding to the eigenvalue λ. This suggests that the series breaks beginning at $j = N + 1$. Moreover, a familiar argument shows that problem (3.2) has eigenvalues

$$\lambda_n = -\frac{1}{\varepsilon}\Big(\frac{\pi}{2}n\Big)^2 + o\Big(\frac{1}{\varepsilon}\Big)$$

for $n = 1, 2, \ldots$, which are simple if ε is small enough. Hence it follows that $N = 0$ and

$$V_0(\omega, \varepsilon) = c_0\,\sin\frac{\pi}{2}n(\omega + 1) + o(1) \tag{3.4}$$

for $\varepsilon \to 0$.

Theorem 3.1. *Suppose $p = 2$. Then an arbitrary formal asymptotic solution of homogeneous problem (2.1) has the form $U(\omega, r, \varepsilon) = r^{\lambda} V_0(\omega, \varepsilon)$, where λ is one of the eigenvalues λ_n.*

Proof. The theorem follows immediately from the above discussion. □

In the original coordinates (x, t) near the point P_3 in $\mathcal{G}$ the formal asymptotic solution proves to be

$$u(x, t, \varepsilon) = c \left(\frac{\varepsilon}{t} \right)^{-\lambda} V_0 \Big(\frac{x}{t^{1/2}}, \varepsilon \Big)$$

for $\varepsilon > 0$. This expansion behaves similarly to boundary layer expansion in singular perturbation problems, since the eigenvalues are negative provided that ε is sufficiently small.

4. Degenerate problem

If $\varepsilon = 0$ then the homogeneous problem corresponding to local problem (1.4) degenerates to

$$\begin{array}{rcl} U''_{\omega,\omega} & = & 0 \quad \text{in} \quad (-1, 1) \times (0, \infty), \\ U & = & 0 \quad \text{at} \quad \{\pm 1\} \times (0, \infty). \end{array} \tag{4.1}$$

Substituting the general solution $U(\omega, r) = U_1(r)\omega + U_0(r)$ of the differential equation into the boundary conditions implies readily $U \equiv 0$ in the half-strip, i.e., (4.1) has only zero solution.

Corollary 4.1. *If $p \geqslant 2$ then the formal asymptotic solution of (2.1) converges to zero uniformly in $t > 0$ bounded away from zero, as $\varepsilon \to 0$. Moreover, for $p > 2$ it vanishes exponentially.*

Proof. This follows immediately from Theorems 2.1 and 3.1. □

On the contrary, if $p < 2$ then the formal asymptotic solution of problem (2.1) hardly converges, as $\varepsilon \to 0$.

5. Generalisation to higher dimensions

The explicit formulas obtained above generalise easily to the evolution equation related to the bth power of the Laplace operator in $\mathbb{R}^n$, where b is a natural number. Consider the first boundary value problem for the operator $\varepsilon\partial_t + (-\Delta)^b$ in a bounded domain $\mathcal{G} \subset \mathbb{R}^{n+1}$. Note that the choice of sign $(-1)^b$ is explained exceptionally by our wish to deal with parabolic (not backward parabolic) equation. By $\varepsilon > 0$ is meant a small parameter.

The boundary of $\mathcal{G}$ is assumed to be C^{∞} except for a finite number of characteristic points. These are those points of $\partial\mathcal{G}$ at which the boundary touches with a hyperplane in $\mathbb{R}^{n+1}$ orthogonal to the t-axis. As above, we restrict our attention to analysis of the Dirichlet problem near a characteristic point like P_3 or P_5 in Figure 1.

The first boundary value problem for the evolution equation in $\mathcal{G}$ is formulated as follows: Let Σ be the set of all characteristic points of the boundary of $\mathcal{G}$. Given any functions f in $\mathcal{G} \to \mathbb{R}$ $u_0, u_1, \ldots, u_{b-1}$ on $\partial\mathcal{G} \setminus \Sigma$, find a function u on $\overline{\mathcal{G}} \setminus \Sigma$ satisfying

$$\begin{array}{rcl} \varepsilon u'_t + (-\Delta)^b u & = & f \quad \text{in} \quad \mathcal{G}, \\ \partial_{\nu}^{j} u & = & u_j \quad \text{at} \quad \partial\mathcal{G} \setminus \Sigma, \end{array} \tag{5.1}$$

for $j = 0, 1, \ldots, b - 1$, where ∂_{ν} is the derivative along the outward unit normal vector of the boundary. We focus upon a characteristic point P_3 of the boundary which is assumed to be the origin in $\mathbb{R}^{n+1}$.

