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SYMPLECTIC TOPOLOGY OF INTEGRABLE DYNAMICAL SYSTEMS. ROUGH TOPOLOGICAL CLASSIFICATION OF CLASSICAL CASES OF INTEGRABILITY IN THE DYNAMICS OF A HEAVY RIGID BODY

1. Formulation of the problem

1.1. Example: classical hamiltonian equations of the motion of a rigid body. The theory of this motion takes it origin from the classical works of Lagrange and Euler. The modern theory of motion of spacecrafts is also based to a considerable extent on the Euler-Poisson dynamic equations which will be described below. Let us "freeze" a rigid body into the Cartesian system of coordinates moving together with the body. We shall denote by x, y, z the coordinates of an arbitrary point in the surrounding space relative to the *moving coordinate system* with the origin denoted by O. We can assume that O is an isolated point in the rigid body at which the origin of the moving coordinate system is located. It is convenient to assume that the x-, y-, and z-axes of the frozen coordinate system are directed along the *principal axes* of the inertia of the rigid body (relative to the point O). We shall denote by A, B, and C the *principal moments of inertia* of the rigid body about the point O. Then *inertia tensor of the rigid body* (relative to the frozen coordinate system chosen in this way) can be written in the form of the diagonal matrix $I = \operatorname{diag}(A, B, C)$. We now denote by P the center of mass of the rigid body and by $\rho = (r_1, r_2, r_3)$ the vector connecting point O with the center of mass P (Fig. 1). Let Ω be the *instantaneous angular velocity of the rigid body*. In the frozen coordinate system, this vector can be written in the form $\Omega = (p, q, r)$. Its coordinates p, q, r are obviously functions of time t.

Let us consider simultaneously another Euclidean system of coordinates which is fixed in the ambient 3-space. We isolate a vertical direction in R^3 and direct a unit vector γ along it (say, "upwards" as in the Fig. 1). Decomposing the vector γ along the x-, y-, and z-axes, we obtain $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, where γ_i are the coordinates of the vector γ, i.e.,

its projections along the axes of the moving coordinate system.

Let us now suppose that an external force field responsible for the motion of the rigid body has an *axial symmetry* with the symmetry axis passing through the center of mass P of the body. Under this assumption, the potential function U depends only on ρ and γ, i.e., $U = U(\rho, \gamma)$. We denote by M the *vector of the kinetic moment* of the body about point O (Fig. 1). Since the frozen coordinate axes are directed along the principal axes of inertia of the body, the vector M can be written in the following simple form: $M = I\Omega$, i.e., has the following components: $M = (Ap, Bq, Cr)$.

The well-known *Euler-Poisson equations* can now be written in the form:

$$\dot{M} = M \times \Omega + \frac{\partial U}{\partial \gamma} \times \gamma,$$

$$\dot{\gamma} = \gamma \times \Omega.$$

Here we denote by x the ordinary vector product of vectors in the three-dimensional Euclidean space. The unknown functions in this problem are the following functions of time: p, q, r and $\gamma_1, \gamma_2, \gamma_3$. We assume that the potential U is known as well as the numerical parameters A, B, C. Thus, the Euler-Poisson equations specify a dynamical system (viz. vector field) on the six- dimensional Euclidean space $R^6(p, q, r, \gamma_1, \gamma_2, \gamma_3)$. The search for a solution of this system is an extremely complicated analytical and geometrical problem.

The first equation (in M) is called the *Euler equation*, while the second equation (in $\dot{\gamma}$) is the *Poisson equation.*

It is expedient to write the dynamic equations for the rigid body in the following coordinate representation:

$$A\dot{p} = (B - C)qr + \gamma_3 \frac{\partial U}{\partial \gamma_2} - \gamma_2 \frac{\partial U}{\partial \gamma_3},$$

$$B\dot{q} = (C - A)pr + \gamma_1 \frac{\partial U}{\partial \gamma_3} - \gamma_3 \frac{\partial U}{\partial \gamma_1},$$

$$C\dot{r} = (A - B)pq + \gamma_2 \frac{\partial U}{\partial \gamma_1} - \gamma_2 \frac{\partial U}{\partial \gamma_2},$$

$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3,$$

$$\dot{\gamma}_2 = p\gamma_3 - r\gamma_1,$$

$$\dot{\gamma}_3 = q\gamma_1 - p\gamma_2.$$

This system is not a Hamiltonian system of equations in the entire six-dimensional space, but it becomes a Hamiltonian system if we confine (restrict) it to some four-dimensional surfaces. The latter are closely connected with the integrals existing in the problem. We shall now consider integrals of motion which form the subject matter of the present section.

It was found long ago that the Euler-Poisson equations have the following three classical integrals:

(1) *energy integral*

$$f_1 = H = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2),$$

(2) *area integral* (which is sometimes called the *angular momentum integral*)

$$f_2 = (I\Omega, \gamma) = Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3,$$

(3) *geometrical integral*

$$f_3 = (\gamma, \gamma) = \gamma_1^2 + \gamma_2^2 + \gamma_3^2.$$

Here (,) denotes the Euclidean scalar product in R^3. It should be recalled once again that all these functions are written in the coordinates of the frozen coordinate system (fixed rigidly to the body, and hence rotating in space together with it). It can be easily verified that all the three integrals are functionally independent (as smooth functions), i.e., their gradients are linearly independent almost at all points of the six-dimensional space R^6.

Let us consider two of the above three integrals, namely, f_2 and f_3. The Euler-Poisson dynamical system (vector field) preserves these integrals. Consequently, the common 4-level surfaces specified by the equations $f_2 = \text{const}$ and $f_3 = \text{const}$ are invariant relative to this vector field. Let us consider a nonsingular (viz., regular) 4-dimensional level surface which will be denoted by M^4. It turns out that M^4 is a symplectic manifold, and the Euler-Poisson dynamical system restricted on M^4 is a Hamiltonian system with the Hamiltonian function $H = f_1$(see, for examples, the papers of S. P. Novikov, V. V. Kozlov, and M. P. Kharlamov). The set of such 4-surfaces M^4 is parameterized by two constants: $f_2 = \alpha$ and $f_3 = \beta$(Fig. 2). Therefore, we shall henceforth assume everywhere that the Euler–Poisson system of dynamic equations has already been reduced to the four-dimensional symplectic manifold M^4. In this case, a function H restricted on M^4 is not constant since all the integrals mentioned above are functionally independent. As before, we shall denote by H the

restriction of H on M^4 to avoid the introduction of new notation and the complication of analysis.

In the present paper we shall consider physical and mechanical systems with two degrees of freedom, i.e., systems specified on four-dimensional symplectic phase manifolds M^4. Since the Hamiltonian function H is always an integral, such a system can always be reduced from a four-dimensional manifold to a three-dimensional surface Q^3_h defined by the equation $H = h(= \text{const})$. We shall consider only the surfaces Q^3_h on which the gradient of the function H differs from zero (at all points), and hence Q^3_h is a smooth 3-manifold. For the sake of simplicity, we can assume that it is compact. This does not restrict in any way the range of applicability of the results obtained below. It should be recalled that Q^3_h is called the *constant-energy* or *isoenergy 3-surface* (corresponding to the given energy level $H = h$).

1.2. Integrability or nonintegrabulity as a manifestation of symmetry or randomness in system evolution. Let us consider the motion of a rigid body described above. The question arises: which bodies (with integrable motion or with nonintegrable one) are encounted more frequently? The intuitive, conceptual meaning of the question is clear: every rigid body (and hence its motion) is defined completely by its shape and initial conditions (at the beginning of the motion). A priori, the shape of the body is arbitrary. We call a body *integrable* if its motion is integrable, say in the Liouville sense, and *nonintegrable* otherwise. It turns out that if the shape of a body is chosen "by chance" or arbitrarily, its motion is "almost certainly" nonintegrable or random. It is clear intuitively that if a rigid body (like an asteroid or a bolid) has no symmetry, its motion looks like a random tumbling in space. On the contrary, the shape of a rocket or spaceship is specially chosen to be symmetric to the highest possible extent to ensure the stability of its flight. This is due to the fact that, roughly speaking, integrability is a manifestation of symmetries in the shape of the body, while nonintegrability is associated with the lack of symmetry. Since the symmetry of a body is a "rare" phenomenon, while the typical case general position is the absence of symmetries, nonintegrability or randomness of the motion of a body is a typical situation.

What is the behavior of a "typical nonintegrable system" on M^4? It was noted above that integrability (in the Liouville sense) indicates, among other things, that every three- dimensional constant-energy 3-surface Q defoliates into Liouville's tori and singular fibers. In particular, each integral trajectory of the general position remains all time on

"its own" two-dimensional torus. On the contrary, if a system is nonintegrable, almost all integral trajectories in the typical case are found to move randomly over the three-dimensional surface Q, each of them filling Q densely everywhere (such movements are sometimes called *ergodic movements*).

