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THE CLOUD DROPLETS EVOLUTION IN VIEW OF THE IMPACT OF FRACTAL ENVIRONMENT: MATHEMATICAL MODELING

T. S. Kumykov

Institute of Applied Mathematics and Automation of Kabardin-Balkar Scientific Center of RAS, 360000, Nalchik, Shortanova st., 89 A, Russia
E-mail: macist20@mail.ru

In this paper, we investigate the effect of the medium with fractal structure on the growth of small cloud droplets at the initial condensation stage of cloud formation using a fractional differential equation. An electrodynamic model of coagulation of droplets under the action of an electric field is constructed in the cloud medium with a fractal structure. Numerical experiments are performed for assessment of the effect of the medium with fractal structure on the growth of cloud particles involving various combinations of microphysical parameters. A general dependence of the growth of cloud particles on different parameters of fractal structure in medium is established.

Key words: cloud droplet, fractal dimension, mathematical model, convective cloud.

Introduction

At present, the fractal environment concept has not yet become clear. Nevertheless, it should be recognized that fractal modeling is a tool for studying the hidden order in the dynamics of disordered systems like geophysical objects. Such objects include clouds, which like many objects of nature present irregular fractals. The study of the effect of fractal properties of the environment on various geo-processes that occur inside the clouds is very topical. Fractal models were proposed for emulation of complex structures due to their self-similarity [1, 2]. It was noted in [3] that clouds with powerful convective currents are of the nontrivial fractal structure. Fractal-dynamic approach in the study of lightning electricity is discribed in [4] An investigation of the dynamics of the electric field when the charge is divided in the fractal structure cloud droplet is presented in [5 - 7]. An adequate description of geologic processes in the clouds is possible only using the medium fractality concept. The apparatus of fractional calculus is widely used for describing processes in the environments with fractal structure [8, 9] it makes possible solving complex systems of integro-differential equations by introducing a new model parameter - the order of the fractional derivative. To a certain extent, the reason to employ fractional calculus is in the appropriate mathematical properties of various linear operators. Thus, fractional integral is a generalization of the n-fold. In the usual sense though not everywhere differentiable, fractal functions can be differentiable in the Riemann-Liouville or Caputo fractional derivative sense, and be applicable for the description of various processes, for example, cloud coagulation. Investigation of the of particles size evolution in droplet clouds during condensation, gravitational and electrodynamic coagulation of droplets in view of the fractal environment is a very important task for weather forecasting methods since the intensity of precipitation as well as radiative properties of clouds that determine solar and thermal emission on the underlying surface depends on the droplet size. In this paper, we study characteristic features of condensation, gravitational, and electrodynamic coagulation mechanisms of cloud droplet growth taking into account the fractal environment. The latter two mechanisms (gravitational and electrodynamic) are the part of the general coagulation mechanism for particle growth.

Droplets growth by condensation in view of the fractality of the medium

Let's consider the case when the drop is in a supersaturated air and the process of diffusion of water vapor proceeds because of the difference in densities. The classical theory was first developed by Maxwell for stationary growth by condensation of a stationary spherical particle. He showed that the square of the droplet radius increases with time at constant supersaturation. In this section, we study the condensation growth model for small droplets, taking into account the effect of the medium with fractal structure.

The small stationary droplet growth rate ($r < 10^{-2}$см) by condensation [10] can be described by equation

$$\frac{dm}{dt} = 4\pi r\varepsilon D\Delta C_r, \tag{1}$$

where m – is the mass of the drop, r–is the droplet radius, $\varepsilon = \frac{B}{B+D/r}$, $B = \gamma\sqrt{\frac{RT}{2\pi\mu}}$, ΔC_r – is the difference between the vapor concentration at a large distance from the drop and equilibrium concentration at the surface, γ – is the condensation coefficient, μ – is the molecular weight of water, R – is the universal gas constant, T – is the temperature, D – is the diffusion coefficient of water vapor in air.