Suppose the domain $\mathcal{G}$ is described in a neighbourhood of the origin by the inequality

$$t > \mathfrak{f}(x), \tag{5.2}$$

where $\mathfrak{f}$ is a smooth function of $x \in \mathbb{R}^n \setminus 0$ homogeneous of degree $p > 0$. We blow up the domain $\mathcal{G}$ at P_3 by introducing new coordinates $(\omega, r) \in D \times (0, 1/\varepsilon)$ with the aid of

$$\begin{array}{rcl} x & = & t^{1/p}\,\omega, \\ t & = & \varepsilon r, \end{array} \tag{5.3}$$

where D is the domain in $\mathbb{R}^n$ consisting of those $\omega \in \mathbb{R}^n$ which satisfy $f(\omega) < 1$. Under this change of variables the domain $\mathcal{G}$ nearby P_3 transforms into the half-cylinder $D \times (0, \infty)$, the cross-section $D \times \{0\}$ blowing down into the origin by (5.3). Note that for $\varepsilon \to 0$ the cylinder $D \times (0, 1/\varepsilon)$ stretches into the whole half-cylinder $D \times (0, \infty)$.

In the domain of coordinates (ω, r) problem (5.1) reduces to an ordinary differential equation with respect to the variable r with operator-valued coefficients. It is easy to see that under transformation (5.3) the derivatives in t and x change by the formulas

$$\begin{array}{rcl} \varepsilon\, u'_t & = & u'_r - \dfrac{1}{p}\dfrac{1}{r}\,(\omega, u'_\omega), \\ u'_{x_k} & = & \dfrac{1}{(\varepsilon r)^{1/p}}\, u'_{\omega_k} \end{array}$$

for $k = 1, \ldots, n$, where $(\omega, u'_\omega) = \displaystyle\sum_{k=1}^{n} \omega_k \frac{\partial u}{\partial \omega_k}$ stands for the Euler derivative. Thus, (5.1) transforms into

$$\begin{array}{rcl} r^Q\, U'_r + \dfrac{1}{\varepsilon^Q}\,(-\Delta_\omega)^b U - \dfrac{1}{p} r^{Q-1}\,(\omega, U'_\omega) & = & r^Q F \quad \text{in} \quad D \times (0, 1/\varepsilon), \\ \partial_\nu^j\, U & = & U_j \quad \text{at} \quad \partial D \times (0, 1/\varepsilon) \end{array} \tag{5.4}$$

for $j = 0, 1, \ldots, b-1$, where $U(\omega, r)$ and $F(\omega, r)$ are pullbacks of $u(x, t)$ and $f(x, t)$ under transformation (5.3), respectively, and

$$Q = \frac{2b}{p}.$$

We are interested in the local solvability of problem (5.4) near the base $r = 0$ in the cylinder $D \times (0, 1/\varepsilon)$. Note that the ordinary differential equation degenerates at $r = 0$, since the coefficient r^Q of the higher order derivative in r vanishes at $r = 0$. The theory of [23] still applies to characterise those problems (5.4) which are locally invertible.

To describe function spaces which give the best fit for solutions of problem (5.4), one constructs formal asymptotic solutions of the corresponding homogeneous problem. That is

$$\begin{array}{rcl} r^Q\, U'_r + \dfrac{1}{\varepsilon^Q}\,(-\Delta_\omega)^b U - \dfrac{1}{p} r^{Q-1}\,(\omega, U'_\omega) & = & 0 \quad \text{in} \quad D \times (0, \infty), \\ \partial_\omega^\alpha\, U & = & 0 \quad \text{on} \quad \partial D \times (0, \infty) \end{array} \tag{5.5}$$

for all $|\alpha| \leqslant b - 1$.

We assume that $p \neq 2b$. Similar arguments apply to the case $p = 2b$, the only difference being in the choice of the Ansatz, see Section 3. We look for a formal solution to (5.5) of the form $U(\omega, r) = e^{S(r)}\, V(\omega, r)$, where S is a differentiable function of $r > 0$ and V expands as a formal Puiseux series with nontrivial principal part

$$V(\omega, r) = \frac{1}{r^{\mathfrak{e} N}} \sum_{j=0}^{\infty} V_{j-N}(\omega)\, r^{\mathfrak{e} j},$$

where N is a complex number and $\mathfrak{e}$ a real exponent to be determined.

