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T. H. Koornwinder, V. B. Kuznetsov, Gauss hypergeometric function and quadratic R -matrix algebras,
Algebra i Analiz, 1994, Volume 6, Issue 3, 161–184

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GAUSS HYPERGEOMETRIC FUNCTION AND QUADRATIC R -MATRIX ALGEBRAS

T. H. Koornwinder, V. B. Kuznetsov¹

Abstract. We consider representations of quadratic R -matrix algebras by means of certain first order ordinary differential operators. These operators turn out to act as parameter shifting operators on the Gauss hypergeometric function and its limit cases and on classical orthogonal polynomials. The relationship with W. Miller's treatment of Lie algebras of first order differential operators will be discussed.

§1. Introduction

The modern approach to finite-dimensional integrable systems uses the language of the representations of R -matrix algebras [3, 5, 7, 9, 10, 12]. There are two quadratic R -matrix algebras appearing in the quantum inverse scattering method (QISM). We will call them QISM I and QISM II, respectively. We restrict ourselves to the simplest case of a 2-dimensional auxiliary space and a rational 4×4 R -matrix of the form

$$R(u) = u + \kappa P, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \kappa, u \in \mathbb{C}. \quad (1.1)$$

Consider a 2×2 matrix

$$\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad (1.2)$$

with a priori noncommuting entries depending on a so-called *spectral parameter* u which is arbitrary complex. The matrix (1.2) is denoted $T(u)$ in the QISM I case and $U(u)$ in the QISM II case. The *QISM I algebra* or *T -algebra* is then defined as the algebra

Key words and phrases. Quadratic R -matrix algebras, Gauss hypergeometric function, classical orthogonal polynomials, recurrence relations.

¹This author was supported by the Netherlands Organisation for Scientific Research (NWO) under the Project # 611-306-540.

generated by all matrix elements of $T(u)$ for all complex values of u subject to the following quadratic relation on $T(u)$ (cf. [1, 5, 13]).

$$R(u-v)T^{(1)}(u)T^{(2)}(v) = T^{(2)}(v)T^{(1)}(u)R(u-v), \quad u, v \in \mathbb{C}. \quad (1.3)$$

Here we use the notation $T^{(1)}(u) = T(u) \otimes I$, $T^{(2)}(v) = I \otimes T(v)$. The *QISM II algebra* or *U-algebra* is the algebra generated by the matrix elements of $U(u)$ for all u subject to a quadratic relation involving two R -matrices [11]:

$$R(u-v)U^{(1)}(u)R(u+v-1)U^{(2)}(v) = U^{(2)}(v)R(u+v-1)U^{(1)}(u)R(u-v), \quad u, v \in \mathbb{C}. \quad (1.4)$$

From now on we assume for both types of algebras that $\kappa = 1$. For $\kappa \neq 0$ in (1.1) this means no loss of generality.

In the present article we construct representations of very simple type (L -operators of rank 1) of both the T - and the U -algebra. In the QISM II case we require moreover a certain symmetry property (unitarity) for the L -operator. We will consider L -operators (1.2) for which certain matrix elements will be realized as first order ordinary differential operators acting as parameter shifting operators on the Gauss hypergeometric function and its limit cases. Specialization then yields shift operator actions on classical orthogonal polynomials. For the QISM I case and for some of the QISM II cases we will point out a close connection of our results with the Infeld-Hull [4] factorization method for second order differential equations and with Miller's [8] treatment of Lie algebras of first order differential operators acting as shift operators on special functions. For the most general QISM II cases, we consider, the connection with Lie algebras of first order differential operators is no longer valid. But then, instead, there is a connection with an action by differential operators (cf. Miller [8]) of the universal enveloping algebra of the Lie algebra $e(3)$.

Our operators will act on special functions $F(u)$ which appear for each $u \in \mathbb{C}$ as a solution of the equation

$$C(u)F(u) = 0, \quad (1.5)$$

i.e., as functions annihilated by one of the two off-diagonal elements (always chosen to be $C(u)$) of an L -operator. The operators $A(u)$ and $D(u)$, for the QISM I algebra, and $A(u)$ and $-A(-u)$, for the QISM II algebra, then give the shifting of the parameter u by ± 1 , respectively:

$$A(u)F(u) = \Delta_-(u - \frac{1}{2})F(u-1), \quad D(u)F(u) = \Delta_+(u + \frac{1}{2})F(u+1), \quad (1.6)$$

for the T -algebra, and

$$A(u)F(u) = \Delta_-(u - \frac{1}{2})F(u-1), \quad -A(-u)F(u) = \Delta_+(u + \frac{1}{2})F(u+1), \quad (1.7)$$

for the U -algebra. Here the $\Delta_{\pm}(u)$ are certain scalars depending on u which factorize the quantum determinant $\Delta(u)$ of an L -operator:

$$\Delta(u) = \Delta_+(u)\Delta_-(u). \quad (1.8)$$

The quantum determinant of a T - or U -algebra is a certain quadratic expression in the generators with the property that it is the generating function for the center of the algebra. So, in an irreducible representation it is, under suitable assumptions, scalar for each u .

The structure of the paper is as follows. In §2 we give further properties of both types of algebras. §3 contains a representative collection of differential recurrence relations for special functions for which we can give an interpretation in terms of L -operators

satisfying (1.3) or (1.4). §4 clarifies the connection of our approach with the factorization method and with Miller's Lie algebra approach. §5 deals with the simplest (rank 1) L -operators for the QISM I algebra and with the corresponding shifting formulas for Gauss hypergeometric functions, etc. In §6 we study rank 1 L -operators for the QISM II algebra and the corresponding differential recurrence relations. In the final §7 we make some concluding remarks on possible applications.

Throughout we use notation like ∂_x for the ordinary derivative d/dx or the partial derivative $\partial/\partial x$ with respect to x .

§2. More about quadratic R -matrix algebras

In the QISM II algebra case we will always add the following relations (symmetry property when changing the sign of u):

$$\begin{aligned} -A(-u) &= D(u) - (A(u) + D(u))/(2u + 1), \\ -D(-u) &= A(u) - (A(u) + D(u))/(2u + 1), \\ B(-u) &= B(u), \\ C(-u) &= C(u). \end{aligned} \tag{2.1}$$

Note that the second equality is implied by the first. Equations (2.1) can be rephrased as the unitarity property $U^{-1}(-u) \sim U(u)$ [11].

The *quantum determinant* of a T - or U -algebra is defined as follows:

$$\begin{aligned} \Delta(u) &= A(u - \frac{1}{2})D(u + \frac{1}{2}) - C(u - \frac{1}{2})B(u + \frac{1}{2}) \\ &= D(u - \frac{1}{2})A(u + \frac{1}{2}) - B(u - \frac{1}{2})C(u + \frac{1}{2}) \\ &= D(u + \frac{1}{2})A(u - \frac{1}{2}) - C(u + \frac{1}{2})B(u - \frac{1}{2}) \\ &= A(u + \frac{1}{2})D(u - \frac{1}{2}) - B(u + \frac{1}{2})C(u - \frac{1}{2}) \end{aligned} \tag{2.2}$$

for the T -algebra, and

$$\begin{aligned} \Delta(u) &= -D(-u + \frac{1}{2})D(u + \frac{1}{2}) - C(u - \frac{1}{2})B(u + \frac{1}{2}) \\ &= -A(-u + \frac{1}{2})A(u + \frac{1}{2}) - B(u - \frac{1}{2})C(u + \frac{1}{2}) \\ &= -D(u + \frac{1}{2})D(-u + \frac{1}{2}) - C(u + \frac{1}{2})B(u - \frac{1}{2}) \\ &= -A(u + \frac{1}{2})A(-u + \frac{1}{2}) - B(u + \frac{1}{2})C(u - \frac{1}{2}) \end{aligned} \tag{2.3}$$

for the U -algebra. The quantum determinant is the generating function for the center of both types of algebras [5, 11].

Relation (1.3) resp. (1.4) can be rewritten in the following extended form as commutators between the algebra generators $A(u), B(u), C(u)$, and $D(u)$:

$$[A, A] = [B, B] = [C, C] = [D, D] = 0, \tag{2.4}$$

$$[A, B] = -(AB - \widetilde{AB})/(u - v), \tag{2.5}$$

$$[B, A] = -(BA - \widetilde{BA})/(u - v), \tag{2.6}$$

$$[A, C] = -(CA - \widetilde{CA})/(u - v), \tag{2.7}$$

$$[C, A] = -(AC - \widetilde{AC})/(u - v), \tag{2.8}$$

$$[B, D] = -(DB - \widetilde{DB})/(u - v), \tag{2.9}$$

$$[D, B] = -(BD - \widetilde{BD})/(u - v), \quad (2.10)$$

$$[D, C] = -(DC - \widetilde{DC})/(u - v), \quad (2.11)$$

$$[C, D] = -(CD - \widetilde{CD})/(u - v), \quad (2.12)$$

$$[A, D] = -(CB - \widetilde{CB})/(u - v), \quad (2.13)$$

$$[D, A] = -(BC - \widetilde{BC})/(u - v), \quad (2.14)$$

$$[B, C] = -(DA - \widetilde{DA})/(u - v), \quad (2.15)$$

$$[C, B] = -(AD - \widetilde{AD})/(u - v), \quad (2.16)$$

for the T -algebra, and

$$[B, B] = [C, C] = 0, \quad (2.17)$$

$$[A, A] = -(BC - \widetilde{BC})/(u + v), \quad (2.18)$$

$$[D, D] = -(CB - \widetilde{CB})/(u + v), \quad (2.19)$$

$$[A, B] = -(AB - \widetilde{AB})/(u - v) - (AB + BD)/(u + v - 1) \\ - (AB + BD - \widetilde{AB} - \widetilde{BD})/(u - v)/(u + v - 1), \quad (2.20)$$

$$[B, A] = -(BA - \widetilde{BA})/(u - v) + (\widetilde{AB} + \widetilde{BD})/(u + v - 1), \quad (2.21)$$

$$[A, C] = -(CA - \widetilde{CA})/(u - v) + (\widetilde{CA} + \widetilde{DC})/(u + v - 1) \\ - (CA + DC - \widetilde{CA} - \widetilde{DC})/(u - v)/(u + v - 1), \quad (2.22)$$

$$[C, A] = -(AC - \widetilde{AC})/(u - v) - (CA + DC)/(u + v - 1), \quad (2.23)$$

$$[D, B] = -(BD - \widetilde{BD})/(u - v) + (\widetilde{AB} + \widetilde{BD})/(u + v - 1) \\ - (AB + BD - \widetilde{AB} - \widetilde{BD})/(u - v)/(u + v - 1), \quad (2.24)$$

$$[B, D] = -(DB - \widetilde{DB})/(u - v) - (AB + BD)/(u + v - 1), \quad (2.25)$$

$$[D, C] = -(DC - \widetilde{DC})/(u - v) - (CA + DC)/(u + v - 1) \\ - (CA + DC - \widetilde{CA} - \widetilde{DC})/(u - v)/(u + v - 1), \quad (2.26)$$

$$[C, D] = -(CD - \widetilde{CD})/(u - v) + (\widetilde{CA} + \widetilde{DC})/(u + v - 1), \quad (2.27)$$

$$[A, D] = -(CB - \widetilde{CB})(u + v + 1)/(u^2 - v^2), \quad (2.28)$$

$$[D, A] = -(BC - \widetilde{BC})(u + v + 1)/(u^2 - v^2), \quad (2.29)$$

$$[B, C] = -(DA - \widetilde{DA})(u + v - 1)/(u^2 - v^2) \\ - (AA - \widetilde{DD})/(u + v), \quad (2.30)$$

$$[C, B] = -(AD - \widetilde{AD})(u + v - 1)/(u^2 - v^2) \\ - (DD - \widetilde{AA})/(u + v), \quad (2.31)$$

for the U -algebra. Here for brevity we use the following notation: $[A, B]$ means the commutator $[A(u), B(v)]$, where the first parameter is u and the second one is v ; DA stands for the noncommutative operator product $D(u)A(v)$; and \widetilde{DA} signifies $D(v)A(u)$

(where v is the first parameter), and so on.