By (1) the rate of change of the droplet radius due to condensation can be written as

$$\frac{dr(t)}{dt} = \frac{3\varepsilon D\Delta C_r}{\rho r(t)}, \tag{2}$$

Where ρ – is the density of water

Similarly to [6] we introduce the effective rate of change of the droplet radius using the apparatus of the theory of fractional differentiation, considering the fractal structure in the medium

$$\left\langle \frac{dr(t)}{dt} \right\rangle = \frac{1}{\tau} D_{0t}^{\alpha-1} \frac{dr(t)}{dt} = \frac{1}{\tau}\partial_{0t}^{\alpha} r(t),\ 0<\alpha<1, \tag{3}$$

hence (2) is of the form

$$\partial_{0t}^{\alpha} r(t) = \frac{3\tau\varepsilon D\Delta C_r}{\rho r(t)}, 0<\alpha<1, \tag{4}$$

where $\partial_{0t}^{\alpha} r(t)$ – is the Caputo fractional derivative, α – is the phenomenological parameter (in our case, the index of the fractal medium), τ – is the characteristic process time.

Consider the Cauchy problem for an ordinary fractional differential equation (4) with initial conditions

$$r(0) = r_0. \tag{5}$$

A Cauchy type problem for differential equations with the Riemann-Liouville fractional derivative is studied in [8].

To find an approximate solution to this problem, we use the difference method proposed in [11]. For this purpose, on the segment $[0,T]$ introduce the grid $\bar{\omega}_h = \{jh, j = 0,1,2...N\}$ with time increment $h = T/N$ where N – is the natural number.

Consider the following difference scheme with differential problem (4) - (5):

$$\frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{j} (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_{t,s} = f(t_j, y_j), \quad j = 1,2,...N, \tag{6}$$

$$y_0 = r_0, \tag{7}$$

where $\Delta_{0t_j}^{\alpha} y = \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{j} (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_{\bar{t},s}$ – is the discrete analog of the fractional derivative, $y_{t,s} = \frac{y_{s+1}-y_s}{h}$ – is the right difference derivative, $f(t_j, y_j) = \frac{3\tau\varepsilon D\Delta C_r}{\rho y_j}$.

Scheme (6)-(7) is explicit:

$$y_{j+1} = y_j + h^{\alpha}\Gamma(2-\alpha)\left[f(t_j, y_j) - \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{j-1} (t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}) y_{t,s} \right], \quad j = 0,1,2,...N-1., \tag{8}$$

Solution to the difference problem y converges to the solution of r for the corresponding differential problem at the rate $O(h^{\alpha})$. Figure 1. The diagram of the approximate solutions

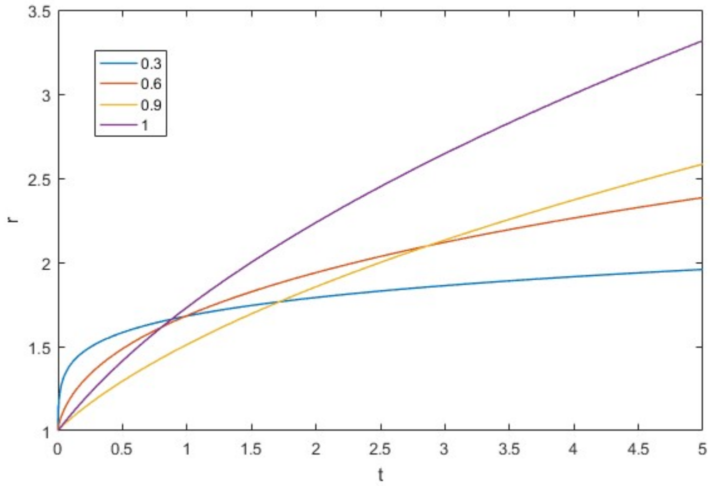

Fig. 1. Solutions chart for problem (4) - (5) for various values of α.

to problem (4) - (5) for different fractality indexes of α. The functions were reduced to a dimensionless form, where the initial value of r_0 is the characteristic length. It can be seen from the figure that for small values of αa sharp jump in the droplet growth is observed, and once passed through the zone of the fractal effect the droplet growth slows down. Fractal effect zone implies the zone where the graphs as $0 < \alpha < 1$ intersect another one as $\alpha = 1$ and fractal properties begin to show up. Thus, it can be suggested that processes in fractal cloud media are much slower than in continuous one. We note that similar results can be achieved with the help of the formula for approximating fractional derivatives obtained in[12]. Equation (3) can be reduced to the form of the following equation [13]

$$\partial_{0t}^{\alpha} r^2(t) = 2P, 0 < \alpha < 1, \tag{9}$$

here $P = \dfrac{3\tau\varepsilon D\Delta C_r}{\rho}$.