On substituting $U(\omega, r)$ into (2.1) we extract the eikonal equation $r^Q S' = \lambda$ for the function $S(r)$, where λ is a (possibly complex) constant to be defined. For $Q \neq 1$ this implies

$$S(r) = \lambda \frac{r^{1-Q}}{1-Q}$$

up to an inessential constant factor. In this way the problem reduces to

$$\begin{array}{rcll} r^Q V'_r + \dfrac{1}{\varepsilon^Q} (-\Delta_\omega)^b V - \dfrac{1}{p} r^{Q-1} (\omega, V'_\omega) & = & -\lambda V & \text{in} \quad D \times (0, \infty), \\ \partial_\omega^\alpha V & = & 0 & \text{on} \quad \partial D \times (0, \infty) \end{array} \tag{5.6}$$

for all $|\alpha| \leqslant b-1$.

Analysis similar to that in Section 2 shows that a right choice of $\mathfrak{e}$ is $\mathfrak{e} = (Q-1)/k$ for some natural number k. On substituting the formal series for $V(\omega, r)$ into (5.6) and equating the coefficients of the same powers of r we get two collections of problems

$$\begin{array}{rcll} \dfrac{1}{\varepsilon^Q} (-\Delta)^b V_{j-N} + \lambda V_{j-N} & = & 0 & \text{in} \quad D, \\ \partial^\alpha V_{j-N} & = & 0 & \text{at} \quad \partial D \end{array} \tag{5.7}$$

for all $|\alpha| \leqslant b-1$, where $j = 0, 1, \ldots, k-1$, and

$$\begin{array}{rcll} \dfrac{1}{\varepsilon^Q} (-\Delta)^b V_{j-N} + \lambda V_{j-N} & = & \dfrac{1}{p} (\omega, V'_{j-N-k}) - \mathfrak{e}(j-N-k) V_{j-N-k} & \text{in} \quad D, \\ \partial^\alpha V_{j-N} & = & 0 & \text{at} \quad \partial D \end{array} \tag{5.8}$$

for all $|\alpha| \leqslant b-1$, where $j = k, k+1, \ldots, 2k-1$, and so on.

Given any $j = 0, 1, \ldots, k-1$, problem (5.7) is essentially an eigenvalue problem for the strongly nonnegative operator $(-\Delta)^b$ in $L^2(D)$ whose domain consists of all functions of $H^{2b}(D)$ vanishing up to order $b-1$ at ∂D. The eigenvalues of the latter operator are known to be all positive and form a nondecreasing sequence $\lambda'_1, \lambda'_2, \ldots$ which converges to ∞. Hence, (5.7) admits nonzero solutions only for

$$\lambda_n = -\frac{1}{\varepsilon^Q} \lambda'_n$$

where $n = 1, 2, \ldots$.

In general, the eigenvalues $\{\lambda'_n\}$ fail to be simple. The generic simplicity of the eigenvalues of the Dirichlet problem for self-adjoint elliptic operators with respect to variations of the boundary have been investigated by several authors, see [20] and the references given there. We focus on an eigenvalue λ'_n of multiplicity 1, in which case the formal asymptotic solution is especially simple. By the above, this condition is not particularly restrictive.

If $\lambda = \lambda_n$, there is a nonzero solution $e_n(\omega)$ of this problem which is determined uniquely up to a constant factor. This yields

$$V_{j-N}(\omega) = c_{j-N}\, e_n(\omega), \tag{5.9}$$

for $j = 0, 1, \ldots, k-1$, where c_{j-N} are constant. Without restriction of generality we can assume that the first coefficient V_{-N} in the Puiseux expansion of V is different from zero. Hence, $V_{j-N} = c_{j-N} V_{-N}$ for $j = 1, \ldots, k-1$. For simplicity of notation, we drop the index n.

On taking the functions $V_{-N}, \ldots, V_{k-1-N}$ for granted, we now turn to problems (2.5) with $j = k, \ldots, 2k-1$. Set

$$f_{j-N} = \frac{1}{p} (\omega, V'_{j-N-k}) - \mathfrak{e}(j-N-k) V_{j-N-k},$$

then for the inhomogeneous problem (5.8) to admit a nonzero solution V_{j-N} it is necessary and sufficient that the right-hand side f_{j-N} be orthogonal to all solutions of the corresponding homogeneous problem, to wit V_{-N}. The orthogonality refers to the scalar product in $L^2(D)$. Let us evaluate the scalar product (f_{j-N}, V_{-N}). We get