Let us now consider a "set of all Hamiltonian systems" on the manifold M, i.e., actually the space of all smooth functions H (since the Hamiltonian field is specified by its Hamiltonian function H). This space is infinite-dimensional. Which systems (integrable or nonintegrable) are encounted more frequently? It is clear intuitively that an integrable systems is a rare event in the myriad of all mathematically conceivable systems. At the same time, integrable cases are encounted frequently in mathematical physics. It can be probably explained by the fact that the real physical Hamiltonians appear in the real world as they become "quite symmetric" for certain values of parameters (which are involved in the formulas for the Hamiltonians).At any rate, classical theoretical mechanics is based to a considerable extent on the ideas about a "certain harmony of the World" propounded in the works by Kepler, Copernicus, and many other outstanding scientists of Middle Ages. The concept of harmony in these works is based on the concept of symmetry. The vast body of experimental data accumulated by scientists over the last few centuries confirms to a certain extent the hypothesis according to which "symmetry reigns the world".

A question arises: is it possible to describe or classify integrable nondegenerate systems and isolate among them a "physical zone" comprising the systems which are encounted in real physics? This problem is very complicated. We shall first consider some important integrable Hamiltonians.

1.3. Examples of physical and mechanical systems integrable in the liouville sense. One of the most vivid examples is associated with the equations of motion of a heavy rigid body described above. We shall take the *potential function* U in the following form:

$$U = -mg(r_1\gamma_1 + r_2\gamma_2 + r_3\gamma_3).$$

It describes the ordinary *gravitational field* (gravity force field). The force of gravity is directed downwards along the vertical axis γ (Fig. 3). It turns out that the heavy rigid body having such a potential admits few important cases of integrability.

1. Euler's integrable case (1750).

If the rigid body is fixed at its center of mass (viz. at point O), i.e., $\rho = 0$(i.e., $r_1 = r_2 = r_3$), such a Hamiltonian H has another (additional) independent integral:

$$f_2 = (I\Omega, I\Omega) = A^2p^2 + B^2q^2 + C^2r^2.$$

In Euler's case, the potential function is zero, and hence the equations of motion are simplified considerably. The Euler equation (i.e., the first of the Euler-Poisson equations) assumes the form

$$\dot{M} = M \times \omega$$

$$\text{or } (I\omega)^{\cdot} = I\omega \times \omega,$$

where ω is the instantaneous angular velocity of the body, and I the diagonal matrix of inertia, $I = \text{diag}\,(A, B, C)$. The latent symmetry of the system is manifested here in the fact that the potential vanishes since the body is fixed at its center of mass and rotates freely about it (although the shape of the body is arbitrary) (Fig. 4).

2. Lagrange's integrable case (1778).

This case is sometimes called the *symmetric top*, or *Lagrange's top*. The symmetry of the system is manifested in the fact that the ellipsoid of inertia is an ellipsoid of revolution (with two equal semiaxes) at point O (i.e., the fixed point about which the body rotates), and the center of gravity of the body lies on the rotational axis (Fig. 5). These conditions are presented analytically in the form

$$A = B \quad \text{and} \quad r_1 = r_2 = 0.$$

A spinning top is a good model of Lagrange's top. According to the Euler equation, the third scalar equation is simplified significally and can be written as $r = 0$. It can be integrated immediately: $r = \text{const}$. Thus, the second integral exists and has the form $f_2 = r$. It has a simple geometrical meaning: the component of the instantaneous angular velocity along the dynamic symmetry axis is preserved. Therefore, the symmetry of the system in this case is manifested clearly.

3. Kovalevskaya integrable case (1889).

Here the integrable Hamiltonian $H$is defined by the following conditions:

$$A = B = 2C \quad \text{and} \quad r_3 = 0.$$

Kovalevskaya obtained the second integral in the form

$$f_2 = (p^2 - q^2 - \nu\gamma_2)^2 + (2pq - \nu\gamma_2)^2,$$

$$\text{where} \quad \nu^2 = \frac{1}{C}(r_1^2 + r_2^2).$$

This case is more complicated as compared to the previous cases since the second integral turned out to be not quadratic, but a *fourth-degree* polynomial (which cannot be apparently replaced by some other quadratic integral). The symmetry of this integrable Hamiltonian is latent and associated with deep-rooted algebraic and geometrical properties of the Euler-Poisson equations.

4. Liouville systems.

A dynamical system is called a *Liouville system* (do not confuse with integrability in the Liouville sense!) if it is possible to choose the coordinates (q_1, q_2) on the compact configuration space M^2, in which the Lagrangian function of the system can be written as

$$L = \frac{1}{2}\lambda_1(q_1) + \lambda_2(q_2)(\dot{q}_1^2 + \dot{q}_2^2) - \frac{V_1(q_1) + V_2(q_2)}{\lambda_1(q_1) + \lambda_2(q_2)},$$

where λ_1, λ_2, V_1 and V_2 are smooth functions. This system turns out to be integrable in the Liouville sense, and the additional integral f_2 has the form

$$f_2 = \frac{1}{2}(\lambda_1 + \lambda_2)(\lambda_1\dot{q}_1^2 + \lambda_2\dot{q}_2^2) + \frac{\lambda_2 V_1 - \lambda_1 V_2}{\lambda_1 + \lambda_2}.$$

Let us list some important Liouville's systems. In view of what has been said earlier, Liouville's systems are integrable in the Liouville sense.

4a. Biharmonic oscillator.

In this case, the Lagrangian has the form

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\alpha x^2 + \beta y^2),$$

where x and y are Cartesian coordinates on the plane, and α, β are constant numbers.

4b. Motion of a point mass in a central force field.

Here the Lagrangian function has the form

$$L = \frac{mr^2}{2}\left(\frac{\dot{r}^2}{r^2} + \dot{\varphi}^2\right) - V(r).$$

Here r and φ are the polar coordinates on the Euclidean plane.

Liouville's systems also cover the following famous problems: Kepler's problem, the problem of motion of a point mass in a plane attracted by two fixed centers (e.g., the motion of a spacecraft in the gravitational field produced by the Earth and the Moon), and so on.

1.4. Classification of all integrable nondegenerate hamiltonian systems (integrable hamiltonians) with two degrees of freedom.

(1) Let us consider two integrable systems (Hamiltonians) of "general position". Are they equivalent (from the topological point of view) or not?

(2) Classify all integrable systems (Hamiltonians) to within topological equivalence. Does a topological invariant "responsible" for this classification exist?

(3) Determine new topological obstacles to integrability.

(4) Define the "complexity" of an integrable system (Hamiltonian).

(5) Describe the "physical zone" in the table of all "mathematically existing" integrable systems (integrable Hamiltonians).

It turns out that in an exact sense, all these problems can be solved. This is one of the results of the theory founded by A. T. Fomenko in [. . . .] and then developed in the series of works by A. T. Fomenko, H. Zieschang, S. V. Matveev, A. V. Brailov, A. V. Bolsinov, V. V. Sharko, V. V. Trofimov and others (see [. . .]).

2. Smooth functions typical on smooth manifolds

2.1. Morse simple functions. Let us consider the space of all smooth functions on a smooth manifold. What is the structure of "typical" general-position functions? What is the difference between these and "exotic" functions? The properties of a function are clearly determined in many respects by the nature of its singularities, i.e., the points at which its differential is zero. Therefore, the question about typical functions can be reduced to the structure of functions with typical singularities.

Let us consider a smooth function $f(x)$ on a smooth manifold X^n and assume that $x_1, \ldots, x_n$ are smooth regular coordinates in the vicinity of a point x. The point x is referred to as *critical* (for the function f) if the differential

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

vanishes at this point. This is equivalent to the condition of vanishing of all partial derivatives of the function (at this point). The critical point is regarded as *nondegenerate* if the second differential

$$d^2 f = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$$

is nondegenerate at this point. This is equivalent to the condition that the matrix of second-order partial derivatives has a nonzero determinant.

According to the *Morse lemma*, we can always choose in the vicinity of every nondegenerate critical point such local coordinates in which the function can be written as a quadratic form (at the same time in the entire open neighborhood of the point):

$$f(x) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

It is important that we are dealing here with an exact equality (there are no additional higher-order terms). The number λ is defined uniquely (for each critical point) and is called its *index*. Fig. 6 shows three possible types of nondegenerate critical points for the function on a two-dimensional surface, viz., the minimum, the maximum, and the saddle point. The level curves of the function in the vicinity of each of these singular points are also indicated in the figure. In appropriate coordinates, the function can be written as follows:

(1) $f = -x^2 - y^2$(maximum, the index is two),
(2) $f = x^2 + y^2$(minimum, the index is zero),
(3) $f = -x^2 + y^2$(saddle point, the index is unity).