The solution to the Cauchy problem for ordinary fractional differential equation (9) with the initial condition

$$r(0) = r_0, \tag{10}$$

is of the form

$$r(t) = \sqrt{\frac{2Kt^{\alpha}}{\Gamma(1+\alpha)} + r_0}. \tag{11}$$

The results obtained allow to perform a comparative analysis of the droplets condensation growth behavior in a fractal cloud medium and a continuous one. By the resulting chart for problem (9) – (10) the droplet growth in the fractal environment is more slowly than in a continuous one for different α. The fractal effect zone is lower and more concentrated in contrast to the resulting zone for problem (4) – (5).

Fractality effect on the droplets growth by gravitational coagulation

We know [14] seven types of coagulation: brownian, gravitational, hydrodynamic, gradient, acoustic, turbulent, electrodynamic. Different types of motion that cause coagulation can

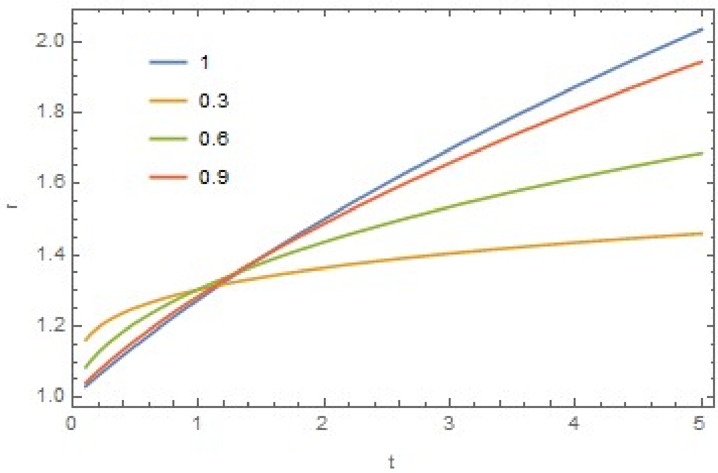

Fig. 2. Solutions chart for problem (9) – (10) for various values of α.

be interrelated but we consider gravitational coagulation in a monodisperse medium. Gravitational coagulation of cloud elements is one of the main processes that affect the growth of the droplet cloud particles initially formed due to homogeneous condensation or due to condensation onto hygroscopic nuclei. Gravitational coagulation is the most important for the sedimentation.

In [14] it is shown that an increase in the droplet radius per unit time when falling through a monodisperse cloud consisting of droplets with the radius r_0 as $r_0 << r$ is described by equation

$$\frac{dr}{dt} = \frac{W r_0^3}{3r^2(t)}, \tag{12}$$

where $W = Esn\Delta v$ is the collision probability between droplets, E is the collision coefficient, s is the effective collision area, n is the number of particles with the radius r per unit volume, Δv is the particle fall rate.

Given the fractality of the cloud environment in which the process takes place, from (12) we come to the following problem

$$\partial_{0t}^{\alpha} r(t) = \frac{\tau W r_0^3}{3r^2(t)}, \quad 0 < \alpha < 1, \tag{13}$$

$$r(0) = r_0, \tag{14}$$

that describes the droplets growth dynamics by gravitational coagulation.

Figure 3. The chart of approximate solutions to problem (13) – (14) for different values of α. It is easily seen with changing the parameter of α the diagrams become identical to the ones obtained by the condensational mechanism. It should be noted that the droplet growth rate by gravitational coagulation is lower than by condensation. It can also be assumed that the fractal effect zones lie at different levels while droplet growing under the action of different mechanisms.