$$(f_{j-N}, V_{-N}) = c_{j-N-k}\Big(\frac{1}{p}\,((\omega, V'_{-N}), V_{-N}) - \mathfrak{e}(j-N-k)\,(V_{-N}, V_{-N})\Big)$$

and, by Stokes' formula,

$$\begin{aligned} ((\omega, V'_{-N}), V_{-N}) &= \int_{\partial D} |V_{-N}|^2 (\omega, \nu)\, ds - \sum_{k=1}^{n} \int_D V_{-N} \frac{\partial}{\partial \omega_k} (\omega_k \overline{V_{-N}})\, d\omega = \\ &= -n\|V_{-N}\|^2 - ((\omega, V'_{-N}), V_{-N}), \end{aligned}$$

the latter equality being due to the fact that V_{-N} is real-valued and vanishes at ∂D. Hence,

$$((\omega, V'_{-N}), V_{-N}) = -\frac{n}{2}\,\|V_{-N}\|^2$$

and

$$(f_{j-N}, V_{-N}) = -c_{j-N-k}\left(\frac{n}{2p} + \mathfrak{e}(j-N-k)\right)\|V_{-N}\|^2 \tag{5.10}$$

for $j = k, \ldots, 2k-1$.

Since $V_{-N} \neq 0$, the condition $(f_{j-N}, V_{-N}) = 0$ fulfills for $j = k$ if and only if

$$\mathfrak{e}N = \frac{n}{2p}. \tag{5.11}$$

Under this condition, problem (5.8) with $j = k$ is solvable and its general solution has the form

$$V_{k-N} = V_{k-N,0} + c_{k-N} V_{-N},$$

where $V_{k-N,0}$ is a particular solution of (5.8) and c_{k-N} an arbitrary constant. Moreover, for $(f_{j-N}, V_{-N}) = 0$ to fulfill for $j = k+1, \ldots, 2k-1$ it is necessary and sufficient that $c_{1-N} = \ldots = c_{k-1-N} = 0$, i.e., all of $V_{1-N}, \ldots, V_{k-1-N}$ vanish. This in turn implies that $f_{k+1-N} = \ldots = f_{2k-1-N} = 0$, whence $V_{j-N} = c_{j-N} V_{-N}$ for all $j = k+1, \ldots, 2k-1$, where c_{j-N} are arbitrary constants. We choose the constants $c_{k-N}, \ldots, c_{2k-1}$ in such a way that the solvability conditions of the next k problems are fulfilled.

More precisely, we consider the problem (5.8) for $j = 2k$, the right-hand side being

$$\begin{aligned} f_{2k-N} &= \Big(\frac{1}{p}(\omega, V'_{k-N,0}) - \mathfrak{e}(k-N)V_{k-N,0}\Big) + c_{k-N}\Big(\frac{1}{p}(\omega, V'_{-N}) - \mathfrak{e}(k-N)V_{-N}\Big) = \\ &= \Big(\frac{1}{p}(\omega, V'_{k-N,0}) - \mathfrak{e}(k-N)V_{k-N,0}\Big) + c_{k-N}\Big(f_{k-N} - \mathfrak{e}kV_{-N}\Big). \end{aligned}$$

Combining (5.10) and (5.11) we conclude that

$$\begin{aligned} (f_{k-N} - \mathfrak{e}k\, V_{-N}, V_{-N}) &= -\mathfrak{e}k\,(V_{-N}, V_{-N}) = \\ &= (1-Q)\,(V_{-N}, V_{-N}) \end{aligned}$$

is different from zero. Hence, the constant c_{k-N} can be uniquely defined in such a way that $(f_{2k-N}, V_{-N}) = 0$. Moreover, the functions $f_{2k+1-N}, \ldots, f_{3k-1-N}$ are orthogonal to V_{-N} if and only if $c_{k+1-N} = \ldots = c_{2k-1-N} = 0$. It follows that V_{j-N} vanishes for each $j = k+1, \ldots, 2k-1$.