Definition 1. *A smooth function is called the Morse function if all its critical points are nondegenerate.*

The following important theorem (M.Morse) holds: *Morse functions are dense everywhere in the space of all smooth functions on a smooth manifold.* In other words, any smooth function can be transformed into a Morse function as a result of even the smallest perturbation. In this case, complex degenerate critical points are scattered to form a union of a certain number of the Morse-type (i.e.,nondegenerate) singularities.

Henceforth, we shall denote by $f^{-1}(r)$ the complete inverse image (preimage) of the value r of the function f. We shall denote by a the regular values of the function, i.e., the values whose inverse image does not have a single critical point. In this case $f^{-1}(a)$ is always a smooth submanifold in X^n(by virtues of the well-known theorem on explicit functions).

We shall denote by c the *critical values* of the function, i.e., such values whose inverse image has at least one critical point.

Further, we can use the slightest perturbation to ensure that every critical level c (i.e., the set of points x for which $f(x) = c$) contains exactly one critical point. In other words, critical points falling on the same level can be moved to closely spaced levels (see Fig. 7). Such Morse functions are sometimes called *simple*.

2.2. Simple atoms and simple molecules. What is the structure of level curves of a simple Morse function defined on a two-dimensional surface X^2? Let us first consider an orientable surface.

If a is a regular value of the function, the relevant level curve consists of a few nonintersecting smooth circles (Fig. 8).

The Case of The Minimum and the Maximum. Let us consider a nonsingular level curve close to the minimum or the maximum of the function. This line is a circle. If the regular value tends to the minimax value, the circle shrinks to a point (Fig. 8). In this case, the two-dimensional disk foliates into concentric circles with a common center (corresponding to the minimax). We shall represent this evolution of level curves and the rearrangement by using the following conditional, but quite visual method. Every component of nonsingular level curve (circle) will be depicted by a point lying at the level a (Fig. 8). Upon a change in a, this point will move and sweep a segment. At the moment when the value of the function becomes critical (equal to c), the circle will shrink to a point. We shall denote this event by the letter A with a segment emerging from it. The segment is directed downwards.

For the minimum, we proceed in exactly the same manner (Fig. 8). In this case, the segment descends from above and terminates (at the lower end) at A.

We shall also assume that A denotes a disk with a point at the center, which foliates into concentric circles.

Saddle Point. If c is the critical saddle-point value, the level curve has a figure-eight shape. When a tends to c, the two circles come closer and merge at a point where the level curve is rearranged. This process is also depicted in Fig. 9. By reversing the direction of motion, we can speak of the inverse process, viz., the decomposition of a circle into two circles. The initial circle is "constricted", and then two circles stick together, after which the figure of eight thus obtained splits into two circles. Proceeding in the same way as in the case of minimax, i.e., presenting every regular circle by a point, and tracing the evolution (during the change in the level), we obtain a graph shown in Fig. 9. This is a "tripod" oriented either upwards, or downwards. We shall denote the corresponding rearrangement by B, assuming at the same time that this letter also describes the pattern depicted in Fig. 10, viz., a plane disk with two holes, which foliates into level curves of the Morse function.

The notation introduced by us is very convenient, for example, for solving the following problem. Let us suppose that a compact closed 2-surface (without a boundary) is defined with a certain simple Morse

function on it. Let all the critical points of this function be known. How can we reconstruct the surface? It turns out to be a simple problem. We must consider all critical points of the function and the rearrangements corresponding to them, draw letters A and B on the relevant levels, and then connect the ends of the *edges* of these graphs. This leads to the certain graph W (Fig. 11). Such graphs are sometimes called *Reeb's graphs*. It is clear that if the surface X^2 is orientable, the graph W defines it unambiguously (to within a diffeomorphism). It should be noted that the graph W is not necessarily plane.

Definition 2. *Two-dimensional surfaces A and B (with a boundary), which are foliated into circles as shown in Figs. 9 and 10, will be referred to as atoms.*

Naturally, the graph W (Fig. 11) will be called a *molecule* (since it consists of atoms). An atom will also be denoted by a letter with a few positive and negative edges emerging from it (upwards and downwards) depending on the number of positive and negative boundary circles of the atom.

We now reject the hypothesis of the orientability of X^2,, i.e., go over to the general case. The minimax rearrangements of the type A have the same construction both in the orientable and nonorientable cases. The difference appears in the case of a saddle point. Let us first recollect how a saddle-point - type rearrangement (with an index equal to unity) actually takes place in the orientable case. It is depicted in Fig. 12. A narrow strip (rectangle) is glued to the pair of boundary circles (representing the boundary of the manifold $f(x) \leqslant c - \epsilon$, where ϵ is a small quantity). The gluing is such that the obtained surface remains orientable. As a result, the boundary is found to be homeomorphic to a circle.

Let us now consider the case when a rearrangement takes place within a nonorientable surface. Some rearrangements (with an index 1, i.e., of the saddle-point type) can occur in this case as for an orientable surface. However, there is always at least one rearrangement which is made according to a completely different principle. This rearrangement is shown in Figs. 13, 14. A twisted (by 180^0) rectangle is glued to the same boundary circle of the oriented surface (with a boundary). As a result, there appears a new Mhuh bius strip within the surface $f \leqslant c+\epsilon$. Clearly, there remains, as before, only one boundary circle after the rearrangement. Thus, as we go over through the critical level c, a circle is transformed again into a circle. Using the symbols introduced by us earlier, i.e., depicting every nonsingular level curve (viz., a circle) by

a point, we must represent the evolution described above as shown in Fig. 14: the edge of the graph with the letter A^* at the middle. This letter denotes conditionally a "nonorientable" rearrangement.

Thus, if f is a simple Morse function on a compact closed surface (which is orientable or nonorientable), we can put it in correspondence with the graph W having vertices of the type A, B, or A^*.

What are the specific features of the rearrangement of circles upon a transition through the critical level in the case of A^*? Figure 15 shows the surface $f^{-1}(c+\epsilon, c-\epsilon)$. It has the form of two Mhuh bius strips glued crosswise.

The "minus" sign marks the circle lying on the level $c-\epsilon$ while the "plus" sign shows the circle on the level $c+\epsilon$. The critical saddle point is at the center.

Let us define a new atom A^* corresponding to the rearrangement. In the orientable case, we simply took the strips $P_c = f^{-1}(c+\epsilon, c-\epsilon)$ for atoms A and B, but now we shall proceed in a different way. We shall take a planar ring for the atom A^* with an asterisk (vertex of the graph) on the axial circle. The multiplicity of the vertex is two since it includes exactly two edges. This atom does not coincide with the strip $f^{-1}(c+\epsilon, c-\epsilon)$.

Fig. 16 shows conditionally a molecule, viz., the graph W of general type including vertices of all the three types: A, A^*, and B.

The atom A has only one boundary circle, and hence only one edge emerges from the vertex A (Fig. 17). Accordingly, two edges emerge from the vertex A^* and three edges from the vertex B.

A molecule can contain cycles and have an intricate structure. Thus, a simple Morse function generates three types of rearrangement. Each atom is denoted by a letter (vertex) with a few edges having free ends emerging from it. Every simple Morse function corresponds to a certain molecule. A molecule is obtained from a certain number of atoms by gluing their free ends. There must be no free end in a molecule.

2.3. Complex Morse functions. The constructions described above can obviously be extended to the functions indicated in the title of this section. The only difference is that the number of atoms increases, and the atoms become more complicated.

Let X^2 be a closed two-dimensional smooth surface (orientable or nonorientable).

Let f be a Morse function of the general type, for which there can be several critical points on a critical level. We know, of course, that we can always "spread" critical points to different levels by applying the

slightest perturbation. However, we shall not do this now, since instead of Morse functions, there will appear in the theory of integrable systems of differential equations similar but more "rigid" functions which preclude such movements. The integral of the Hamiltonian system in the general case cannot be perturbed, since otherwise it stops being an integral. It is just for this reason that the theory of atoms and molecules for the Morse functions on X^2, which will be considered below, has not been discussed (to our knowledge) in the mathematical literature.

2.4. Complex atoms and complex molecules. If c is the minimax value of the function f, the rearrangement of a nonsingular level curve (consisting of several nonintersecting circles) occurs as in the case of a simple function, i.e., consists of a few type A rearrangements (Fig. 18).

Let us first assume that the surface X^2 is *orientable* and f is a Morse function on X^2.