Similarly to problem (9) – (10) and applying Lemma 1 [13] we bring the problem (13) – (14) to the form

$$\partial_{0t}^{\alpha} r^3(t) = 3U, 0 < \alpha < 1, \tag{15}$$

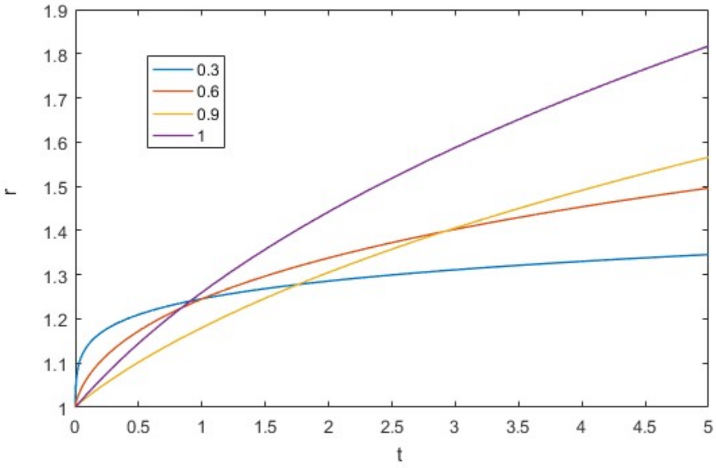

Fig. 3. Solutions chart for problem (13) – (14) for various α.

$$r(0) = r_0, \tag{16}$$

where $U = \dfrac{\tau W r_0^3}{3}$.

The solution to problem (15)–(16) has the form

$$r(t) = \sqrt[3]{3U \int_0^t \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt + r_0}. \tag{17}$$

Figure 4. The solution charts to problem for different are developing more slowly than

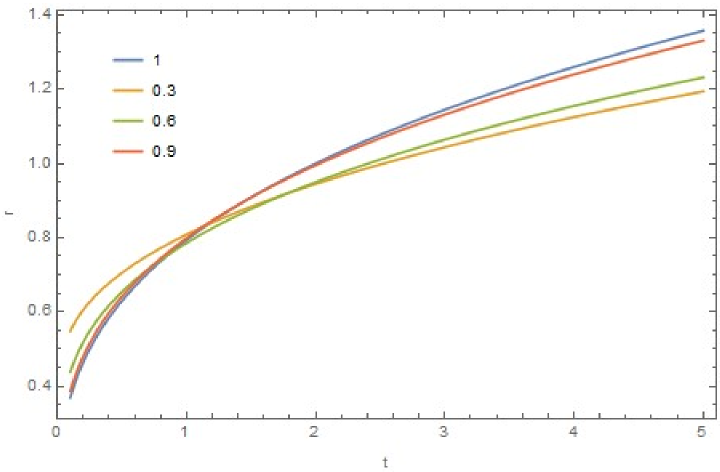

Fig. 4. Solutions chart for problem (15)–(16) for various α.

solution diagrams to (13) – (14). By the resulting chart for problem (15) – (16) the droplet growth in the fractal environment is more slowly than in a continuous one for different values of α. The zone of the fractal effect is also lower in contrast to the zone we have gotten for problem (13) – (14) and more concentrated.

As $r_0 \leq r$ equation (9) becomes

$$\frac{dr}{dt} = W(r(t) - r_0). \tag{18}$$

Next we consider the Cauchy problem:

$$\partial_{0t}^{\alpha} r(t) = \tau W(r(t) - r_0), \quad 0 < \alpha < 1, \tag{19}$$

$$r(0) = r_0. \tag{20}$$

By introducing the effective velocity and fractality of the medium we derive equation (19) – (20).

The solution to problem (19) – (20) has the form of [9]

$$r(t) = r_0 t^0 E_{\alpha,1}(\tau W t^{\alpha}) + \tau W r_0 \int_0^t (t-\theta)^{\alpha-1} E_{\alpha,\alpha}\left(\tau W (t-\theta)^{\alpha}\right) d\theta, \tag{21}$$

where $E_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)}$ is a Mittag -Leffler type function.