Continuing in this manner we construct a sequence of functions $V_{j-N}(\omega, \varepsilon)$, for $j = 0, 1, \ldots$, satisfying equations (5.7) and (5.8). The functions $V_{j-N}(\omega, \varepsilon)$ are defined uniquely up to a common constant factor c_{-N}. They depend smoothly on the parameter ε^p. Moreover, V_{j-N} vanishes identically unless $j = mk$ with $m = 0, 1, \ldots$. Therefore,

$$\begin{aligned} V(\omega, r, \varepsilon) &= \frac{1}{r^{\mathfrak{e}N}} \sum_{m=0}^{\infty} V_{mk-N}(\omega, \varepsilon)\, r^{\mathfrak{e}mk} = \\ &= \frac{1}{r^{n/2p}} \sum_{m=0}^{\infty} \tilde{V}_m(\omega, \varepsilon)\, r^{(Q-1)m} \end{aligned}$$

is a unique (up to a constant factor) formal asymptotic solution of problem (5.6) corresponding to $\lambda = \lambda_n$. Summarising, we arrive at the following generalisation of Theorem 2.1.

Theorem 5.1. *Let $p \neq 2b$. Then an arbitrary formal asymptotic solution of homogeneous problem (5.5) has the form*

$$U(\omega, r, \varepsilon) = \frac{c}{r^{n/2p}} \exp\Big(\lambda\, \frac{r^{1-Q}}{1-Q}\Big) \sum_{m=0}^{\infty} \frac{\tilde{V}_m(\omega, \varepsilon)}{r^{(1-Q)m}},$$

where λ is one of eigenvalues $\lambda_n = -\dfrac{1}{\varepsilon^Q} \lambda'_n$.

Thus, the construction of formal asymptotic solution U of general problem (5.1) follows by the same method as in Section 2.

In the original coordinates (x, t) close to the point P_3 in $\mathcal{G}$ the formal asymptotic solution looks like

$$u(x, t, \varepsilon) = c\, \Big(\frac{\varepsilon}{t}\Big)^{n/2p} \exp\Big(-\frac{\lambda'}{\varepsilon}\, \frac{t^{1-Q}}{1-Q}\Big) \sum_{m=0}^{\infty} \tilde{V}_m\Big(\frac{x}{t^{1/p}}, \varepsilon\Big) \Big(\frac{\varepsilon}{t}\Big)^{(1-Q)m} \qquad (5.12)$$

for $\varepsilon > 0$. If $1 - Q > 0$, i.e., $p > 2b$, expansion (5.12) behaves in much the same way as boundary layer expansion in singular perturbation problems, since the eigenvalues are all negative. The threshold value $p = 2b$ is a turning contact order under which the boundary layer degenerates.

The computations of this section extend obviously both to eigenvalues λ_n of higher multiplicity and arbitrary self-adjoint elliptic operators $A(x, D)$ in place of $(-\varDelta)^b$. When solving nonhomogeneous equations (5.8), one chooses the only solution which is orthogonal to all solutions of the corresponding homogeneous problem (5.7). This special solution actually determines what is known as Green operator. However, formula (5.12) becomes less transparent. And so we omit the details.

6. Parameter dependent norms

For $p < 2b$, expansion (5.12) fails to be asymptotic in small $\varepsilon > 0$, even if (x, t) is bounded away from the boundary of D. An asymptotic character of this series can only be revealed on using parameter dependent norms. Indeed, if $\varepsilon \to 0$, then the summands on the right-hand side of (5.12) increase unless the quotient t/ε does not exceed 1. Hence, ε is allowed to tend to zero only under the condition that $t/\varepsilon < 1$. Then expansion (5.12) still reveals certain asymptotic character. Within the framework of analysis on manifolds with singularities one exploits the weighted norms

$$\Big(\int_D \exp\Big(2\gamma\, \frac{1}{t^Q}\, \frac{t}{\varepsilon}\Big) \Big(\frac{t}{\varepsilon}\Big)^{-2\mu} |u(x, t, \varepsilon)|^2 dx dt \Big)^{1/2}$$

on functions defined near the singular point, where γ and μ are real numbers, cf. [2].

This research was supported by the Russian Foundation for Basic Research, grant 11-01-91330-NNIO_ a, and German Research Society (DFG), grant TA 289/4-2.

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Вырождение граничного слоя в окрестности сингулярных точек

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Мы изучаем задачу Дирихле в ограничеcкой плоской области для уравнения теплопроводности с малым параметром, умноженным на производную по t. Поведение решения вблизи характеристических точек границы представляет особый интерес. Поведение хорошо изучено, если характеристичекая прямая является касательной к границе с порядком касания не меньше 2. Мы разрешаем границе иметь не только порядок касания не меньше 2, но и быть точке каспидальной сингулярностью в характеристической точке. Мы не только строим асимтотическое решение задачи вблизи характеристической точки, но и описываем, как граничный слой вырождается.

Ключевые слова: уравнение теплопроводности, задача Дирихле, характеристические точки, граничный слой.