Definition 3. *We shall call a connected component of the 2- surface $f^{-1}(c-\epsilon, c+\epsilon)$ an atom corresponding to the critical value* c *(there can be a few atoms corresponding to* c *if they lie on the same level of the function). We shall denote the atom by P_c^2(or P_c). The surface P_c is always orientable because X^2 is supposed to be orientable. The critical points of the function f lying on P_c^2 are called the vertices of the atom. In the case when the surface X^2 is orientable, these vertices can only have a multiplicity of 0 (the isolated minimax point of the function), or* 4 *(the saddle point is at the center of the "cross"; four edges converge to this vertex).*

The surface P_c^2 has a boundary consisting of a certain number of circles. The circles lying at the level $c-\epsilon$ will be called *negative,* while those lying at the level $c+\epsilon$ will be referred to as *positive.* Their number can be different. A singular level curve, viz., the connected component of the curve $f^{-1}(c)$(which will be denoted by K_c) is a connected closed curve with singularities. The singularities of K_c are exactly the *vertices of the atom.* The graph K_c can be naturally called the *skeleton,* or *spine of the atom* (see example in Fig. 18).

It should be noted that different atoms may have same spine K_c.. Therefore, K_c itself does not define the atom uniquely. The surface P_c obviously contracts to its own spine. A spine is the *deformational retract* of its atom (i.e., the spine remains stationary all the time when the atom contracts to it). This contraction can be carried out along the gradient lines of the function f defined on the atom.

Let us now suppose that the surface X^2 is nonorientable. Then its atoms can be obtained from the atoms defined above by adding to the graphs K_c the star-vertices (asterisks) at some edges, which will be called *vertices of multiplicity two.* The number of such asterisks and their arrangement can be arbitrary (in the case of an arbitrary surface X).

Let us describe some properties of atoms. If we omit the graph K_c from the atom, the latter will disintegrate into a union of a few rings. Near every edge of the graph K_c there is exactly one positive and exactly one negative circle. The vertices of the atom, viz., the singular points (vertices) of the graph K_c, may only have a multiplicity of 0, 2, or 4. The graph K_c cannot be a "pure" circle without vertices since by definition it is a singular level curve of the function f passing through critical points (at least through one). The critical points are just the vertices of the graph. By the way, we could complete the picture by including a simple circle (loop without vertices) into the list of graphs K_c by considering the regular level curve of the function f.

Figure 18 shows an example of an atom which will be denoted by D_1(the meaning of this notation will be explained below). This atom is a plane surface (i.e., can be realized in the form of a domain on a plane). However, not all atoms are plane. An example of a different atom is shown in Fig. 19. We can easily construct a Morse function realizing this atom. Such an atom cannot be embedded into a plane. However, it can be *immersed* in the plane by allowing selfintersections. This immersion is shown in Fig. 19.

Clearly, the number of atoms is infinitely large. On the other hand, they can be easily ordered and classified as their complexity increases. Let us describe a useful and graphical method of depicting atoms. It is known from topology that any two-dimensional orientable surface with a boundary can be immersed in a two-dimensional sphere. Therefore, any atom can be visualized as immersed in a sphere. Naturally, we responding atoms by edges. As a result, we obtain the graph W with atoms as its vertices.

3. Bott functions as 'typical' integrals of integrable systems

3.1. Bott functions.

Definition 4. *A smooth function on a manifold Q^n is called a Bott function if its critical points form nondegenerate critical submanifolds in Q.*

Let us clarify the definition. A critical submanifold L consists (by definition) of critical points of the function f, i.e., $df(x) = 0$ for any point x from L. In particular, the function f is constant along L. At every point x we consider a disk $D^{n-k}(x)$ which is transversal (normal) to the submanifold L^k, where k is the dimension of L. A critical submanifold is considered to be nondegenerate if the second-order differential d^2f is nondegenerate on each normal disk D^{n-k}. In other words, it is required that the restriction of the function f on each normal disk be a Morse function (on the disk) (see Fig. 23).

Clearly, Bott functions are not general position function (in the space of smooth functions). Their number is smaller that that of the Morse functions, but they are extremely important for applications. Let, for example, a Lie group acts on the manifold and the function $f$be invariant to this action. If x is a critical point of the function f, all the points obtained from it as a result of action of the group are clearly also critical. Consequently, the entire orbit of the point x ("growing" from x upon the action of the group) consists of critical points. If the function is of the Morse type in directions transversal to the orbit, $f$is the Bott function on the entire manifold. Thus, "naturally nondegenerate" *functions with symmetries* are Bott functions. This is the reason behind the considerable interest aroused by Bott functions in various branches of geometry and physics. For example, Bott functions appear quite naturally in the theory of integrable systems considered below.

Definition 5. *The index of the critical submanifold of the Bott function $f$is defined as the index of its restriction to an arbitrary normal disk D^{n-k}.*

It should be recalled that the restriction of the function to such a disk is a Morse function, and hence the index is defined for it uniquely. It can be proved that this index does not depend on the choice of point x on a connected critical submanifold L. Consequently, nondegenerate critical submanifolds can be minimax and of saddle-point type relative to a certain index λ(where $0 \leqslant \lambda \leqslant n-k$).

3.2. Integrals which are 'typical' in the hamiltonian physics. The following experimentally established fact (see the review of experimental results in [5, 11]) is very important: *the overwhelming majority of integrals f of real integrable Hamiltonian systems appearing in mathematical physics and mechanics are Bott functions* (on almost all isoenergy 3- surfaces). In this rough sense, we can state that Bott integrals form a class of "typical" functions (of general position) in the integrable

Hamiltonian mechanics.

We will call an integrable system *nondegenerate* if it has a Bott integral.

Naturally, the following question arises immediately: are Bott integrals the integrals of general position in strictly mathematical sense (in the class of all integrals on isoenergy 3-surfaces)? A complete answer to this question has not been obtained so far, and this problem seems to be very interesting. Probably, its formulation requires a refinement.

Let us formulate an interesting result obtained recently by V. V. Kalashnikov (Jr.).

Statement. *Let us consider on isoenergy 3-surface Q^3 the class of all integrable SMOOTH Hamiltonian systems such that:*

1) *bifurcation diagram of momentum mapping is the union of a finite number of a smooth curves,*

2) *all critical submanifolds of the integral f are circles, tori and Klein bottles*

3) *each critical level $f^{-1}(c)$ of the integral f is the union of a rings glued to the finite number of critical circles (i.e., boundary of each ring consists of a critical circles).*

Then the nondegenerate integrable smooth systems (i.e., which admit Bott integrals) on a given isoenergy 3-surface Q are dense in the set of all such integrable smooth systems (with properties 1-3*) on Q.*

Let us clarify the above-mentioned experimental fact which is a result of investigation by many mathematicians including, for example, R. Cushman, H. Knorrer, A. Iacob, M. P. Kharlamov, V. V. Kozlov, A. V. Bolsinov, A. A. Oshemkov, and also E. N. Selivanova, L. S. Polyakova, B. S. Kruglikov, V. V. Kalashnikov (Jr.), T. I. Pogosyan, E. V. Anoshkina, P. Topalov, O. E. Orel, D. V. Shuvaev, A. V. Brailov, A. Kuznetzov and others.

In the present paper we are mainly dealing with systems having two degrees of freedom, viz., Hamiltonian fields on four- dimensional phase spaces. It was noted above that such a systems can be always reduced to a three-dimensional constant-energy surfaces Q which will henceforth be regarded as a smooth compact submanifolds in M^4. For example, we can consider only such 3- surfaces Q on which *grad* $H \neq 0$. Then the second integral f restricted on Q is a smooth function which always has certain critical points.

Lemma 1. *An integral f cannot have any isolated critical points on Q.*

Proof. According to the assumption concerning nonsingularity of Q, we have $grad\ H \neq 0$. It follows hence that the vector field $v = sgrad\ H$ differs from zero at each point on Q. Consequently, if x is a critical point of f, an integral trajectory of the field v on which the function f is constant (since f is an integral) emerges from x. Therefore, if $df(x) = 0$, the differential is equal to zero at all points of the integral trajectory, i.e., the latter consists completely of critical points. The lemma is proved.

Thus, the integral $f$can never be a Morse function on a nonsingular constant-energy 3-surface. To what class does it actually belong? It was found that the situation is typical when its critical submanifolds in Q are *nondegenerate* (see above). This means exactly that $f$is a Bott integral "almost in all cases".

It turns out (see [1, 9]) in this case that any connected critical submanifold of the integral $f$is diffeomorphic to a circle, or torus, or a Klein bottle.

In actual practice, only circles are most significant in this case. As a matter of fact, while considering constant- energy 3-surfaces Q to within two-sheeted coverings, we can always (see [1, 9]) get rid of critical Klein bottles by replacing their critical tori. To be more precise, by going over (when required) to a two-sheeted covering Q over Q, we can always "unfold" the critical Klein bottle into a critical torus. After that, using the ambiguity in the choice of the integral f (if it exists), we can always replace it (locally, in the neighborhood of the critical torus) by another integral f for which the given torus is not critical any longer.