Once transformed representation of (21) can be reduced to the form

$$r(t) = r_0 t^0 E_{\alpha,1}(\tau W t^{\alpha}) + \tau W r_0 t^{\alpha} E_{\alpha,\alpha+1}(\tau W t^{\alpha}). \tag{22}$$

Figure 5. According to the approximate solution charts to problem (15) – (16) for

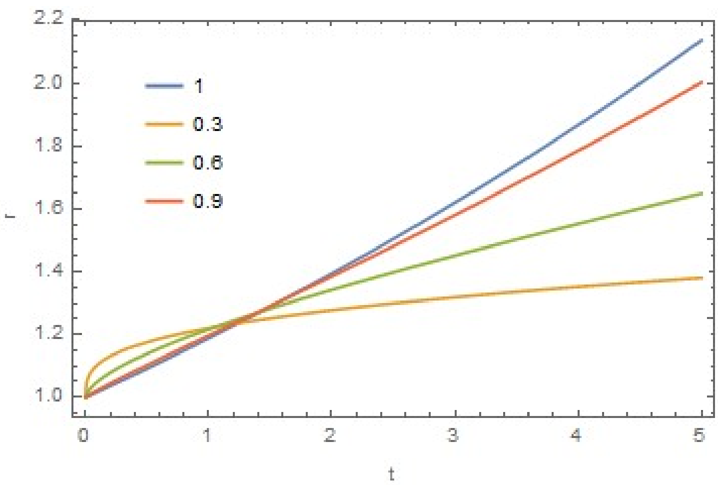

Fig. 5. Solutions chart for problem (15)–(16) for various α.

different values of α(the same as in Fig. 3 and 4), the tendency of droplet growth at different values of $\alpha$is observed. The zone of the fractal effect is at the same level as in Fig. 3.

Modeling of droplet coagulation process in a cloudy medium with fractal structure: Electrodynamic approach

Let's consider now how the Earth's gravitational and electrical forces act on the immobile charged droplet. Under the action of these two forces, the droplet starts moving and collides with the molecules of different gases in the surrounding air. Such collisions impede the motion of the droplets. This implies that we can introduce some resistance force that depends on the velocity of the droplet: the faster the droplet moves, the greater the resistance to its movement. Thus, three forces act on the charged drop in the atmosphere (Figure 6): gravitational, electric, resistance forces

$$F = F_g + F_e + F_r, \tag{23}$$

where $F_g = mg$, $F_e = qE$, q – is the droplet charge, E – is the electric field strength, $F_r = 6\pi\eta rv$, η – is the dynamic viscosity in the medium, r – is the radius of the droplet, v – is the droplet drop rate. The droplet moving velocity can be determined using (23) as follows

$$v = \frac{mg + qE}{6\pi\eta r}, \tag{24}$$

provided the velocity is constant, that implies the steady motion $\frac{dv}{dt} = 0$.
By analysis of formula (24) it can be seen that the droplet velocity is representable as the sum of the droplet moving velocity under the Earth attraction and the velocity under the action of the electric field:

$$v = v_g + v_e. \tag{25}$$

The difference in the droplet speed cause collisions and coagulation of droplets. If the difference in speed is due to the difference in the droplet masses, then it is gravitational coagulation, and if the difference is due to the difference in the electric charges of the droplets, then it is the electrical coagulation.