Theorem 2.1. *Let W be a complex vector space on which the QISM I algebra acts by an algebra representation. Suppose \mathcal{D} is a subset of \mathbb{C} of the form $\{u_0 + m \mid m \in \mathbb{Z}, j_- < m < j_+\}$, where $u_0 \in \mathbb{C}$ and $j_{\pm} = \pm\infty$ or integer, such that*

- (i) $\{w \in W \mid C(u)w = 0\}$ is 1-dimensional for any $u \in \mathcal{D}$,
- (ii) if $u \in \mathcal{D}$, $0 \neq w \in W$ and $C(u)w = 0$ then

$$A(u)w \begin{cases} \neq 0, & u \neq u_0 + j_- + 1, \\ = 0, & u = u_0 + j_- + 1, \end{cases} \tag{2.32}$$

$$D(u)w \begin{cases} \neq 0, & u \neq u_0 + j_+ - 1, \\ = 0, & u = u_0 + j_+ - 1. \end{cases} \tag{2.33}$$

For each $u \in \mathcal{D}$ choose $0 \neq F(u) \in W$ such that $C(u)F(u) = 0$. Then

$$A(u)F(u) = \Delta_-(u - \frac{1}{2})F(u - 1), \quad u \in \mathcal{D}, \quad u \neq u_0 + j_- + 1, \tag{2.34}$$

$$D(u)F(u) = \Delta_+(u + \frac{1}{2})F(u + 1), \quad u \in \mathcal{D}, \quad u \neq u_0 + j_+ - 1, \tag{2.35}$$

for certain scalar functions $\Delta_{\pm}(u \pm \frac{1}{2})$. Furthermore, the operator $\Delta(u)$, when acting on $\text{Span}\{F(v) \mid v \in \mathcal{D}\}$, is scalar for $u \in \mathcal{D} \pm \frac{1}{2}$ and it satisfies

$$\Delta(u) = \begin{cases} \Delta_+(u)\Delta_-(u), & u \in \mathcal{D} \pm \frac{1}{2}, \quad u \neq u_0 + j_{\pm} \mp \frac{1}{2}, \\ 0, & u = u_0 + j_{\pm} \mp \frac{1}{2}. \end{cases} \tag{2.36}$$

Proof. The matrix $L(u)$ satisfies all relations (2.2)–(2.16). By substitution of $u = v - 1$ in (2.8), we get

$$C(v - 1)A(v) = A(v - 1)C(v).$$

Apply both sides to $F(v)$ ($v \in \mathcal{D}$, $v \neq u_0 + j_0 + 1$), then we get the equation

$$C(v - 1)A(v)F(v) = 0.$$

$F(v - 1)$ spans the zero space of $C(v - 1)$ in W . Thus

$$A(v)F(v) \sim F(v - 1). \tag{2.37}$$

When we handle the commutator (2.11) in a similar way we get $D(u)F(u) \sim F(u + 1)$. We write the proportionality factors as in (2.34)–(2.35), by scalar factors Δ_{\pm} depending on u . Now apply the quantum determinant $\Delta(u - \frac{1}{2})$, expressed by the second formula of (2.2), to $F(u)$ ($u \in \mathcal{D}$) and use (2.32) or (2.33) or (2.34)–(2.35). This yields (2.36), with u replaced by $u - \frac{1}{2}$, and with both sides acting on $F(u)$. Since $\Delta(u - \frac{1}{2})$ commutes with $A(v)$ and $D(v)$, the general case of (2.36) then follows. •

Theorem 2.2. *Let W be a complex vector space on which the QISM II algebra acts by an algebra representation. Keep the other assumptions of Theorem 1, except that $D(u)$ in (2.33) is replaced by $-A(-u)$. Then the conclusions of Theorem 1 remain valid, except that $D(u)$ in (2.35) is replaced by $-A(-u)$.*

Proof. Analogous to the proof of Theorem 2.1. Equation (2.37) is now obtained from (2.23), while we get from (2.26) the equation $C(u + 1)(2uD(u) - A(u))F(u) = 0$. In view of (2.1) this implies $-A(-u)F(u) \sim F(u + 1)$. Use the second formula of (2.3) for the proof of (2.36). •

§3. Some formulas for the classical special functions

For special functions of hypergeometric type there exists a large number of formulas in which a (usually first order) differential operator acting on the special function yields a special function of similar type but with some parameters shifted. Usually such formulas occur in pairs, with shifting of parameters in opposite directions. Below we list some pairs of shift operator actions for which we will later give interpretations in the framework of QISM I or II algebras. Throughout we use u for the parameter which is shifted. We give the formulas for the case of infinite power series. For terminating power series the formulas can be rewritten in terms of Jacobi polynomials, etc.

3.1. Gauss hypergeometric function, Legendre function, and Jacobi polynomials. The *Gauss hypergeometric function* [2, Ch. 2]

$${}_2F_1(a, b; c; x) \equiv F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \quad |x| < 1, \quad c \neq 0, -1, -2, \dots, \quad (3.1)$$

is, up to a constant factor, the only analytic solution $f(x)$ in a neighbourhood of 0 of the equation

$$(x(1-x)\partial_x^2 + (c - (a+b+1)x)\partial_x - ab)f(x) = 0. \quad (3.2)$$

The solution is normalized by $f(0) = 1$. The function (3.1) has a unique analytic continuation to $\mathbb{C} \setminus [1, \infty)$. A second solution to (3.2) is given by

$$f(x) = F(a, b; a+b-c+1; 1-x), \quad c-a-b \neq 1, 2, \dots \quad (3.3)$$

Note also

$$F(a, b; c; x) = (1-x)^{-a} F(a, c-b; c; x/(x-1)). \quad (3.4)$$

Jacobi polynomials:

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_n}{n!} F(-n, n+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-x)) = (-1)^n P_n^{(\beta, \alpha)}(-x), \\ n = 0, 1, 2, \dots, \quad \alpha, \beta \neq -1, -2, \dots \quad (3.5)$$

Legendre function [2, Ch. 3]:

$$P_\nu^\mu(x) = \frac{2^\mu (x^2-1)^{-\mu/2}}{\Gamma(1-\mu)} F(1-\mu+\nu, -\mu-\nu; 1-\mu; \frac{1}{2} - \frac{1}{2}x), \\ x \in \mathbb{C} \setminus (-\infty, 1], \quad \mu \neq 1, 2, \dots, \quad (3.6)$$

$$P_\nu^\mu(x) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(x), \quad \mu = 1, 2, \dots \quad (3.7)$$

Shift operator pairs:

$$(x(1-x)\partial_x - bx + c - a - u)F(a+u, b; c; x) = (c-a-u)F(a+u-1, b; c; x), \quad (3.8)$$

$$(x\partial_x + a+u)F(a+u, b; c; x) = (a+u)F(a+u+1, b; c; x); \quad (3.9)$$

$$\begin{aligned} & \left(x\partial_x - \frac{b-c}{1-x} + b-1+u\right) [(1-x)^{a+u}F(a+u, b+u; c+u; x)] \\ & = (c+u-1)(1-x)^{a+u-1}F(a+u-1, b+u-1; c+u-1; x), \end{aligned} \tag{3.10}$$

$$\begin{aligned} & ((1-x)\partial_x + a+u) [(1-x)^{a+u}F(a+u, b+u; c+u; x)] \\ & = \frac{(a+u)(b+u)}{c+u} (1-x)^{a+u+1}F(a+u+1, b+u+1; c+u+1; x); \end{aligned} \tag{3.11}$$

$$\begin{aligned} & \left(x(1-x)\partial_x + \left(\frac{1}{2}-a\right)x + \frac{1}{2}c - \frac{1}{2} + \left(u - \frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{\delta}{u - \frac{1}{2}}\right) F(a+u, a-u; c; x) \\ & = \frac{(a-u)(a-c+u)}{2\left(u - \frac{1}{2}\right)} F(a+u-1, a-u+1; c; x), \end{aligned} \tag{3.12}$$

$$\begin{aligned} & \left(-x(1-x)\partial_x - \left(\frac{1}{2}-a\right)x - \frac{1}{2}c + \frac{1}{2} + \left(u + \frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{\delta}{u + \frac{1}{2}}\right) F(a+u, a-u; c; x) \\ & = \frac{(a+u)(a-c-u)}{2\left(u + \frac{1}{2}\right)} F(a+u+1, a-u-1; c; x), \end{aligned} \tag{3.13}$$

where

$$\delta = \frac{1}{2}(a-c + \frac{1}{2})(a - \frac{1}{2}); \tag{3.14}$$

$$\left((x^2-1)^{1/2}\partial_x + \frac{ux}{(x^2-1)^{1/2}}\right)P_\nu^u(x) = (\nu+u)(\nu-u+1)P_\nu^{u-1}(x), \tag{3.15}$$

$$\left((x^2-1)^{1/2}\partial_x - \frac{ux}{(x^2-1)^{1/2}}\right)P_\nu^u(x) = P_\nu^{u+1}(x). \tag{3.16}$$

The following special pair of shift operator actions can be derived from (3.10)–(3.11). Fix $n \in \{1, 2, \dots\}$. Define functions F_k ($k \in \mathbb{Z}$) by

$$F_k(x) = \frac{(n+k)!}{k!} (1-x)^{-n+k} F(-n+k, n+k+1; k+1; x), \quad k = 0, 1, 2, \dots, \tag{3.17}$$

$$F_{-k}(x) = (-1)^k x^k (1-x)^{-k} F_k(x), \quad k = 1, 2, \dots \tag{3.18}$$

Then

$$\left(x\partial_x - \frac{n}{1-x} + n+k\right) F_k(x) = (n+k) F_{k-1}(x), \tag{3.19}$$

$$\left((1-x)\partial_x - n+k\right) F_k(x) = -(n-k) F_{k+1}(x). \tag{3.20}$$

3.2. Confluent hypergeometric function and Laguerre polynomials. The *confluent hypergeometric function* [2, §6.3]

$${}_1F_1(a; c; x) \equiv \Phi(a, c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k k!} x^k, \quad c \neq 0, -1, -2, \dots, \tag{3.21}$$

is, up to a constant factor, the only entire analytic solution $f(x)$ of the equation

$$(x\partial_x^2 + (c-x)\partial_x - a) f(x) = 0. \tag{3.22}$$

The solution is normalized by $f(0) = 1$. The confluent hypergeometric function can be obtained as a limit case of the Gauss hypergeometric function

$${}_1F_1(a; c; x) = \lim_{b \rightarrow \infty} {}_2F_1(a, b; c; b^{-1}x). \tag{3.23}$$

The other special functions we will discuss can also be obtained as limits of the ${}_2F_1$ -function. Accordingly, all further shift operator pairs listed below are limit cases of shift operator pairs in §3.1.