Therefore, we shall henceforth assume, without any loss of generality, that all critical submanifolds of the Bott integral $f$on Q are circles. In other words, we assume that there are neither critical tori nor critical Klein bottles. It is expedient to note that on every critical circle, a direction which is the direction of the field $v = \mathrm{sgrad} H$ is defined uniquely. Therefore, all critical circles are oriented.

The critical circles on a three-dimensional manifold can be only of two types: minimax, and of the saddle-point type. In the latter case, they have an index of unity, the restriction of the integral $f$to a normal two-dimensional disk relative to such a circle has a saddle point of the type $-x^2+y^2$(Fig. 24). In the case of a minimax circle, the restriction of the integral f on a normal 2-disk has either a minimum, or a maximum at the center of the disk.

4. Rough and fine topological equivalence of integrable systems

Let us consider two nonresonance inegrable Hamiltonian systems v_1 and v_2 on constant-energy nonsingular 3-surfaces Q_1 and Q_2 respectively. Any symplectic manifold is orientable. Therefore, every constant-energy submanifold is also orientable. Each integrable system defines Liouville's foliation on "its own" 3-manifold. What systems should be regarded as equivalent?

Definition 6. *(see* [1, 9]*). Two integrable systems are referred to as finely topologically equivalent if there exists a diffeomorphism* $\lambda : Q_1 - - - > Q_2$*, which preserves the orientation of constant-energy manifolds and transforms Liouville foliation of the system* v_1 *into Liouville foliation of the system* v_2 *so that the orientation of isolated critical circles is preserved (Fig.* 25*).*

Since "almost all" the Liouville tori are the closures of their integral trajectories, finely equivalent systems actually have "identical closures" of all their solutions. The solutions themselves (trajectories) can be different, but they have the same closures.

Let us describe *rough equivalence.* We shall consider Liouville foliation of a system v and an arbitrary regular Liouville torus. Cutting a 3-manifold Q along it, we obtain a new manifold whose boundary consists of two tori (two banks of the cut). We reglue these tori by using an arbitrary diffeomorphism. This gives generally a new manifold with Liouville foliation. We shall call this operation *Liouville twisting* (along the Liouville torus) (see Fig. 26). This operation is reversible.

Definition 7. *(see* [1, 9]*). Two integrable systems* v_1 *and* v_2 *are referred to as roughly topologically equivalent if the Liouville foliations corresponding to them can be transformed into each other by Liouville twistings.*

Finely equivalent systems are obviously roughly equivalent as well. The converse is not true. Therefore, the number of rough equivalence classes is "smaller" than the number of fine equivalence classes. In this sense, the rough classification of integrable systems must be simpler than the fine classification.

How can we explain the interest towards the fine and rough classification of integrable systems? In the first place, there is the problem of establishment of equivalence (or nonequivalence) of real physical systems. At the moment, quite a large number of integrable systems is

available. However, all of them are defined analytically, their Hamiltonians being written in specific coordinates determined by the physical aspect of the problem. In their appearance, two systems might look quite different. It cannot be ruled out, however, that this nonsimilarity is only due to an inappropriate choice of the coordinates. Is it possible that Liouville tori of the two systems can be made to coincide by a change in the coordinates? If such a transformation can be found explicitly, the problem of equivalence is naturally solved. In some physical problems, such a transformation of variables could be actually found (in most cases in a nontrivial way).

If, however, no transformation was found, it is hard to state whether the two systems are equivalent or not. For example, it would be interesting to find the number of "really different" cases of integrability among those that have already been observed in the theory of a heavy rigid body. It cannot be ruled out that at some energy levels, the Hamiltonian flow of Lagrange case is the equivalent of the Hamiltonian flow of Kovalevskaya case or Euler case (on same isoenergy surfaces). It turns out that this is true indeed, and the table of such crossed equivalences and the list of nonequivalent Hamiltonians (at different energy levels) will be given below.

Obviously, in order to prove nonequivalence, we require a *topological invariant* which is preserved upon a change of variables. If the values of this invariant are different for the two systems being compared, these systems are nonequivalent. But such an invariant should be found first, which is a nontrivial problem. We shall now go over to the solution of this problem.

5. Theorem of rough and fine classification of integrable Hamiltonian systems with two degrees of freedom. Applications in physics and mechanics

5.1. Formulation of the main theorem. In the following analysis (as before), we shall consider nonresonance Bott-type Hamiltonian systems. We define *a non- degenerate system* (on a constant-energy 3-surface) as one whose Hamiltonian has an additional independent Bott integral (on this isoenergy 3-surface).

Usually, we shall assume that such an integral f exists not only on a given 3-surface Q, but also in its certain open neighborhood, i.e., on all constant-energy 3-surfaces which are close to Q. We could easily consider the case of the isolated 3- surface Q, but we shall not go into these details here.

Fir the sake of simplicity, we shall consider integrable systems on *stable 3-surfaces* Q^3_h. This means that upon a small change h, the Hamiltonian system remains finely topologically equivalent to the initial system. Investigations of specific physical systems reveal (see the list of publications given above) that *practically all constant-energy 3-surfaces are stable.*

The problem is formulated as follows: *Classify such systems to within rough and fine topological equivalence.* In other words, we must classify Liouville's foliations on constant- energy 3-surfaces Q. In "fine case", the list of all pairwise nondiffeomorphic Liouville's foliations should be constructed in Q-type manifolds.

The following two versions of the problem are possible:

(a) classify the systems (foliations) on constant-energy 3- surfaces Q_h, and

(b) classify the systems (foliations) on the 4-manifolds M^4 themselves.

We shall begin with the classification on constant-energy 3-surfaces. It should be recalled that we shall consider systems v having no critical Klein bottles on Q^3_h(this can always be done if we consider q to within two-sheeted coverings, see above).

We shall formulate the general classification theorem in the form of a sequence of individual theorems. While formulating the main result, we shall first use the formal definition of topological invariants in the form of a molecule and a marked molecule. However, we shall then clarify the actual meaning of these invariants (in terms of an integrable system) and demonstrate the actual method of calculating molecules W and W^* for specific physical systems.

Theorem 1. *(Theorem of existence and computability of an invariant. See* [1, 5, 7, 9], *A. T. Fomenko and H. Zieschang).*

Let $v = sgradH$ be an integrable Hamiltonian nonresonant Bott system (Liouville foliation) on a constant-energy nonsingular compact 3-manifold Q^3_h. Then there exists a canonical effective algorithm putting in correspondence to such a system v two objects: a molecule W(v) and a marked (equipped) molecule $W^*(v)$. The structure of the molecule W was described above. The structure of the marked (equipped) molecule W^* and the algorithm of computation of the molecules W and W^* for the given system v will be described below. This algorithm is effective in the sense that the molecules W(v) and $W^*(v)$ can actually (and quite easily) be calculated for many concrete physical and mechanical systems (defined analytically). The molecules W and W^* are determined completely by the nonresonant Hamiltonian H itself (i.e., they do not depend on the form

of an additional independent integral f).

The molecule W is a graph with atoms as its vertices. It is shown schematically in Fig. 27. Let us now define the *marked molecule.* For this purpose, we assign to each edge of the molecule a rational number r_i which can vary in the limits $o \leqslant r_i < 1$, or can be equal to infinity. These rational marks placed on various edges are completely independent from one another. Further, we mark every edge by the number ϵ_i which can assume values $+1$ or -1. Let us now consider the set of all the edges of the molecule with marks ∞(Fig. 27). These edges together with their vertices form a certain subgraph in W. It consists of a certain number of connected components (there are two such components in Fig. 27). We introduce an important notion of *family.*

Family is on the following objects:

a) or maximal connected subgraph consisting of a saddle atoms (i.e., different of A, where A is a minimax atom) connected by "infinite edges", i.e., edges with marks ∞,

b) or isolated saddle atom without "infinite edges".

We shall mark each such connectedness component (which we call a *family*) by an integer number n_j. These integral-valued parameters are completely independent for different families. As a result we obtain the following object:

$$W^* = (W, \{r_i\}, \{\epsilon_i\}, \{n_j\}),$$

where $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant s$; further, $0 \leqslant r_i < 1$ or $r_i = \infty; \epsilon_i = \pm 1$ and n_j are arbitrary integers. Here n is the number of edges of the graph (molecule) W.