Under "good weather"conditions, the electric field of the atmosphere is directed toward the Earth's surface (Figure 6), that is, positively charged droplets are attracted to the Earth's surface, while negatively charged drops tend to move upward. The greater the difference in the electric charges acquired by droplets, the greater the difference in velocities, the more frequent the collision of droplets, hence, the growth of droplets becomes more intensive. To calculate the electrodynamic coagulation velocity, we can

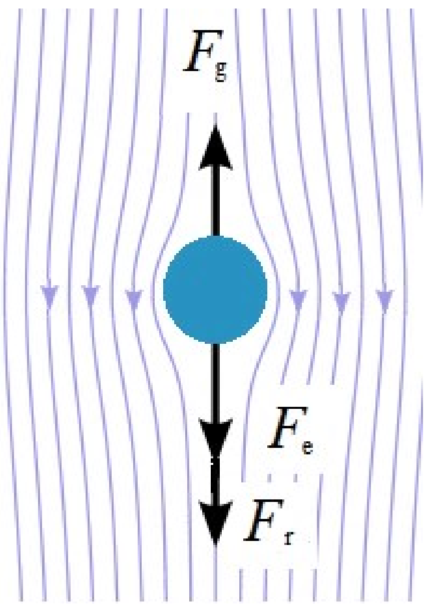

Fig. 6. Charged droplet in the electric field

employ the gravitational coagulation formula and substitute the gravitational field effect by the electric field effect on droplet velocity as follows

$$\frac{dr(t)}{dt} = \frac{K\omega v_e}{4\rho}, \tag{26}$$

where K is the average of the capture coefficient of the rest drops by a drop with the radius r, ω is cloud liquid water content.

With taking into account the effective velocity of dimensional changes of the droplet, formula (26) becomes

$$\partial_{0t}^{\alpha} r(t) = \frac{\tau K\omega v_e}{4\rho}, \tag{27}$$

or

$$\partial_{0t}^{\alpha} r(t) = \frac{\tau K\omega qE}{24\rho\pi\eta r(t)}, \quad 0 < \alpha < 1. \tag{28}$$

Consider the Cauchy problem for an ordinary differential equation of fractional order (28) with the initial condition

$$r(0) = r_0. \tag{29}$$

Employing the difference scheme proposed above, we find an approximate solution to problem (28)– (29).

Figure 7. The numerical solution chart for the problem with different values of α ($\alpha = 0,3$; $\alpha = 0,5; \alpha = 0,7$) using the MATLAB R2016B. We can observe a sharp jump in the droplet growth for small values of α, and after passing through the fractal effect zone the droplet growth slows down. Therefore, it can be assumed, that processes in the fractal environment proceed much more slowly than in the continuous one. Figure 7.

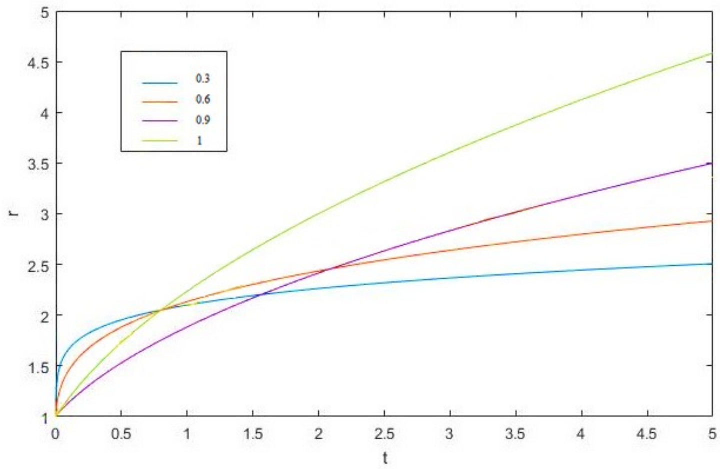

Fig. 7. Solutions chart for problem (28)– (29) for various values of α

Now we can conclude the proposed model is in good agreement with the exact solution. By Lemma 1 [13] problem (28)– (29) becomes

$$\partial_{0t}^{\alpha} r^2(t) = 2H, \quad 0 < \alpha < 1, \tag{30}$$

$$r(0) = r_0, \tag{31}$$

where $H = \dfrac{\tau K \omega q E}{24 \rho \pi \eta}$.
The solution of problem (30)– (31) is of the form:

$$r(t) = \sqrt{\frac{Ht^{\alpha}}{\Gamma(1+\alpha)} + r_0}. \tag{32}$$

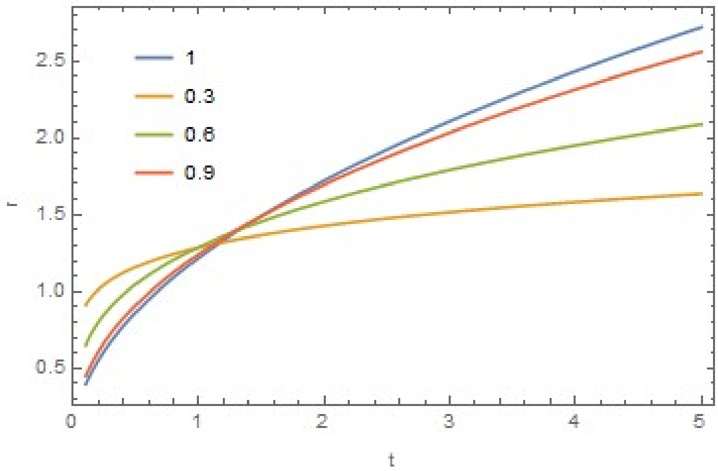

Fig. 8. Solution charts for problem (30)– (31) for various values of α

As we can see from Figure 8, for different values of α the droplet growth in the fractal environment proceeds more slowly than in a continuous one. The resulting fractal zone for (30)–(31) is lower in contrast to the zone we obtained for problem (28)–(29). After passing through the fractal zone, a zone of the dynamic fractal growth begins. In this zone as $0 < \alpha < 1$ lines lie below the exact solution, and the process undoubtedly proceeds more slowly.

Conclusion

This paper is devoted to the influence of the fractality of the medium on the cloud droplet growth using various mechanisms: condensation growth, gravitational and electrodynamic coagulation of cloud drops. The average radius of droplets in the cloud is the main sedimentary factor, and its evolution is associated with the gravitational coagulation. The obtained formulas can be used to calculate the dimensional change in the cloud particles under the action of condensation and coagulation of droplets with given parameters and with the fractality of the medium taken into account. The numerical experiments carried out to evaluate the effect of the fractality of the medium on the growth of cloud particles with microphysical parameters in different combinations proved the general relationship between the cloud particles growth and the fractal properties of the medium. This is expressed in a sharp jump and then decrease in the droplet growth velocity for different values of the fractality. The zones of the onset of the action of the fractal effect under various droplet growth mechanisms are revealed. It can be noted that the numerical results for electrodynamic coagulation of droplets are in good agreement with the exact solution.

From the results obtained, it can be seen that under the condensation and gravitational growth mechanisms the fractal bands are at the same levels, and the graphs obtained using the difference scheme [11], lie above while the zone of the fractal effect is less concentrated than the zone resulted from using the lemma 1 [13].

In the cases of the growth by electrical coagulation, it can be noted that the resulting fractal effect zones are located relatively higher than in the cases of the other growth mechanisms considered. During all three approaches, the dynamic fractal growth begins to show up after passing through the zone of the fractal effect. Once having passed this zone the growth process proceeds more slowly than in a continuous medium.

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УДК 517.98

РАЗВИТИЕ ОБЛАЧНЫХ КАПЕЛЬ В РЕЗУЛЬТАТЕ ВОЗДЕЙСТВИЯ ФРАКТАЛЬНОЙ СРЕДЫ: МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ

Т. С. Кумыков

Институт прикладной математики и автоматизации КБНЦ РАН,
360000, г. Нальчик, ул. Шортанова, 89 А
E-mail: macist20@mail.ru

В настоящей работе мы исследуем влияние среды с фрактальной структурой на рост малых облачных капель на начальной стадии конденсации образования облаков с использованием дробного дифференциального уравнения. В облачной среде с фрактальной структурой построена электродинамическая модель коагуляции капель под действием электрического поля. Проведены численные эксперименты для оценки влияния среды на фрактальную структуру на рост облачных частиц с использованием различных комбинаций микрофизических параметров. Установлена общая зависимость роста частиц облака от разных параметров фрактальной структуры в среде.

Ключевые слова: облачная капля, фрактальная размерность, математическая модель, конвективное облако.