For arbitrary a and c , *Tricomi's Ψ -function* $\Psi(a, c; x)$ can be defined, for instance, by the contour integral representation [2, 6.11(9)]. It is, up to a constant factor, the only analytic solution $f(x)$ of equation (3.22) on $(0, \infty)$ such that $f(x)$ is of at most polynomial growth as $x \rightarrow \infty$. This characterization can be extracted from [2, §6.7 and §6.13.1]. All derivatives of this function f have the same growth property as f . The solution is normalized by $f(x) = x^{-a} + \mathcal{O}(|x|^{-a-1})$ as $x \rightarrow \infty$.

Laguerre polynomials:

$$L_n^\alpha(x) = \frac{(\alpha + 1)_n}{n!} \Phi(-n, \alpha + 1; x) = \frac{(-1)^n}{n!} \Psi(-n, \alpha + 1; x), \quad n = 0, 1, 2, \dots \quad (3.24)$$

Shift operator pairs:

$$(x\partial_x - x + c - a - u) \Phi(a + u, c; x) = (c - a - u) \Phi(a + u - 1, c; x), \quad (3.25)$$

$$(x\partial_x + a + u) \Phi(a + u, c; x) = (a + u) \Phi(a + u + 1, c; x); \quad (3.26)$$

$$(x\partial_x - x + c + u - 1) \Psi(u, c + u; x) = -\Psi(u - 1, c + u - 1; x), \quad (3.27)$$

$$\partial_x \Psi(u, c + u; x) = -u \Psi(u + 1, c + u + 1; x); \quad (3.28)$$

$$\begin{aligned} & \left(\partial_x + \frac{u}{x} + \frac{\delta}{u - \frac{1}{2}} \right) \left[x^u e^{-\frac{x}{2}} \Psi\left(2\delta + \frac{1}{2} + u, 2u + 1; x\right) \right] \\ &= \frac{2\delta + \frac{1}{2} - u}{2u - 1} x^{u-1} e^{-\frac{x}{2}} \Psi\left(2\delta + \frac{1}{2} + u - 1, 2u - 1; x\right), \end{aligned} \quad (3.29)$$

$$\begin{aligned} & \left(-\partial_x + \frac{u}{x} + \frac{\delta}{u + \frac{1}{2}} \right) \left[x^u e^{-\frac{x}{2}} \Psi\left(2\delta + \frac{1}{2} + u, 2u + 1; x\right) \right] \\ &= \frac{2\delta + \frac{1}{2} + u}{2u + 1} x^{u+1} e^{-\frac{x}{2}} \Psi\left(2\delta + \frac{1}{2} + u + 1, 2u + 3; x\right). \end{aligned} \quad (3.30)$$

3.3. Parabolic cylinder function and Hermite polynomials. The *parabolic cylinder function* [2, §8.2] can be defined by

$$D_\nu(x) = 2^{\frac{1}{2}(\nu-1)} e^{-x^2/4} x \Psi\left(\frac{1}{2} - \frac{1}{2}\nu, \frac{3}{2}; \frac{1}{2}x^2\right). \quad (3.31)$$

The function $f(x) = e^{\frac{1}{4}x^2} D_\nu(x)$ is, up to a constant factor, the only entire analytic solution of the equation

$$(\partial_x^2 - x\partial_x + \nu) f(x) = 0 \quad (3.32)$$

such that $f(x)$ is on $(0, \infty)$ of at most polynomial growth as $x \rightarrow \infty$. All derivatives of this function f have the same growth property as f . The solution is normalized by $f(x) = x^\nu + \mathcal{O}(x^{\nu-2})$ as $x \rightarrow \infty$.

Hermite polynomials (polynomials of degree n):

$$H_n(x) = 2^{\frac{1}{2}n} e^{\frac{1}{2}x^2} D_n(2^{\frac{1}{2}}x), \quad n = 0, 1, 2, \dots \quad (3.33)$$

Shift operator pairs:

$$(\partial_x - x)(e^{\frac{1}{4}x^2} D_{-u}(x)) = -e^{\frac{1}{4}x^2} D_{-u+1}(x), \quad (3.34)$$

$$\partial_x(e^{\frac{1}{4}x^2} D_{-u}(x)) = -u e^{\frac{1}{4}x^2} D_{-u-1}(x). \quad (3.35)$$

3.4. Bessel function. The *Bessel function* [2, Ch. 7]

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}x)^{2m+\nu}}{m! \Gamma(m + \nu + 1)} = \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu + 1)} {}_0F_1(-; \nu + 1; -\frac{1}{4}x^2),$$

$$\nu \neq -1, -2, \dots, \tag{3.36}$$

$$J_\nu(x) = (-1)^\nu J_{-\nu}(x), \quad \nu = -1, -2, \dots, \tag{3.37}$$

is a solution $f(x)$ of the equation

$$((x\partial_x)^2 + x^2 - \nu^2) f(x) = 0. \tag{3.38}$$

Hence the function $f(x) = {}_0F_1(-; 1 + \nu; \frac{1}{4}x^2)$ ($\nu \neq -1, -2, \dots$) satisfies the equation

$$(\partial_x^2 + (2\nu + 1)x^{-1}\partial_x - 1) f(x) = 0. \tag{3.39}$$

Another solution to (3.39) (for any ν) is given by $f(x) = x^{-\nu} K_\nu(x)$ ($x > 0$), where K_ν is the *modified Bessel function of the third kind* defined for instance by the integral representation [2, 7.12(23)]. It is, up to a constant factor, the unique analytic solution $f(x)$ of (3.39) on $(0, \infty)$ which tends to 0 faster than any inverse power as $x \rightarrow \infty$ (cf. [2, 7.13(7)]). All derivatives of this function f have the same growth property as f . The solution is normalized by $f(x) = (\pi/2)^{\frac{1}{2}} x^{-\nu-\frac{1}{2}} e^{-x} (1 + \mathcal{O}(x^{-1}))$ as $x \rightarrow \infty$.

Shift operator pairs:

$$(x\partial_x + 2u) [x^{-u} K_u(x)] = -x^{-u+1} K_{u-1}(x), \tag{3.40}$$

$$x^{-1} \partial_x [x^{-u} K_u(x)] = -x^{-u-1} K_{u+1}(x); \tag{3.41}$$

$$(\partial_x + ux^{-1}) J_u(x) = J_{u-1}(x), \tag{3.42}$$

$$(-\partial_x + ux^{-1}) J_u(x) = J_{u+1}(x). \tag{3.43}$$

§4. Lie algebras of first order differential operators

In the QISM I case we obtain from (2.14) that

$$D(u - 1)A(u) - A(u)D(u - 1) = B(u - 1)C(u) - B(u)C(u - 1),$$

$$D(u)A(u + 1) - A(u + 1)D(u) = B(u)C(u + 1) - B(u + 1)C(u).$$

In combination with (2.2) (fourth resp. second formula) this yields

$$D(u - 1)A(u) = B(u - 1)C(u) + \Delta(u - \frac{1}{2}), \tag{4.1}$$

$$A(u + 1)D(u) = B(u + 1)C(u) + \Delta(u + \frac{1}{2}). \tag{4.2}$$

Assume now that we have a representation of the QISM I algebra on a space of functions in one variable, analytic on a certain region, such that (i) $B(u) = B_0$ is independent of u , (ii) $\Delta(u)$ is scalar for all u , (iii) $A(u)$ and $D(u)$ are first order differential operators. Then equations (4.1)–(4.2) show that the second order operator $B_0C(u)$ has a suitable form for the factorization method, which was originated by Schrödinger and was due in its definitive form to Infeld and Hull [4].

Similarly, in the QISM II case we obtain from (2.18) that

$$A(u + 1)A(-u) - A(-u)A(u + 1) = -B(u + 1)C(-u) + B(-u)C(u + 1),$$

$$A(u)A(-u + 1) - A(-u + 1)A(u) = -B(u)C(-u + 1) + B(-u + 1)C(u).$$

Assume that $B(-u) = B(u)$ and $C(-u) = C(u)$ (part of the symmetry properties (2.1)). Then it follows in combination with (2.3) (fourth resp. second formula) that

$$-A(-u+1)A(u) = B(u-1)C(u) + \Delta(u - \frac{1}{2}), \quad (4.3)$$

$$-A(u+1)A(-u) = B(u+1)C(u) + \Delta(u + \frac{1}{2}). \quad (4.4)$$

So assume that we have a representation of the QISM II algebra on a space of functions in one variable, analytic on a certain region, such that (i) $B(u) = B_0$ is independent of u , (ii) $C(u) = C(-u)$ for all u , (iii) $\Delta(u)$ is scalar for all u , (iv) $A(u)$ is a first order differential operator. Then equations (4.3)-(4.4) show that the second order operator $B_0C(u)$ has a suitable form for the Infeld-Hull factorization method.

The factorization method is summarized in Miller [8, Ch. 7]. Under some special assumptions on the type of factorizing operators (first order part not depending on u , zero order part of degree at most one in u), a complete classification of all possibilities is given.

These factorizing operators of special type give rise to a Lie algebra of first order differential operators in two variables. Indeed, assume that we have elements $A(u), D(u), K(u), \Delta(u)$ ($u \in \mathbb{C}$) of an associative algebra \mathcal{A} such that $\Delta(u)$ is scalar and

$$D(u-1)A(u) = K(u) + \Delta(u - \frac{1}{2}), \quad (4.5)$$

$$A(u+1)D(u) = K(u) + \Delta(u + \frac{1}{2}). \quad (4.6)$$

Assume that $A(u)$ and $D(u)$ are of degree at most one in u :

$$A(u) = A_0 + A_1u, \quad D(u) = D_0 + D_1u. \quad (4.7)$$

Consider now the algebra spanned by elements of the form $Bt^k\partial_t^l$, where $B \in \mathcal{A}$. Define in this algebra the elements

$$J^+ = t(D_0 + D_1t\partial_t) = tD(t\partial_t), \quad J^- = t^{-1}(A_0 + A_1t\partial_t) = t^{-1}A(t\partial_t). \quad (4.8)$$

Then, in view of (4.5)-(4.6), we have

$$J^+J^- = K(t\partial_t) + \Delta(t\partial_t - \frac{1}{2}), \quad J^-J^+ = K(t\partial_t) + \Delta(t\partial_t + \frac{1}{2}).$$

Hence

$$[J^-, J^+] = \Delta(t\partial_t + \frac{1}{2}) - \Delta(t\partial_t - \frac{1}{2}). \quad (4.9)$$

Assume furthermore that A_1 and D_1 commute. Then it follows from (4.7) and (4.9) that, for certain complex constants Q_2 and Q_1 , we have

$$[J^-, J^+] = 2Q_2t\partial_t + Q_1. \quad (4.10)$$

Clearly, we have also the commutators

$$[t\partial_t, J^\pm] = \pm J^\pm. \quad (4.11)$$

Thus the elements $J^+, J^-, t\partial_t$ span, together with the central element 1, a four-dimensional complex Lie algebra denoted $\mathcal{G}(a, b)$ in Miller [8, §2-5]. Here $a^2 = -Q_2$ and $b = Q_1$. They fall apart into three isomorphism classes: (i) $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ ($a \neq 0$), (ii) complexification of the harmonic oscillator algebra, i.e. of the semidirect sum of \mathbb{R} with the Heisenberg Lie algebra ($a = 0, b \neq 0$), (iii) $\mathfrak{e}(2, \mathbb{C}) \oplus \mathbb{C}$ ($a = b = 0$), where $\mathfrak{e}(2, \mathbb{C})$ is the complexified Lie algebra of the group of plane motions.