Definition 8. *The graph W^* with numerical marks indicated above is called a marked or equipped molecule.*

Definition 9. *Two molecules W_1 and W_2 are called identical (coinciding) if there exists a homeomorphism transforming the graph W_1 (i.e., its atoms and edges) into the graph W_2 (into the corresponding atoms and edges). The homeomorphism establishes the identity of the relevant atoms. Two marked molecules are considered to be identical (coinciding) if there exists a homeomorphism satisfying the above condition and such that it makes the numerical marks of the molecules coincide.*

All the numerical parameters (marks)r_i, ϵ_i, and n_j may vary independently. The number s does nor exceed the number of atoms in the molecule W. Each molecule W therefore generates a countable number of marked molecules W^*.

Below, we shall describe the computational algorithm for a molecule if a concrete physical system is specified. It is usually convenient to choose for this purpose any of its additional integrals f, but the final result does not depend on its choice.

Theorem 2. *(Theorem on rough and fine classification; see* [4, 5, 6]*; A. Fomenko, H. Zieschang, S. Matveev, and A. Bolsinov).*

Let v_1 and v_2 be two integrable systems (Liouville foliations), $v_1 - - > W(v_1)$ and $v_2 - - > W(v_2)$ the molecules corresponding to them, and $v_1 - - > W^*(v_1)$ and $v_2 - - > W^*(v_2)$ the marked molecules corresponding to them. Then

a) The systems (Liouville foliations)v_1 *and* v_2 *are roughly topologically equivalent if and only if their molecules coincide, i.e.,* $W(v_1) = W(v_2)$.

b) The systems (Liouville foliations)v_1 *and* v_2 *are finely topologically equivalent if and only if their marked molecules coincide, i.e.,* $W^*(v_1) = W^*(v_2)$.

Consequently, a molecule and a marked molecule are topological invariants of the integrable systems.

Theorem 3. *(Theorem of realization; see* [6]*).*

Let $\{W\}$ and $\{W^*\}$ be the sets of all possible molecules and all possible marked molecules (i.e., when numerical marks run through all admissible values). Then there exists a one-to-one correspondence between the following sets:

$$\left(\begin{array}{c}\text{all classes of rough equivalence}\\ \text{of integrable systems}\end{array}\right) < - - - > \{W\},$$

$$\left(\begin{array}{c}\text{all classes of fine equivalence}\\ \text{of integrable systems}\end{array}\right) < - - - > \{W^*\}.$$

The sets $\{W\}$ and $\{W^*\}$ are discrete. Consequently, $\{W\}$ gives us a discrete list of all integrable systems to within rough equivalence, while $\{W^*\}$ gives us a discrete list of all integrable systems to within fine equivalence.

In other words, the discrete set $\{W^*\}$ defines the complete list of all pairwise non-diffeomorphic Liouville foliations (on all possible constant-energy compact 3-surface Q). The discreteness of the list of all integrable systems (considered up to fine of rough equivalence) is a remarkable (and not quite obvious beforehand) fact. It turned out that there are *no continuous parameters* in the set of such systems, i.e., the topological type

of an integrable nondegenerate system can change only "jump wise", or discretely. It should be recalled that there are the continuous parameters for foliations of the general type. Let us consider the following well-known example.

We consider an arbitrary orientable foliation F of codimension 1 on a three-dimensional orientable manifold M^3(consequently, the general position layers in this foliation are two-dimensional). It turns out that the foliation F can be put in correspondence with a three-dimensional real-valued cohomology class of the manifold M. Since the foliation is orientable, it can be defined globally by the 1-form τ which does not vanish. This form is defined by the foliation to within multiplication by a function that differs from zero (at all points). The form τ satisfies the integrability condition which is equivalent to the existence of the 1-form η such that $\eta \wedge \tau = d\tau$. It turns out that the 3-form $\eta \wedge d\eta$ is closed, independent of the arbitrariness in the choice of the forms τ and η, and is completely defined by the initial foliation F. The cohomology class of this form is known as the *characteristic class* of a foliation F. Taking the value of this class on the fundamental three-dimensional cohomology class of the given 3-manifold, we obtain a certain number. Thurston constructed a smooth family of two-dimensional foliations on a standard three-dimensional sphere S^3, for which the characteristic class changes continuously with a continuous change of the parameter. It follows hence that the foliations of this family are pairwise non-diffeomorphic (and even pairwise non-concordant and pairwise non-cobordant). Thus, here we have an example of a *continuous family* of pairwise non-diffeomorphic foliations. Naturally, they are not of the Liouville type (in our sense).

Theorem 3 gives us a complete list of all mathematically feasible and existing integrable systems (Liouville foliations). Using the expressive language of the Hamiltonian mechanics, we obtain a discrete list of all mathematically feasible and realizable cases of integrability (for systems with 2 degrees of freedom), i.e., the list of all mathematically feasible and realizable integrable Hamilton functions. Naturally, we are not in a position so far to write explicit analytical formulas for all integrable Hamiltonians, but we describe their topological structure since we give the list of all possible atoms, and hence the algorithm for constructing an arbitrary marked molecule.

Theorem 4. *(Algorithmic enumeration and identification of integrable systems; see* [6]*).*

a) *There exists an effective algorithm for enumerating of all cases of (finely or roughly) equivalent integrable systems.*

b) *There exists an effective algorithm for identifying of topologically equivalent (and non-equivalent) integrable systems.*

c) *There exists an effective algorithmical classification of integrable systems (up to rough and fine equivalence).*

Let $W^*(v)$ be a marked molecule of an arbitrary integrable system. Each of this molecule contains a certain number of vertices (isolated asterisks or crosses). The number of vertices of an atom can be naturally called its *atomic weight.*

We shall denote by m the total number of vertices in all the atoms of a molecule. It is natural to refer to this number as the *molecular weight.* It is equal to the sum of all atomic weights of the molecule.

Clearly, m is a nonnegative integer. The case $m = 0$is possible only for the simplest molecule shown in Fig. 28. Without going into details here, we only note that it describes an integrable geodesic flow of a flat two-dimensional torus (i.e., the torus supplied with a locally Euclidean metric).

Further, we denote by $n$the number of edges of the molecule W^*. Clearly, $n \geqslant 1$. The number n can be naturally called the *valence of the molecule.*

Definition 10. *The complexity of a molecule W^* (or W) is defined by the pair of integers (m, n).*

The complexity of a molecule (of integrable system) can be depicted by a point on a two-dimensional plane (with integer coordinates).

Theorem 5. *(Interpretation of complexity; see* [1, 5, 6]*).*

Let (m,n) be the complexity of an integrable system v (i.e., of its marked molecule). Then: a) *The number m coincides with the number of critical periodic solutions of the system v (i.e., equal to the number of isolated singular layers of the Liouville foliation which are homeomorphic to the circle).*

b) *The number n coincides with the number of open "cylinders" $T^2 \times D^1$(here a "cylinder" is the direct product of a 2-torus by a segment), which can be obtained from the 3-manifold Q^3_h by omitting all the singular layers of the Liouville foliation from it (Fig. 29).*

Thus, the pair of numbers (m, n) supplies much information about the topology of the system (foliation). In particular, m is the total number of minimax and saddle-type critical circles in the foliation.

Let us consider the first quadrant on a plane and an integral-valued *lattice* on it. The lattice divides the plane into cells of side 1 (Fig. 30).

Let us agree to depict any integrable system by a point located in the cell with the coordinates (m, n), where (m, n) is its complexity. Then each cell will contain an infinite (countable) set of classes of fine equivalence of integrable systems.

We shall denote by $\lambda(m, n)$ the number of classes of rough equivalence of integrable systems of complexity (m, n). In other words, $\lambda(m, n)$ is the number of non-marked molecules W of complexity (m, n). We shall denote by $\Lambda(m)$ the upper boundary of a nonzero domain filled with integrable systems (on a quadrant). In other words, $\Lambda(m)$ has the property: $\lambda(m, \Lambda(m)) \neq 0$ and $\lambda(m, n) = 0$ for $n > \Lambda(m)$. Further, let $\lambda_0(m, n)$ be the number of all non-marked molecules of complexity (m, n), which have no asterisks (i.e., are composed of atoms without asterisks).

Theorem 6. *(Distribution of integrable systems according to their complexity; see*[6], *A. Bolsinov, S. Matveev, A. Fomenko).*

a) *The number $\Lambda(m, n)$ is always finite.*

b) *The number $\lambda_0(m, n) = 0$ for all odd m. In other words, an integrable system without asterisks can exist only for an even m (i.e., for an even number of critical periodic solutions of the system).*

c) *The following equality holds: $\Lambda(m) = [3m/2]$, where denotes the integral part (Fig.* 31*).*

d) *The number $\lambda(m, n)$ is always positive for all cells (m, n), for which $n \leqslant \Lambda(m)$. In other words, the function λ differs from zero on the entire domain bounded from above by the straight line $n = [3m/2]$.*

e) *Since the classification of molecules is algorithmised completely, the number of molecules of a low complexity, namely, for $m \leqslant 4$, was calculated with the help of a computer. The result is presented in Fig.* 32.