Conversely, if J^\pm are of the form (4.8) and if (4.10) holds then

$$A(u+1)D(u) - D(u-1)A(u) = 2Q_2u + Q_1.$$

If we then put

$$\Delta(u) = Q_2 u^2 + Q_1 u + Q_0$$

for some constant Q_0 then $\Delta(u + \frac{1}{2}) - \Delta(u - \frac{1}{2}) = 2Q_2 u + Q_1$ and

$$A(u + 1)D(u) - \Delta(u + \frac{1}{2}) = D(u - 1)A(u) - \Delta(u - \frac{1}{2}).$$

So, if $K(u)$ is put equal to the left hand side of the above identity, we recover (4.5)–(4.6).

Let us return to equations (4.5)–(4.6). Suppose that the algebra \mathcal{A} acts on some linear space W and that, for some $u_0 \in \mathbb{C}$ and some $F(u_0) \in W \setminus \{0\}$, we have that $K(u_0)F(u_0) = 0$. Then it follows from (4.5)–(4.6) that $K(u_0 - 1)(A(u_0)F(u_0)) = 0$ and $K(u_0 + 1)(D(u_0)F(u_0)) = 0$. Moreover, $D(u_0 - 1)A(u_0)F(u_0) = \Delta(u_0 - \frac{1}{2})F(u_0)$ and $A(u_0 + 1)D(u_0)F(u_0) = \Delta(u_0 + \frac{1}{2})F(u_0)$. Thus, for $k = 1, 2, \dots$, we can recursively define $F(u_0 + k) = \text{const.} D(u_0 + k - 1)F(u_0 + k - 1)$ and $F(u_0 - k) = \text{const.} A(u_0 - k + 1)F(u_0 - k + 1)$, as long as these vectors are nonzero. In that way we obtain strings of vectors $F(u)$ ($u \in \mathcal{D}$) as in Theorem 2.1, where $K(u)F(u) = 0$ for $u \in \mathcal{D}$ and (2.34)–(2.35) are valid for a certain choice of $\Delta_{\pm}(u)$. If, moreover, the operators J^{\pm} are defined by (4.8), then

$$J^{\pm}(t^u F(u)) = \Delta_{\pm}(u \pm \frac{1}{2})t^{u \pm 1} F(u \pm 1), \quad u \in \mathcal{D}.$$

So we have a Lie algebra acting on the elements $t^u F(u)$. The algebra acting on the elements $F(u)$ is not a Lie algebra, in general, but we will see later in this paper that this algebra action can often be extended to a QISM I algebra action. The crucial point in making this extension is to find an operator C_0 acting on W such that $C_0 F(u) = -u F(u)$ for $u \in \mathcal{D}$. A necessary condition for finding such an operator will be that the elements $F(u)$ ($u \in \mathcal{D}$) are linearly independent. Note that this is not yet guaranteed by the above assumptions. But, of course, we do know that the elements $t^u F(u)$ ($u \in \mathcal{D}$) are linearly independent.

In [8, §2-7] Miller assumes that

$$J^+ = t(\partial_x - k(x)t\partial_t + j(x)), \tag{4.12}$$

$$J^- = t^{-1}(-\partial_x - k(x)t\partial_t + j(x)) \tag{4.13}$$

for certain analytic functions k and j . Then he shows that (4.10) holds if and only if

$$k'(x) + k(x)^2 = Q_2, \tag{4.14}$$

$$j'(x) + k(x)j(x) = -\frac{1}{2}Q_1. \tag{4.15}$$

The general solution of equations (4.14)–(4.15) will depend on two parameters, but one parameter is trivial because the equations are invariant under translation. Solution of the equations yields six different cases, two for each isomorphism class of the Lie algebra $\mathcal{G}(a, b)$, depending on whether $k(x)$ is constant or not. The list of solutions is as follows (cf. [8, p. 272], we do not give the trivial translation parameter):

Type (A) $k(x) = a \cot ax, \quad j(x) = \frac{b}{2a} \cot ax + \frac{q}{\sin ax}. \tag{4.16}$

Type (B) $k(x) = ia, \quad j(x) = \frac{ib}{2a} + qe^{-iax}. \tag{4.17}$

Type (C') $k(x) = x^{-1}, \quad j(x) = -\frac{1}{4}bx + qx^{-1}. \tag{4.18}$

Type (D') $k(x) = 0, \quad j(x) = -\frac{1}{2}bx. \tag{4.19}$

Type (C'') $k(x) = x^{-1}, \quad j(x) = qx^{-1}. \tag{4.20}$

Type (D'') $k(x) = 0, \quad j(x) = q. \tag{4.21}$

Here a, b are the parameters from $\mathcal{G}(a, b)$ and q is another parameter. For types A and B we have $a \neq 0$, for types C' and D' we have $a = 0$ and $b \neq 0$, and for types C'' and D'' we have $a = b = 0$. For types A, C' and C'' $k(x)$ is not constant, but for types B, D' and D'' it is. In the following we do not consider the trivial case D'' because it does not give any shift operator pair.

The operators (4.12)–(4.13) are in a certain normal form. We want to transform them into another normal form which is better adapted to the QISM I algebra. This is done in the following lemmas, which can be proved by straightforward computation.

Lemma 4.1. *Let J^\pm be given by (4.12)–(4.13) and assume that (4.10), and so (4.14)–(4.15), hold. Make a transformation of the variables x, t by replacing t with $(\psi(x))^{-1}t$, where ψ is such that*

$$(\psi'(x)/\psi(x))^2 = -Q_2 + k(x)^2.$$

Then

$$J^+ = t(D_{01}(x)\partial_x + D_{00}(x) + \delta t\partial_t), \quad (4.22)$$

$$J^- = t^{-1}(A_{01}(x)\partial_x + A_{00}(x) + \alpha t\partial_t), \quad (4.23)$$

where $A_{00}, A_{01}, D_{00}, D_{01}$ are certain analytic functions with A_{01} and D_{01} not identically zero, and α, δ are complex constants such that $\alpha\delta = Q_2$.

Furthermore, $A_{01}(x)/D_{01}(x)$ is constant or not depending on whether $k(x)$ is constant or not.

Lemma 4.2. *Let J^\pm be given by (4.22)–(4.23). Write*

$$A_0 = A_{01}(x)\partial_x + A_{00}(x), \quad D_0 = D_{01}(x)\partial_x + D_{00}(x). \quad (4.24)$$

Then (4.10) holds if and only if $\alpha\delta = Q_2$ and

$$[D_0, A_0] = \delta A_0 + \alpha D_0 - Q_1. \quad (4.25)$$

Furthermore, (4.25) holds if and only if

$$D_{01}(x)A'_{01}(x) - A_{01}(x)D'_{01}(x) = \delta A_{01}(x) + \alpha D_{01}(x), \quad (4.26)$$

$$D_{01}(x)A'_{00}(x) - A_{01}(x)D'_{00}(x) = \delta A_{00}(x) + \alpha D_{00}(x) - Q_1. \quad (4.27)$$

Lemma 4.3. *Let J^\pm be given by (4.22)–(4.23) with A_{01} and D_{01} not identically zero and assume that (4.10), and so (4.26)–(4.27), hold. Make a transformation of the variables x, t by replacing t with $(\varphi(x))^{-1}t$, where φ is such that*

$$2\varphi(x)\varphi'(x) = \frac{\delta - \alpha\varphi(x)^2}{A_{01}(x)}.$$

Then

$$J^+ = t(\chi(x)\partial_x + j^+(x) - k(x)t\partial_t),$$

$$J^- = t^{-1}(-\chi(x)\partial_x + j^-(x) - k(x)t\partial_t),$$

for certain analytic functions χ (not identically zero), j^\pm , and k . Furthermore $k(x)$ is constant or not according to whether $A_{01}(x)/D_{01}(x)$ is constant or not. Finally, for a

suitable analytic function f , not identically zero, and after a suitable transformation of the x -variable, the operators $f(x)^{-1} J^\pm \circ f(x)$ take the form (4.12)–(4.13).

Equations (4.24) and (4.25) will not change if the operators A_0 and D_0 are replaced by operators $f(x)^{-1} A_0 \circ f(x)$ and $f(x)^{-1} D_0 \circ f(x)$, respectively, for a suitable analytic function f , not identically zero. We call such transformations *gauge transformations*. Neither do the equations change when we make an analytic transformation of the x -variable. We will consider solutions to the equations as equivalent if they can be obtained from each other by the two types of transformations just described.

In the formulas below we list operators $A(u) = A_0 + \alpha u$, $D(u) = D_0 + \delta u$ such that A_0 and D_0 are of the form (4.24) and such that they satisfy the equivalent conditions of Lemma 4.2. These formulas can be derived either from (4.16)–(4.20) by use of the above lemmas, or by a straightforward verification that the conditions of Lemma 4.2 are satisfied.

$$\begin{aligned} \text{Type (A)} \quad A(u) &= x(1-x)\partial_x - bx + c - a - u, \quad D(u) = x\partial_x + a + u, \\ [D_0, A_0] &= -x^2\partial_x - bx, \quad Q_1 = c - 2a. \end{aligned} \tag{4.28}$$

$$\begin{aligned} \text{Type (B)} \quad A(u) &= x\partial_x - x + c - a - u, \quad D(u) = x\partial_x + a + u, \\ [D_0, A_0] &= -x, \quad Q_1 = c - 2a. \end{aligned} \tag{4.29}$$

$$\begin{aligned} \text{Type (C')} \quad A(u) &= x\partial_x - x + c - 1 + u, \quad D(u) = \partial_x, \\ [D_0, A_0] &= \partial_x - 1, \quad Q_1 = 1. \end{aligned} \tag{4.30}$$

$$\begin{aligned} \text{Type (D')} \quad A(u) &= \partial_x - x, \quad D(u) = \partial_x, \\ [D_0, A_0] &= 1, \quad Q_1 = -1. \end{aligned} \tag{4.31}$$

$$\begin{aligned} \text{Type (C'')} \quad A(u) &= x\partial_x + 2u, \quad D(u) = x^{-1}\partial_x, \\ [D_0, A_0] &= 2x^{-1}\partial_x, \quad Q_1 = 0. \end{aligned} \tag{4.32}$$

For all these types we can give functions $F(u)$ on which $A(u)$ and $D(u)$ act as shifting operators. See equations (3.8)–(3.9), (3.25)–(3.26), (3.27)–(3.28), (3.34)–(3.35), (3.40)–(3.41), respectively. These functions $F(u)$ are not uniquely determined. We might write down similar formulas with another choice $F(u)$ for the solution of the corresponding second order equation.

Next we discuss a form of the operators J^\pm (cf. (4.12)–(4.13)) which we will meet in the case of the QISM II algebra. Put

$$J^+ = t(-A_{01}(x)\partial_x - A_{00}(x) + A_1(x)(t\partial_t + \frac{1}{2})), \tag{4.33}$$

$$J^- = t^{-1}(A_{01}(x)\partial_x + A_{00}(x) + A_1(x)(t\partial_t - \frac{1}{2})), \tag{4.34}$$

where A_{00}, A_{01}, A_1 are certain analytic functions with A_{01} not identically zero. Observe that (4.12)–(4.13) is of the form (4.33)–(4.44) if and only if $j(x) = 0$ (then necessarily, by (4.15), $Q_1 = 0$). This occurs nontrivially in (4.16)–(4.21) ($k(x)$ is not constant and $j(x) = 0$) precisely for Types A ((4.16) with $b = q = 0$) and C'' ((4.20) with $q = 0$). On the other hand, the operators J^\pm of the form (4.33)–(4.34) can be brought in the form (4.12)–(4.13) by suitable gauge transformation and x -transformation.

Write $A_0 = A_{01}(x)\partial_x + A_{00}(x)$ as before. The following lemma can be proved by a straightforward computation.

Lemma 4.4. *Let J^\pm be given by (4.33)–(4.34). Then (4.10) holds if and only if $Q_1 = 0$ and*

$$[A_0, A_1] + A_1^2 = Q_2. \tag{4.35}$$

Furthermore, (4.35) holds if and only if

$$A_{01}A'_1 + A_1^2 = Q_2. \tag{4.36}$$

In the formulas below we list operators $A(u) = A_0 + A_1u$ with A_0 an analytic operator of the form (4.24) and A_1 an analytic function such that A_{00}, A_{01}, A_1 satisfy the equivalent conditions of Lemma 4.4.

$$\begin{aligned} \text{Type (A)} \quad A(u) &= x(1-x)\partial_x + (u-a)(x-\frac{1}{2}), \\ [A_1, A_0] &= -x(1-x). \end{aligned} \tag{4.37}$$

$$\begin{aligned} \text{Type (C''')} \quad A(u) &= ux^{-1} + \partial_x, \\ [A_1, A_0] &= x^{-2}. \end{aligned} \tag{4.38}$$

For these two types we can give functions $F(u)$ on which $A(u)$ and $-A(-u)$ act as shifting operators. See equations (3.12)–(3.13) ($c = a + \frac{1}{2}$) and (3.42)–(3.43), respectively.

§5. Rank 1 L -operators for the QISM I algebra

The T -algebra is the algebra with matrix elements of $T(u)$ as generators and with relation (1.3). We pass to a quotient algebra by adding the relation $T(u) = uT(1) + (1-u)T(0)$. In other words, we make the ansatz that $T(u)$ is of the form

$$L(u) = \begin{pmatrix} A_1u + A_0 & B_1u + B_0 \\ C_1u + C_0 & D_1u + D_0 \end{pmatrix} = L_1u + L_0. \tag{5.1}$$

Substitute (5.1) in relation (1.3). Then we get the algebra with $A_i, B_i, C_i, D_i, i = 0, 1$ as generators and with relations

$$L_1^{(1)}L_1^{(2)} = L_1^{(2)}L_1^{(1)}, \quad L_1^{(1)}L_0^{(2)} = L_0^{(2)}L_1^{(1)}, \tag{5.2}$$

$$[L_0^{(1)}, L_0^{(2)}] = -[P, L_1^{(1)}L_0^{(2)}]. \tag{5.3}$$

Relations (5.2) imply that the entries of the L_1 -matrix are in the center of the algebra. Let us pass once more to a quotient algebra by adding the relation

$$L_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{5.4}$$

for certain $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. A representation of the algebra with relations (5.2) and (5.3) which has the property that all elements in the center of the algebra are represented as scalars, can also be viewed as a representation of the algebra with relations (5.3) and (5.4) for a certain choice of $\alpha, \beta, \gamma, \delta$.

Remark 5.1. Consider for a moment the more general situation of a QISM I algebra relation (1.3) with n -dimensional auxiliary space, i.e., $R(u) = u + P$, where $P(x \otimes y) = y \otimes x$ for $x, y \in \mathbb{C}^n$. Just as in the 2×2 case we make the ansatz that $T(u) = uM + L$, where M and L are $n \times n$ matrices with a priori noncommuting matrix elements. Then the matrix entries of M are in the center of the algebra. We add the relations $M_{ij} = \mu_{ij}$ for certain $\mu_{ij} \in \mathbb{C}$. Then we obtain the algebra with the L_{ij} as generators and with relations

$$[L^{(1)}, L^{(2)}] = -[P, \mu^{(1)}L^{(2)}], \tag{5.5}$$

hence

$$[L_{ir}, L_{js}] = -\mu_{jr}L_{is} + \mu_{is}L_{jr}. \tag{5.6}$$

Alternatively, we may consider the linear space \mathfrak{g} with the L_{ij} ($i, j = 1, \dots, n$) as basis vectors (so they are linearly independent) and with antisymmetric bilinear product defined by (5.6). We claim that, for any choice of the μ_{ij} , the space \mathfrak{g} equipped with this product becomes a Lie algebra, i.e., the product satisfies the Jacobi identity. Indeed, we have

$$[[L_{ir}, L_{js}], L_{kt}] = \mu_{jr}\mu_{ks}L_{it} - \mu_{jr}\mu_{it}L_{ks} - \mu_{is}\mu_{kr}L_{jt} + \mu_{is}\mu_{jt}L_{kr}$$

and two similar identities obtained by cyclic permutation of the indices (ijk) and (rst) . Addition of the three identities yields 0 on the right hand side.

The fact that \mathfrak{g} is a Lie algebra is equivalent to saying that the L_{ij} are linearly independent in the algebra with generators L_{ij} and relations (5.6). Then this algebra is the universal enveloping algebra of the Lie algebra \mathfrak{g} .

Let us return to the case $n = 2$. The commutator (5.3) yields a Lie algebra spanned by A_0, B_0, C_0 , and D_0 . The componentwise form of the commutator (5.3) is as follows:

$$\begin{aligned} [A_0, B_0] &= \beta A_0 - \alpha B_0, & [A_0, C_0] &= \alpha C_0 - \gamma A_0, \\ [A_0, D_0] &= \beta C_0 - \gamma B_0, & [B_0, C_0] &= \alpha D_0 - \delta A_0, \\ [B_0, D_0] &= \beta D_0 - \delta B_0, & [C_0, D_0] &= \delta C_0 - \gamma D_0. \end{aligned} \tag{5.7}$$

The quantum determinant (2.2) now is of the form

$$\Delta(u) = (\alpha\delta - \beta\gamma)u^2 + Q_1u + Q_0, \tag{5.8}$$

$$Q_1 = \alpha D_0 - \gamma B_0 + \delta A_0 - \beta C_0, \tag{5.9}$$

$$Q_0 = A_0D_0 - B_0C_0 + \frac{1}{2}(\alpha D_0 + \gamma B_0 - \delta A_0 - \beta C_0) - \frac{1}{4}(\alpha\delta - \beta\gamma). \tag{5.10}$$

Here the right hand sides of (5.9) and (5.10) give operators in the center of the algebra. We consider (5.9) and (5.10) as added relations, for a certain choice of $Q_1, Q_2 \in \mathbb{C}$. So a representation of the algebra with relations (5.2) and (5.3) which has the property that all elements in the center of the algebra are represented as scalars, can also be viewed as a representation of the algebra with relations (5.3), (5.4), (5.9), and (5.10) for a certain choice of $\alpha, \beta, \gamma, \delta, Q_1, Q_2$.

From now on we assume that

$$\beta = 0, \quad \gamma \neq 0. \tag{5.11}$$

Remark 5.2. We will determine the type of Lie algebra given by (5.7) with (5.9), (5.11). These commutators can be equivalently written as

$$[-\gamma^{-1}C_0, A_0 - \alpha\gamma^{-1}C_0] = -(A_0 - \alpha\gamma^{-1}C_0),$$

$$[-\gamma^{-1}C_0, D_0 - \delta\gamma^{-1}C_0] = D_0 - \delta\gamma^{-1}C_0,$$

$$[A_0 - \alpha\gamma^{-1}C_0, D_0 - \delta\gamma^{-1}C_0] = 2\alpha\delta(-\gamma^{-1}C_0) + Q_1.$$

These three equations have the same structure as equations (4.10), (4.11), which we took from Miller's book [8] and which give rise to a Lie algebra $\mathcal{G}(a, b)$ with $a^2 = -\alpha\delta$ and $b = Q_1$. Thus we find the same three types of Lie algebras spanned by A_0, C_0, D_0 , and the central element 1 as in the discussion after (4.11).

In the following we will obtain realizations of these Lie algebras as operators acting on functions of one variable. Here A_0 and D_0 will be first order differential operators,

but C_0 a second order differential operator or an integro-differential operator. It is interesting to compare this with Miller [8], whose only realizations of these Lie algebras by operators acting on functions of one variable are by first order differential operators.

The following lemma can be proved in a straightforward way. It shows that equations (5.7), (5.9), and (5.10), with (5.11), and under the assumption that B_0 is injective, can be equivalently written in a much more simple form.

Lemma 5.3. *Let $\alpha, \gamma, \delta, Q_0, Q_1$ be scalars, with $\gamma \neq 0$. Let A_0, B_0, C_0, D_0 be operators acting on some linear space W . Let B_0 be injective. Then the following three statements are equivalent:*

- (a) $\begin{pmatrix} A_0 + \alpha u & B_0 \\ C_0 + \gamma u & D_0 + \delta u \end{pmatrix}$ is a representation of the QISM I algebra with quantum determinant $\Delta(u) = \alpha \delta u^2 + Q_1 u + Q_0$;
- (b) The six commutators (5.7) and formulas (5.9), (5.10) are valid with $\beta = 0$;
- (c) The following three equalities are valid:

$$[D_0, A_0] = \alpha D_0 + \delta A_0 - Q_1, \tag{5.12}$$

$$\gamma B_0 = \alpha D_0 + \delta A_0 - Q_1, \tag{5.13}$$

$$B_0 C_0 = (A_0 + \alpha) D_0 - \left(\frac{1}{4} \alpha \delta + \frac{1}{2} Q_1 + Q_0\right). \tag{5.14}$$

Moreover, if $\{A_0, B_0, C_0, D_0, \alpha, \gamma, \delta, Q_0, Q_1\}$ satisfy these equivalent conditions, then so do $\{\lambda \mu A_0, \lambda \mu \nu^{-1} B_0, \lambda \nu C_0, \lambda D_0, \lambda \mu \alpha, \lambda \nu \gamma, \lambda \delta, \lambda^2 \mu Q_0, \lambda^2 \mu Q_1\}$ (where λ, μ, ν are nonzero scalars) and $\{D_0, B_0, C_0, A_0, -\delta, -\gamma, -\alpha, Q_0, -Q_1\}$.