Thus, the table of complexities of integrable systems has zeros above straight line $n = [3m/2]$ since there is no integrable system in this region, while there is always at least one integrable system (of a given complexity) on the straight line itself and in each cell (m, n) below it. All the molecules lying on vertical straight lines passing through add m must have asterisks. It can be seen from Fig. 32 that with increasing m, the number of molecules having the complexity (m, n) increases rapidly.

This theorem indicates the existence of deep-rooted regularities in the complexity distribution of integrable systems.

Thus, the list (table) of all mathematically feasible integrable systems has the following structure. We must consider (Fig. 33) the lattice (m, n) on a plane and place all rough equivalence classes of integrable systems a given complexity into each cell of the lattice. It is convenient to depict

these systems by points in a cell. The number $\lambda(m, n)$ of points within a cell turns out to be always *finite.* In order to obtain the list of all classes of fine equivalence, we must "raise" from each such point a countable set of points "along the normal to the plane" (Fig. 33), including rational and integral-valued parameters r_i, n_j, and ϵ_i. The three-dimensional lattice (Fig. 33) obtained in this way depicts the set of all integrable systems (with 2 degrees of freedom) up to fine equivalence.

It is clear from what has been said above that fine classification can be obtained from rough classification by including rational parameters, i.e., is of algebraic nature. On the other hand, rough classification describes mostly geometrical and topological properties of the system. In a certain sense, the most massive *topological* part of the classification is concentrated in atoms and molecules (without marks), and the marks only multiply and transform this topology *algebraically.* These remarks are naturally of associative nature and do not claim to a better clarity at this stage.

Theorem 7. *(Calculation of topological invariants for real physical and mechanical systems).*

Rough topological invariants W and almost all fine topological invariants W^* are calculated completely for integrable cases of the following well-known physical systems with two degrees of freedom. These include:

a) *all main cases of integrability known today in the dynamics of a heavy three-dimensional rigid body, namely: Euler's, Lagrange's, Kovalevskaya's, Goryachev-Chaplygin, Sretenskii's, Clebsch's, Steklov's, and Zhukovskii's cases;*

b) *all known cases of integrability in the dynamics of four-dimensional rigid body (i.e., the system of equations on the Lie algebra of the $SO(4)$ group);*

c) *integrable Toda lattices and their modern algebraic analogs;*

d) *integrable geodesic flows of Riemannian metrics on a 2-sphere and on 2-torus (having a quadratic or linear additional integral);*

e) *some important cases of integrability of chemical kinetic equations;*

f) *integrable cases of Hamiltonian systems describing the A-phase and the B-phase in the theory of superfluid Helium-3* (He^3); *and*

g) *Kepler's problem (in celestial mechanics).*

The exact description of the above cases of integrability and the tables of calculated molecules (and marked molecules) can be found in the series of works by A. Bolsinov, A. Oshemkov, G. Okuneva, L. Polyakova, E. Selivanova, B. Kruglikov, T. Nguen, and also V. Matveev, E. Anoshk-

ina, V. Kalashnikov (Jr.), P. Topalov, O. Orel, A. Kuznetsov, and others (see References). Each of the integrability cases listed above normally involves several molecules corresponding to various energy levels. Figure 34 shows some examples of marked molecules.

We have obtained a topological classification of all mathematically feasible integrable systems (Hamiltonians). It is clear, however, that not all of them can be realized in physics and mechanics. Therefore, it would be very interesting to describe (at least approximately, in the informal sense) the "physical zone" in the table of all mathematical systems, i.e., the zone filled with "real physical systems". What is the shape of this physical zone? Does it contain "holes", periodicities, etc.? In other words, which part of the mathematical list is realized in nature?

With this aim in view, the following program was developed and realized partially (work on this program is being continued even today).

At the first stage, a complete (to the highest possible extent) list of the best known cases of integrability in real physics and mechanics was compiled. Obviously, this "physical list" is not complete due to the following factors. First, not all cases of physical integrability have been discovered so far. Second, the division of the known cases of integrability into physical and mathematical is disputable to a considerable extent. It is sometimes not clear whether an integrable Hamiltonian should be referred to "real physics", or it should be regarded as a "mathematical model". Nevertheless, such a physical list was compiled (among other things, it includes integrable Hamiltonians indicated in Theorem 7). It was found that it ultimately embraces a few tens of series of integrable physical systems (for different energy levels).

At the second stage of the program, molecules and marked molecules of the systems included in the "physical list" were calculated. This work has not been completed so far, but a considerable fraction of integrable systems can be assumed to have been already investigated. Theorem 7 is just one of the results obtained in this direction.

At the third stage, all the molecules obtained were arranged in a complete table of all "mathematical cases of integrability". As a result, it has become possible to outline the required physical zone in the complexity table. The cells (m, n) containing physical systems were marked, and the number of such systems was calculated. It should be recalled that we always consider a pair: viz., the systems v on the constant-energy 3-surface Q. An analysis of the obtained experimental pattern revealed some remarkable regularities which form the basis of the following statement.

Experimental description of the shape and nature of the physical zone in the mathematical table.

It was found that:

. a) *The complexities (m,n) of all investigated integrable physical systems fill a quite narrow region (strip) stretching along the straight line described by the equation* $n = m - 1$ *(Fig. 35). None of the investigated systems deviates (as regards its complexity) from this remarkable straight line by more that* 2 *units.*

b) *In all investigated physical cases of integrability, only the following quite simple three-dimensional manifolds are encountered as constant-energy 3-surfaces: sphere* S^3, *torus* T^3, *direct product* $S^1 \times S^2$, *projective space* RP^3, *and connected sums of the 3-manifolds listed above.*

c) *In all investigated physical systems, only the following quite simple atoms are encountered:* A, A^*, B, C_2, *and* D_k *(where* k *is any number starting from* 1*) (Fig. 36).*

d) *In all investigated systems, only the numerical marks* $r_i = 0, 1/2,$ *and* ∞, *as well as the marks* $n_j = 0, 1,$ *and* 2 *are encountered.*

We should recall the definition of the connected sum of the manifolds M_1^n and M_2^n. We remove open disks D^n from the manifolds M_1^n and M_2^n. As a result, we obtain two manifolds with a boundary S^{n-1}. We shall identify these boundaries by using a diffeomorphism and smoothen the "corners" in order to obtain a smooth manifold. Similarly, we can assume that we have connected the two manifolds with the help of the cylinders $S^{n-1} \times D^1$ (Fig. 37). The obtained manifold is called the *connected sum* and is denoted by $M_1 \# M_2$. The connected sums of 3-manifolds listed above are encountered in actual practice in Hamiltonian mechanics.

We considered above the distribution of physical systems among mathematical systems on all possible constant-energy 3- manifolds. It would be interesting to find such a distribution in the case when the manifold Q^3 is fixed. In real physical systems, constant-energy 3-surfaces diffeomorphic to the sphere S^3 are encountered most frequently. This is not surprising. Indeed, the following situation is typical: the Hamiltonian H is defined on the phase 4-space M^4 and attains its minimum (or maximum) value at an isolated point x_0(Fig. 38). In such a case, close constant-energy 3-surfaces are obviously diffeomorphic to a standard 3-dimensional sphere S^3. As the energy increases, the point x_0 is "blown" into a 3-sphere. Thus, the simplest 3- manifold S^3 is the "carrier" of many Hamiltonian systems. For example, the remarkable cases of integrability like Euler's, Lagrange's, or Kovalevskaya's (in the dynamics of

a heavy rigid body), some cases of Toda lattices, etc. can be realized on S^3.