The next proposition is, in a certain sense, an inverse to Theorem 2.1. If operators $A(u) = A_0 + \alpha u$ and $D(u) = D_0 + \delta u$ act on basis vectors $F(u)$ as in (2.34)–(2.35) (part of the conclusion of Theorem 2.1), then we can define actions of operators B_0 and C_0 such that condition (c) of Lemma 5.3 is satisfied for certain Q_0 and Q_1 . So we then have obtained a representation of the QISM I algebra.

Proposition 5.4. *Let \mathcal{D} be a subset of \mathbb{C} of the same form as in Theorem 2.1. Let W be a complex vector space spanned by linearly independent vectors $F(u)$ ($u \in \mathcal{D}$). Let α and δ be scalars. Let A_0 and D_0 be linear operators on W such that $A(u) = A_0 + \alpha u$ and $D(u) = D_0 + \delta u$ act on $F(u)$ as in (2.34)–(2.35). Let $\Delta(u)$ ($u \in \mathcal{D} \pm \frac{1}{2}$) be defined by (2.36) and assume that it is of the form*

$$\Delta(u) = \alpha \delta u^2 + Q_1 u + Q_0$$

for certain scalars Q_0 and Q_1 . Then (5.12) is valid. Now define B_0 by (5.13) (with $\gamma = 1$) and C_0 by $C_0 F(u) = -u F(u)$ ($u \in \mathcal{D}$). Then (5.14) is satisfied and also the other commutators in (5.7) (with $\beta = 0, \gamma = 1$). Then condition (a) of Lemma 5.3 is also satisfied. (However, B_0 is not necessarily injective.)

Proof. For $u \in \mathcal{D}$ we find

$$A_0 D_0 F(u) = (u(Q_1 - \alpha D_0 - \delta A_0) - \alpha D_0 + \left(\frac{1}{4} \alpha \delta + \frac{1}{2} Q_1 + Q_0\right)) F(u),$$

$$D_0 A_0 F(u) = (u(Q_1 - \alpha D_0 - \delta A_0) + \delta A_0 + \left(\frac{1}{4} \alpha \delta - \frac{1}{2} Q_1 + Q_0\right)) F(u).$$

Hence (5.12) is satisfied when both sides act on $F(u)$. For B_0 and C_0 as defined in the proposition, it then follows that (5.24) is satisfied when both sides act on $F(u)$. The other commutators in (5.7) (with $\beta = 0, \gamma = 1$) can now be proved by using (5.12), (5.13) if B_0 is involved, and by letting both sides act on $F(u)$ if C_0 is involved. •

We want to find a realisation of our QISM I algebra as in Theorem 2.1. From this point of view the transformations of A_0, B_0 , etc. as given in the last statement of the Lemma 5.3 do not mean any essential change. Thus, without lack of generality we may assume that $\gamma = 1$ and we may restrict our attention to three special choices for the pair α, δ : one with $\alpha, \delta \neq 0$, one with $\alpha \neq 0 = \delta$ and one with $\alpha = \delta = 0$.

We now make the restrictive assumption that A_0 and D_0 are first order differential operators of the form (4.24), analytic on a certain region:

$$A_0 = A_{00}(x) + A_{01}(x)\partial_x, \quad D_0 = D_{00}(x) + D_{01}(x)\partial_x. \quad (5.15)$$

We want to classify solutions of equations (5.12), (5.13), (5.14) such that A_0 and D_0 are of the form (5.15). Note that (5.12) coincides with (4.25). Essentially, up to equivalence under gauge transformations and transformations of the x -variable, all operators A_0 and D_0 satisfying (4.25) are given in the list (4.28)–(4.32). The corresponding B_0 , which equals $[D_0, A_0]$ by (5.12) and (5.13), is also given there. From (5.14) there follows now an expression for $B_0 C_0$. It depends yet on the constant Q_0 . If $\alpha\delta$ and Q_1 are not both zero (all but the last case), then we may fix Q_0 , after a possible translation of u such that $\Delta(u_0) = 0$ for a certain u_0 .

There are now two methods to proceed. The first method tries to obtain C_0 from the known expression for $B_0 C_0$, which may involve the taking of the inverse of a first order differential operator. The second method uses Proposition 5.4. For an explicit pair of shift operator actions on functions $F(u)$ as in Section 3, we may verify the assumptions of that proposition. Next we define C_0 by $C_0 F(u) = -uF(u)$. Sometimes it is not evident that the functions $F(u)$ are linearly independent. Without this property, it is of course not possible to define C_0 as in Proposition 5.4.

Below we follow the first method first in a formal way. We will give, parallel to the list (4.28)–(4.32), formal expressions for C_0 , or rather for $C(u) = C_0 + u$, such that (5.14) is satisfied. Afterwards we will specify a space W on which the operators act, such that the formal inverse can be understood rigorously. The expressions for $C(u)$ are as follows.

$$\begin{aligned} \text{Type (A)} \quad C(u) &= -(x\partial_x + b)^{-1} (x(1-x)\partial_x^2 \\ &\quad + (c - (b+a+u+1)x)\partial_x - ab - bu). \end{aligned} \quad (5.16)$$

$$\text{Type (B)} \quad C(u) = -(x\partial_x^2 + (c-x)\partial_x - a - u). \quad (5.17)$$

$$\text{Type (C')} \quad C(u) = (\partial_x - 1)^{-1} (x\partial_x^2 + (c+u-x)\partial_x - u). \quad (5.18)$$

$$\text{Type (D')} \quad C(u) = \partial_x^2 - x\partial_x + u. \quad (5.19)$$

$$\text{Type (C'')} \quad C(u) = \frac{1}{2}\partial_x^{-1} (x\partial_x^2 + (2u+1)\partial_x - x), \quad Q_0 = 1. \quad (5.20)$$

For each of the five nontrivial types above we will now give a space W , on which the operators A_0, B_0, C_0, D_0 act, such that B_0 is injective, the inverses of differential operators for types A, C' and C'' can be rigorously understood, and the conditions of Theorem 2.1 are satisfied. For this last task we have to give suitable subsets \mathcal{D} of \mathbb{C} such that the equation $C(u)w = 0$ has a one-dimensional solution in W for each $u \in \mathcal{D}$.

Type A. See (4.28), (5.16), (3.1), (3.2), (3.8)–(3.9). Assume that $b, c \neq 0, -1, -2, \dots$. Let W be the set of all analytic functions on the open unit disk in \mathbb{C} . Then A_0, B_0, D_0 act on W , the operator B_0 is moreover injective on W and $(x\partial_x + b)^{-1}$ acts on convergent power series by termwise application according to the rule

$$(x\partial_x + b)^{-1} (x^k) = (b+k)^{-1} x^k, \quad k = 0, 1, 2, \dots,$$

so $C(u)$ acts on W . Thus the three equivalent conditions of Lemma 5.3 are satisfied. Then the conditions of Theorem 2.1 are satisfied with $F(u)(x) = {}_2F_1(a+u, b; c; x)$ and we can take $\mathcal{D} = u_0 + \mathbb{Z}$ if $u_0 \notin (-a + \mathbb{Z}) \cup (c - a + \mathbb{Z})$ or we can take $\mathcal{D} = \{-a, -a-1, \dots\}$, for which $F(u)$ becomes a Jacobi polynomial.

Type B. See (4.29), (5.17), (3.21), (3.22), (3.25)–(3.26). Assume that $c \neq 0, -1, \dots$. Let W be the set of all entire analytic functions on \mathbb{C} . Then the conditions of Theorem 2.1 are satisfied with $F(u)(x) = {}_1F_1(a+u; c; x)$ and we can take $\mathcal{D} = u_0 + \mathbb{Z}$ if $u_0 \notin (-a + \mathbb{Z}) \cup (c - a + \mathbb{Z})$, or we can take $\mathcal{D} = \{-a, -a-1, -a-2, \dots\}$, for which $F(u)$ becomes a Laguerre polynomial.

Type C'. See (4.30), (5.18), (3.22), (3.27)–(3.28). Let W consist of all analytic functions f on $(0, \infty)$ such that $f(x)$ and all its derivatives $f^{(p)}(x)$ are of at most polynomial growth as $x \rightarrow \infty$. For $f \in W$ define

$$(\partial_x - 1)^{-1} f(x) = -e^x \int_x^\infty e^{-y} f(y) dy.$$

Then $C(u)$ acts on W . Then the conditions of Theorem 2.1 are satisfied with $F(u)(x) = \Psi(u, c+u; x)$ and we can take $\mathcal{D} = u_0 + \mathbb{Z}$ if $u_0 \notin \mathbb{Z}$, or we can take $\mathcal{D} = \{0, -1, -2, \dots\}$, for which $F(u)$ becomes a Laguerre polynomial.

Type D'. See (4.31), (5.19), (3.31), (3.32), (3.34)–(3.35). Let W be the set of all entire analytic functions f on \mathbb{C} such that $f(x)$ and all its derivatives $f^{(p)}(x)$ are on $(0, \infty)$ of at most polynomial growth as $x \rightarrow \infty$. Then the conditions of Theorem 2.1 are satisfied with $F(u)(x) = e^{\frac{1}{2}x^2} D_{-u}(x)$ and we can take $\mathcal{D} = u_0 + \mathbb{Z}$ if $u_0 \notin \mathbb{Z}$, or we can take $\mathcal{D} = \{0, -1, -2, \dots\}$, for which $F(u)$ becomes a Hermite polynomial.

Type C''. See (4.32), (5.20), (3.39), (3.40)–(3.41). Let W consist of all analytic functions f on $(0, \infty)$ such that $f(x)$ and all its derivatives $f^{(p)}(x)$ tend to 0 faster than any inverse power of x as $x \rightarrow \infty$. For $f \in W$ define

$$\partial_x^{-1} f(x) = - \int_x^\infty f(y) dy.$$

Then $C(u)$ acts on W . Then the conditions of Theorem 2.1 are satisfied with $F(u)(x) = x^{-u} K_u(x)$ and we can take $\mathcal{D} = u_0 + \mathbb{Z}$ for any u_0 .

Remark 5.5. In a sense, the Type A case is the generic case, since the other cases can be obtained from it by suitable limit transitions. This can be seen on the level of formulas for the special functions (cf. (3.23)), of Lie algebras and of QISM I algebra representations.

In Subsection 3.1 we gave several other shift operator pairs of Type A. The pair (3.10)–(3.11) is a variant of the pair (3.8)–(3.9). Indeed, first replace $F(a+u, b; c; x)$ by a second solution to (3.2) as in (3.3), then make a transformation as in (3.4), next a transformation of the independent variable, and finally a gauge transformation. If $a = 0$ and $c \notin \mathbb{Z}$ in (3.10)–(3.11) then we can take for W the set of all polynomials in $(1-x)^{-1}$ and for \mathcal{D} the set $\{0, -1, -2, \dots\}$. For $u \in \mathcal{D}$, $F(u)(x) = (1-x)^u F(u, b+u; c+u; x)$ is a polynomial of exact degree $-u$ in $(1-x)^{-1}$. Thus the functions $F(u)$ are linearly independent elements

of W and we realize on the span of the $F(u)$'s a Type A representation of the QISM I algebra because of Proposition 5.4.

The shift operator pair (3.19)–(3.20) can be obtained by a specialization of (3.10)–(3.11) (for negative k also apply a gauge transformation). Superficially one would say that equations (3.19)–(3.20) realize a finite-dimensional representation of the QISM I algebra on the span of F_k for $k = -n, \dots, n$. However, $(1 - x)^n F_k(x)$ is a polynomial in x of exact degree n . So the $2n + 1$ functions F_k can never be linearly independent and it is impossible to have an operator C_0 with $C_0 F_k = -kF_k$.

§6. Rank 1 L -operators for the QISM II algebra

The (unitary) U -algebra is the algebra with matrix elements of $U(u)$ as generators and with relations (1.4) and (2.1). Let us pass to a quotient algebra by adding relations stating that $(u - \frac{1}{2})U(u)$ is a polynomial of degree ≤ 3 in u . In other words, we make the ansatz that $U(u)$ is of the form

$$L(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = (u - \frac{1}{2})^2 L_2 + (u - \frac{1}{2})L_1 + L_0 + (u - \frac{1}{2})^{-1}L_{-1}, \tag{6.1}$$

where

$$L_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix}, \quad L_1 = \begin{pmatrix} A_1 & B_2 \\ C_2 & A_1 - 2A_2 \end{pmatrix}, \tag{6.2}$$

$$L_0 = \begin{pmatrix} A_0 & B_0 + \frac{1}{4}B_2 \\ C_0 + \frac{1}{4}C_2 & -A_0 + 2A_1 - 2A_2 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} A_{-1} & 0 \\ 0 & A_{-1} \end{pmatrix}.$$

Substitute (6.1) in relation (1.4). Then we get the algebra with matrix elements of L_i ($i = 2, 1, 0, -1$) as generators and with relations

$$L_2^{(1)}L_{2,1,0,-1}^{(2)} = L_{2,1,0,-1}^{(2)}L_2^{(1)}, \quad L_{-1}^{(1)}L_{2,1,0,-1}^{(2)} = L_{2,1,0,-1}^{(2)}L_{-1}^{(1)}, \tag{6.3}$$

$$[L_1^{(1)}, L_0^{(2)}] = -[P, L_2^{(1)}L_0^{(2)}] - L_2^{(1)}PL_0^{(2)} + L_0^{(2)}PL_2^{(1)}, \tag{6.4}$$

$$[L_0^{(1)}, L_0^{(2)}] = -\{[P, L_1^{(1)}], L_0^{(2)}\} - 2A_{-1}[P, L_2^{(1)}] + [L_0, L_2]^{(2)}. \tag{6.5}$$

Here curved brackets mean anticommutator. Relations (6.3) imply that the entries of the L_2 and L_{-1} matrices are in the center of the algebra. Let us pass once more to a quotient algebra by adding the relations

$$L_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}, \quad A_{-1} = \delta, \tag{6.6}$$

for certain $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. We thus obtain an algebra with generators A_0, A_1, B_0, C_0 and relations

$$[A_1, A_0] = \gamma B_0 - \beta C_0,$$

$$[A_1, B_0] = -2\alpha B_0 + \beta (2A_0 - 2A_1 + \frac{3}{2}\alpha),$$

$$[A_1, C_0] = 2\alpha C_0 + \gamma (-2A_0 + 2A_1 - \frac{3}{2}\alpha),$$

$$[A_0, B_0] = -\{A_1, B_0\} + \beta (2A_0 - \frac{5}{2}A_1 + 2\alpha + 2\delta),$$

$$[A_0, C_0] = \{A_1, C_0\} + \gamma (-2A_0 + \frac{5}{2}A_1 - 2\alpha - 2\delta),$$

$$[B_0, C_0] = -2\{A_0, A_1\} + 4(A_1 - \alpha)^2 + 4\alpha A_0 + 4\alpha\delta. \tag{6.7}$$

The quantum determinant (2.3) now is of the form

$$\Delta(u) = -(\alpha^2 + \beta\gamma)u^4 + Q_2u^2 + Q_0 + \delta^2u^{-2}, \tag{6.8}$$

$$Q_2 = A_1^2 - 2\alpha A_0 - \gamma B_0 - \beta C_0 + \frac{1}{2}\beta\gamma, \tag{6.9}$$

$$Q_0 = -A_0^2 - B_0C_0 + 2\delta A_1 - \frac{1}{4}\gamma B_0 - \frac{1}{4}\beta C_0 - \frac{1}{16}\beta\gamma. \tag{6.10}$$

Here the right hand sides of (6.9) and (6.10) give operators in the center of the algebra. We consider (6.9) and (6.10) as added relations, for a certain choice of $Q_2, Q_0 \in \mathbb{C}$. So a representation of the algebra with relations (6.3)–(6.5) which has the property that all elements in the center of the algebra are represented as scalars, can also be viewed as a representation of the algebra with relations (6.7), (6.9), and (6.10) for a certain choice of $\alpha, \beta, \gamma, \delta, Q_0, Q_2$.

Let us assume that

$$\alpha = \beta = 0, \quad \gamma = 1. \tag{6.11}$$

First we make an observation somewhat analogous to Remark 5.2.

Remark 6.1. Consider the algebra \mathcal{A} with generators $A_0, A_1, B_0, C_0, \delta$ and with two sets of relations: relations (6.7) under assumption (6.11), and relations stating that δ is in the center of the algebra. There is a homomorphism of this algebra into the universal enveloping algebra $\mathcal{U}(\mathfrak{e}(3))$ of the Lie algebra $\mathfrak{e}(3)$. A Lie group corresponding to $\mathfrak{e}(3)$ is the group of motions of 3-dimensional Euclidean space. The Lie algebra $\mathfrak{e}(3)$ is 6-dimensional. It can be described by a basis $\mathcal{P}^\pm, \mathcal{P}^3, \mathcal{J}^\pm, \mathcal{J}^3$, and the commutation relations

$$\begin{aligned} [\mathcal{J}^3, \mathcal{J}^\pm] &= \pm \mathcal{J}^\pm, & [\mathcal{J}^3, \mathcal{P}^\pm] &= [\mathcal{P}^3, \mathcal{J}^\pm] = \pm \mathcal{P}^\pm, \\ [\mathcal{J}^+, \mathcal{P}^+] &= [\mathcal{J}^-, \mathcal{P}^-] = [\mathcal{J}^3, \mathcal{P}^3] = 0, \\ [\mathcal{J}^+, \mathcal{J}^-] &= 2\mathcal{J}^3, & [\mathcal{J}^+, \mathcal{P}^-] &= [\mathcal{P}^+, \mathcal{J}^-] = 2\mathcal{P}^3, \\ [\mathcal{P}^3, \mathcal{P}^\pm] &= [\mathcal{P}^+, \mathcal{P}^-] = 0. \end{aligned}$$

The center of the universal enveloping algebra is generated by two Casimir elements:

$$C = (\mathcal{P}^3)^2 + \mathcal{P}^+\mathcal{P}^-, \quad \tilde{C} = \frac{1}{2}(\mathcal{P}^+\mathcal{J}^- + \mathcal{P}^-\mathcal{J}^+) + \mathcal{P}^3\mathcal{J}^3.$$

It is now straightforward to verify that the relations for the generators of \mathcal{A} are satisfied when we put these generators equal to the following elements of $\mathcal{U}(\mathfrak{e}(3))$.

$$\begin{aligned} A_0 &= \frac{1}{2}(\mathcal{P}^+\mathcal{J}^- - \mathcal{P}^-\mathcal{J}^+), & A_1 &= \mathcal{P}^3, & B_0 &= -\mathcal{P}^+\mathcal{P}^-, \\ C_0 &= -\frac{1}{2}\{\mathcal{J}^+, \mathcal{J}^-\} - (\mathcal{J}^3)^2 - \frac{1}{4}, & \delta &= -\tilde{C}\mathcal{J}^3. \end{aligned}$$

This yields the announced algebra homomorphism of \mathcal{A} into $\mathcal{U}(\mathfrak{e}(3))$. We do not yet know if this homomorphism is injective.

The following lemma can be proved in a straightforward way. It shows that equations (6.7), (6.9) and (6.10), with (6.11), and under the assumption that B_0 is injective, can be equivalently written in a much more simple form.

Lemma 6.2. *Let δ, Q_0, Q_2 be scalars. Let A_0, A_1, B_0, C_0 be operators acting on some linear space. Let B_0 be injective. Then the following three statements are equivalent:*

(a) $\left(\begin{matrix} (u-\frac{1}{2})A_1 + A_0 + \delta(u-\frac{1}{2})^{-1} & B_0 \\ u^2 + C_0 & (u+\frac{3}{2})A_1 - A_0 + \delta(u-\frac{1}{2})^{-1} \end{matrix} \right)$ is a representation of the QISM II algebra with quantum determinant $\Delta(u) = Q_2u^2 + Q_0 + \delta^2u^{-2}$.

(b) The six commutators (6.7) and formulas (6.9), (6.10) are valid with $\alpha = \beta = 0$ and $\gamma = 1$.

(c) The following three equations are valid:

$$[A_1, A_0] = A_1^2 - Q_2, \tag{6.12}$$

$$B_0 = A_1^2 - Q_2, \tag{6.13}$$

$$B_0 C_0 = 2\delta A_1 - A_0^2 - \frac{1}{4} B_0 - Q_0. \tag{6.14}$$

Moreover, if $\{A_0, A_1, B_0, C_0, \delta, Q_0, Q_2\}$ satisfy these equivalent conditions, then so do $\{\lambda A_0, \lambda A_1, \lambda^2 B_0, C_0, \lambda \delta, \lambda^2 Q_0, \lambda^2 Q_2\}$ (where λ is a nonzero scalar).

We want to find a realisation of our QISM II algebra as in Theorem 2.2. From this point of view the transformations of A_0, A_1 , etc. as given in the last statement of Lemma 6.2 do not mean any essential change. Thus, without lack of generality we may restrict our attention to two special choices for the Q_2 : one with $Q_2 \neq 0$ and one with $Q_2 = 0$.

We now make the restrictive assumption that A_0 is a first order differential operator and A_1 is a scalar function of x :

$$A_0 = A_{00}(x) + A_{01}(x)\partial_x, \quad A_1 = A_{10}(x). \tag{6.15}$$

The following approach should now be followed. Find all operators of the form (6.15) such that (6.12) is satisfied for some number Q_2 . (It is sufficient to find one solution in each equivalence class formed by gauge transformations and transformations of the x -variable.) Then define B_0 by (6.13) and try to define C_0 by (6.14). Fix some function space W on which these operators act. Then the equivalent conditions of Lemma 6.2 are satisfied. Finally check if the conditions of Theorem 2.2 are satisfied for some choice of \mathcal{D} . Note that the analog of the second method described in §5 cannot be used here, since we were not able to formulate an analog of Proposition 5.4 for the QISM II case.

For the case $\delta = 0$ a classification (up to equivalence) of all operators of the form (6.15) such that (6.12) is satisfied was already given (in the nontrivial cases) by (4.37) and (4.38). Only types A and C'' showed up. We generalize these results for the case of general δ in the short list below. It is immediately verified that equation (6.12) is satisfied for A_0 and A_1 given there. It turns out that B_0 is a function, so (6.14) defines C_0 without problems. We can show that the possibilities for $A(u), B_0$ and $C(u)$ listed below are the only ones up to gauge transformations and transformations of the x -variable, but we do not include the proof here.

Generalized Type (A)

$$A(u) = (u - \frac{1}{2})(x - \frac{1}{2}) + x(1 - x)\partial_x + (\frac{1}{2} - a)x + \frac{1}{2}c - \frac{1}{2} + \frac{\delta}{u - \frac{1}{2}}, \tag{6.16}$$

$$B_0 = [A_1, A_0] = -x(1 - x),$$

$$C(u) = x(1 - x)\partial_x^2 + [c - (2a + 1)x]\partial_x - a^2 + u^2;$$

Generalized Type (C'')

$$\begin{aligned}
 A(u) &= (u - \frac{1}{2})x^{-1} + \partial_x + \frac{1}{2}x^{-1} + \frac{\delta}{u - \frac{1}{2}}, \\
 B_0 &= [A_1, A_0] = x^{-2}, \\
 C(u) &= -x^2\partial_x^2 - x\partial_x - x^2 + 2\delta x + u^2.
 \end{aligned}
 \tag{6.17}$$

For these two L -operators we can give functions $F(u)$ on which $A(u)$ and $-A(-u)$ act as shifting operators. See equations (3.12)–(3.13), (3.29)–(3.30), respectively. Below we give a space W and suitable subsets \mathcal{D} of \mathbb{C} such that the equation $C(u)w = 0$ has one-dimensional solution in W for each $u \in \mathcal{D}$, so such that the conditions of Theorem 2.2 are satisfied.

Generalized Type A. See (6.16), (3.1), (3.2), (3.12)–(3.13). Assume that $c \neq 0, -1, -2, \dots$. Let W be the set of all analytic functions on the open unit disk in \mathbb{C} . Then $A(u), -A(-u), B_0, C(u)$ act on W . Then the conditions of Theorem 2.2 are satisfied with $F(u)(x) = {}_2F_1(a+u, a-u; c; x)$ and we can take $\mathcal{D} = u_0 + \mathbb{Z}$ if $u_0 \notin (\pm a + \mathbb{Z}) \cup (\pm(c-a) + \mathbb{Z})$, or we can take $\mathcal{D} = \{-a, -a-1, \dots\}$ if $-2a, -2a+c \neq 0, 1, 2, \dots$, or we can take $\mathcal{D} = \{a, a+1, \dots, -a\}$ if $a = 0, -1, -2, \dots$. In the second and third case $F(u)$ becomes a Jacobi polynomial with fixed parameters, while the shift only affects the degree.

Generalized Type C'' . See (6.17), (3.22), (3.29)–(3.30). Let W consist of all analytic functions f on $(0, \infty)$ such that $f(x)$ and all its derivatives $f^{(p)}(x)$ tend to 0 faster than any inverse power of x as $x \rightarrow \infty$. Then $A(u), -A(-u), B_0, C(u)$ (see (6.17)) act on W . Then the conditions of Theorem 2.2 are satisfied with $F(u)(x) = x^u e^{-x/2} \Psi(2\delta + \frac{1}{2} + u, 2u + 1; x)$ and we can take $\mathcal{D} = u_0 + \mathbb{Z}$ if $u_0 \notin (\pm(2\delta + \frac{1}{2}) + \mathbb{Z})$, or we can take $\mathcal{D} = \{-2\delta - \frac{1}{2}, -2\delta - \frac{1}{2} - 1, \dots\}$ if $-4\delta - 1 \neq 0, 1, 2, \dots$, or we can take $\mathcal{D} = \{2\delta + \frac{1}{2}, 2\delta + \frac{1}{2} + 1, \dots, -2\delta - \frac{1}{2}\}$ if $2\delta + \frac{1}{2} = 0, -1, -2, \dots$. In the second and third case $F(u)$ becomes a Laguerre polynomial in x^{-1} multiplied by an exponential and a power, while the shift affects both the degree and the parameter.

Remark 6.3. Write the operator in the left hand side of (3.15) as $A(u) = (u - \frac{1}{2})A_1 + A_0$. Then equation (6.12) is satisfied. In fact, this operator is equivalent to the Type (A) case given in (4.37). However, the Legendre functions on which the shift operator pair in (3.15)–(3.16) acts, cannot be obtained generally from the hypergeometric functions in the $\delta = 0$ case of (3.12)–(3.13) by just making a gauge transformation and a change of x -variable. We have to pass also to another solution of the corresponding second order differential equation. The choice of an appropriate space W is not so clear now. But in the case of a finite dimensional representation we can pass from (3.12)–(3.13) to (3.15)–(3.16) without passing to another solution of the differential equation.

For the case $\delta = 0$ of (3.29)–(3.30) the operators coincide with the operators in (3.42)–(3.43). However, the functions in (3.29)–(3.30) do not specialize for $\delta = 0$ to the Bessel functions in (3.42)–(3.43) but to other solutions of the corresponding second order differential equation. For the functions in (3.42)–(3.43) there may be a problem of a good choice of W .

Note that, in the cases of a finite dimensional representation of the QISM II algebra we met above, the functions $F(u)$ and $F(-u)$ ($u \in \mathcal{D}$) are proportional and certainly not

linearly independent. This is compatible with the fact that $F(u)$ must be an eigenfunction of C_0 with eigenvalue $-u^2$.

Remark 6.4. Let us give more comments on the homomorphism of the quadratic algebra \mathcal{A} coming from the QISM II algebra into $\mathcal{U}(\mathfrak{e}(3))$ (see Remark 6.1). Miller [8] defines the following two operators in the universal enveloping algebra of the algebra $\mathfrak{e}(3)$:

$$X(u, +) = \mathcal{P}^- \mathcal{J}^+ + \mathcal{P}^3 \mathcal{J}^3 + (u + 1) \mathcal{P}^3 - \frac{\tilde{C}}{u + 1} \mathcal{J}^3 - \tilde{C},$$

$$X(u, -) = -\mathcal{P}^- \mathcal{J}^+ - \mathcal{P}^3 \mathcal{J}^3 + u \mathcal{P}^3 - \frac{\tilde{C}}{u} \mathcal{J}^3 + \tilde{C},$$

Then he gets the following actions for these operators on the basis vectors $f_m^{(u)}$ of the representation space for the algebra $\mathfrak{e}(3)$:

$$X(u, +) f_m^{(u)} = \frac{\omega(u - q + 1)}{u + 1} f_m^{(u+1)},$$

$$X(u, -) f_m^{(u)} = \frac{\omega(u + m)(u - m)(u + q)}{u} f_m^{(u-1)},$$

where the basis functions $f_m^{(u)}$ are fixed by the diagonal action of the following four mutually commuting operators:

$$\mathcal{J}^3 f_m^{(u)} = m f_m^{(u)}, \quad \left(\frac{1}{2} \{ \mathcal{J}^+, \mathcal{J}^- \} + (\mathcal{J}^3)^2 \right) f_m^{(u)} = u(u + 1) f_m^{(u)},$$

$$C f_m^{(u)} = \omega^2 f_m^{(u)}, \quad \tilde{C} f_m^{(u)} = \omega q f_m^{(u)}.$$

Notice that the operators $X(u, \pm)$ shift the parameter u of the basis (while the \mathcal{J}^\pm shift m and constitute, together with \mathcal{J}^3 , the sub-algebra $\mathfrak{sl}(2)$). The operators $X(u - \frac{1}{2}, \pm)$ coincide with our operators $\mp A(\mp u) = (u \pm \frac{1}{2}) A_1 \mp A_0 + \delta(u \pm \frac{1}{2})^{-1}$ and they give some shift operator actions for Gauss and confluent hypergeometric functions (see generalized Types A and C'' (6.16)–(6.17), which are Types E and F , respectively, in [8]). It would be natural to try to find any analogous homomorphism of the algebra \mathcal{A} with general commutation relations (6.7) (when α and $\beta \neq 0$) into the universal enveloping algebra of a Lie algebra. We found that it is possible to do so with $\mathcal{U}(\mathfrak{o}(4))$ in the case $\beta \neq 0$, $\alpha = 0$. Then the case $\beta = 0$ corresponds to the contraction of $\mathfrak{o}(4)$ to $\mathfrak{e}(3)$. The case of $\alpha \neq 0$ is still unsolved.

§7. Concluding remarks

In this section we would like to give some comments on the possible applications of the above results in the theory of finite-dimensional quantum integrable systems. By use of the comultiplication operation we get the monodromy matrix for the quantum integrable chain as a product of L -operators each being associated with a particular site of a chain. The main question is to study the spectral problem for the complete set of commuting integrals of motion. In such a way the special functions appear as common eigenfunctions of those commuting operators.

In the present paper we have constructed a lot of new L -operators for both T - and U -algebras connecting each particular L -operator with a particular recurrence relation for the corresponding special function. In this approach we have got an interpretation for the spectral parameter u appearing as an argument of operator-valued entries of the

matrix $L(u)$. The meaning of the spectral parameter u is that it is a parameter (like a , b , or c in ${}_2F_1(a, b, c; x)$) of the special function $F(u)$ defined by the following rule:

$$C(u)F(u) = 0. \quad (7.1)$$

This equation looks like one appearing in the so-called algebraic Bethe ansatz (ABA) technique [5, 12, 13]. In this analogy, $F(u)$ is a pseudovacuum state. But the crucial difference now is that our "pseudovacuum" might depend on the spectral parameter, so it can exist in the situations where the standard ABA does not work. The method of variable separation or functional Bethe ansatz (FBA) [6, 7, 10, 12] was developed to overcome the obstacles in application of ABA. This method has dealt with operator zeros of the equation

$$C(u) = 0. \quad (7.2)$$

For $F(u)$ defined by (7.1) we have been able to find the operators acting in the spectral parameter space (see Theorems 2.1 and 2.2). These operators act as shifting operators to the function $F(u)$ considered now as a function of spectral parameter. We have work in progress on further applications of this idea. In particular, we are preparing a paper with a further generalisation of this technique for q -special functions.

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Поступило 23 декабря 1993 г.