Theorem 8. *(Distribution of the well-known physical systems in the mathematical list of all integrable systems on a 3-sphere. See* [8], *A. Fomenko and T. Nguen).*

a) *If* v *is an integrable Bott Hamiltonian system of a low complexity (m,n) on a constant-energy 3-sphere* S^3, *where* $m \leqslant 4, n \leqslant 4$, *this system is roughly equivalent to one of the following four systems:* V_1, V_2, V_3, *and* V_4 *whose complexities are* $(2,1),(3,2),(4,3)$ *and* $(4,3)$ *respectively (Fig.* 39*). These four systems are defined respectively by the following molecules:*

$$W_1 = A\text{---}A$$

$$W_2 = A\text{---}A^*\text{---}A$$

$$W_3 = \begin{array}{ccccc} A & -- & B & -- & A \\ & & | & & \\ & & A & & \end{array}$$

$$W_4 = A\text{---}A^*\text{---}A^*\text{---}A$$

b) *Three of these systems, namely,* V_1, V_2, *and* V_3, *have already been discovered in physics. They are encountered in the following well-known cases of integrability (on certain constant-energy* 3*-surfaces diffeomorphic to* S^3*):*

System V_1 *corresponds to Euler's, Lagrange's, Kovalevskaya's, Goryachev–Chaplygin, and Clebsch's cases, Toda lattices, and four-dimensional rigid body;*

System V_2 *corresponds to the Goryachev-Chaplygin case,*

System V_3 *to the Kovalevskaya, and Clebsch cases, and a four-dimensional rigid body.*

c) *Consequently, all "mathematical" integrable non- degenerate Hamiltonians H (i.e. with a Bott integral) on the sphere* S^3, *which have a low complexity* (m,n), *namely,* $m \leqslant 3$ *and* $n \leqslant 3$*(and considered up to rough equivalence) are found to be "physical" systems that have already been discovered in the Hamiltonian mechanics. This part of the mathematical table contains no other physical Hamiltonians that have not been discovered so far.*

d) *The complete picture of the distribution of the number of mathematical integrable systems on the sphere* S^3 *at the beginning of the mathematical table is shown in Fig. 40. The number of nonmarked molecules on the 3-sphere with a complexity* $m \leqslant 8$ *is indicated.*

e) *The "physical part" of the mathematical table (for $m \leqslant 8$) is presented in Fig.* 41. *The figures I, II, III, IV and V mark the cells in which physical systems have been discovered. Here,*

cell I corresponds to the Euler, Lagrange, Kovalevskaya, Goryachev-Chaplygin and Clebsch cases, Toda lattices, and a four-dimensional rigid body;

cell II corresponds to the Goryachev-Chaplygin case,

cell III to the Kovalevskaya and Clebsch cases and to a four-dimensional rigid body,

cell IV to the Kovalevskaya and Goryachev-Chaplygin cases and a four-dimensional rigid body,

cell V to the Kovalevskaya case.

The asterisks mark the cells in which no physical systems have been discovered so far.

f) *The following interesting fact is worth nothing: all physical integrable systems on the sphere S^3 known today have the complexity (m,n) which satisfies the equation $n = m - 1$. In other words, all these systems lie exactly on the remarkable straight line, along which the experimental general physical zone described above extends (as along the axis).*

g) *An explicit description of all molecules of mathematical integrable systems on the constant-energy 3-sphere is obtained. Namely, an non-marked molecule W is realized on the 3-sphere if and only if:*

1. *The genus of the molecule is zero, i.e., its carrier, viz., the surface P^2, is a two-dimensional sphere;*

2. *Each atom in the molecule W has not more than one asterisk.*

h) *A marked molecule W^* can be realized on the 3-sphere if and only if it satisfies all the conditions of item (g), and the numerical marks in W^* satisfy certain algebraic conditions (which are described explicitly in* [8]*).*

In the above analysis, we mainly assumed that the constant- energy 3-surface Q is fixed. However, while analyzing physical Hamiltonian systems, we must consider different energy levels corresponding to different constant-energy 3-surfaces. If the energy changes in the vicinity of its regular value, a molecule is naturally preserved (to be more precise, it is replaces by a topologically equivalent molecule in a stable case). A change may occur only at the moment of passage through the critical value of the Hamiltonian H or the bifurcational value (in a non- stable case). However, we are dealing with only stable systems (see above) for which there are no bifurcational values.

By varying the energy monotonically (for example, increasing it), we obtain a discrete family of molecules. Depicting them by points on the complexity lattice (each of whose cells corresponds to a certain complexity (m, n)), and connecting them consecutively by segments, we obtain a polygonal trajectory. Moving along it, we can visualize the evolution of the molecule upon an increase in energy.

Fig. 42 shows polygonal trajectories calculated for various physical systems. Evolution curves for the Kovalevskaya and Goryachev–Chaplygin–Sretenskii cases are presented in the figure. Why do several evolution polygons correspond to the Kovalevskaya case instead of one? As a matter of fact, the initial phase space in the dynamics of a heavy rigid body is a six-dimensional manifold. A reduction to a four-dimensional symplectic manifold is carried out by fixing the values of two classical integrals. Fixing different values, we obtain different manifolds M^4. Consequently, the Kovalevskaya case, for example, is realized on different symplectic 4-manifolds. Each of them corresponds to its own evolution curve. In the Kovalevskaya case, different polygonal lines correspond to different values of so-called area integral, while in the Goryachev-Chaplygin-Sretenskii case they correspond to different values of hyrostatic moment. Thus, a polygonal line indicates the topological behavior of a given system on the entire phase space M^4 (if M^4 is fixed).

The experimental result described above gives rise to many new problems and hypothesis since we have discovered extremely interesting regularities in the distribution of physical systems in the table of all "mathematically feasible" systems.

5.2. Relation between invariants W, W^* it and the topology of an integrable system. Substantial interpretation of atoms and molecules.

The main object, viz., a molecule W and a marked molecule W^*, were introduced above formally. It would be appropriate here to explain their actual meaning from the point of view of an integrable system. The invariant W^* is closely connected with the Liouville foliation, this connection being quite simple. This allows us, among other things, to calculate effectively the topological invariants of Hamiltonian systems.

Let Q_h^3 be a constant-energy 3-manifold with an integrable system v, i.e., supplied with Liouville foliation. For the sake of simplicity, we fix an additional integral f (it is immaterial which integral is fixed since the theory does not depend on the choice of the specific form of f in view of the nonresonant nature of the system).

Let c be the critical value for f, and f_c denote the connected compo-

nent of the complete inverse image (preimage)$f^{-1}(c)$(Fig. 43) containing the singular fiber of the Liouville foliation. The dimension of f_c in this case is either one (when critical circle is minimax), or two (when critical circle is saddle). We shall denote by ΔQ_c^3 the connected component of the total inverse image $f^{-1}(c-\epsilon, c+\epsilon)$, containing f_c(Fig. 43). The 3-dimensional manifold ΔQ_c^3 has a boundary consisting of a few Liouville tori.

It is well known that the Liouville foliation is defined on the manifold ΔQ_c^3.. Its fibers are defined by the equation $f = \text{const}$. This foliation has only one singular fiber (namely, f_c). All the remaining fibers are nonsingular Liouville tori. As the value of the constant varies, these tori are somehow transformed and rearranged at the moment of intersection of the critical level $f = c$.

However, along with this obvious foliation, there exists on the 3-manifold ΔQ_c^3 one more remarkable foliation which is not so easy to see. But it is the foliation that plays the leading role in the theory of topological classification of integrable systems.

The Liouville foliation is two-dimensional, i.e., almost all its fibers (with the exception of isolated minimax circles) are 2-dimensional. The fibers of the other foliation are 1- dimensional (they are simply circles and lie on the fibers of the former foliation). The first projection (the mapping of f) projects ΔQ_c^3 onto the 1-dimensional segment $(c+\epsilon, c-\epsilon)$, the fiber of the projection f being 2-dimensional. The following important theorem forms the basis of the first step in the theory of classification.

Theorem 9. *(Substantial interpretation of atoms in terms of a Hamiltonian system. A. T. Fomenko* [5]*).*

a) *Each 3-manifold ΔQ_c^3 is a Seifert fibration with a circular fiber above a certain 2-dimensional base which we shall denote by P_c^2.Singular fibers of Seifert fibration can only be of the type* (2,1) *(it should be recalled that in the general Seifert fibration singular fibers of any type (p,q) can be encountered) (see Fig. 44).*

b) *As the manifold ΔQ_c^3 is projected on the base P_c^2, the singular fiber f_c is projected onto a certain graph K_c. It turns out that the surface P_c^2 with the graph K_c lying on it coincides with a certain atom. Conversely, any atom can be obtained by the method described above.*

c) *If an atom has no asterisks as vertices, the Seifert foliation corresponding to it is the direct product of the base and a circle, i.e., $Q_c^3 = P_c^2 \times S^1$. If, however, an atom has asterisks as vertices, then ΔQ_c^3 is no longer a direct product and is a nontrivial Seifert fibration. In this case, the asterisk vertices are in one-to-one correspondence with singular*

fibers of a foliation of the type (2,1).

d) *Each atom* (P_c^2, K_c) *exactly describes the pattern of interaction and bifurcation of Liouville tori in the isoenergy 3-manifold* Q_h^3 *during their transition through the critical fiber f=c. Events happening to the tori are completely described by the evolution of the atom corresponding to level curves of the Morse function determining the foliation of the atom into circles and singular curve. In a certain exact sense (see below), these level curves are the "traces" of the Liouville tori, which are cut by the latter on a 2-dimensional atom during their evolution within the constant-energy 3-surface.*

e) *Therefore, the classification of atoms is exactly the classification of all possible types of bifurcations of the Liouville tori in the vicinity of a critical value. Ultimately, this gives us the description of all possible bifurcations of the solutions (their closures) of an integrable system.*

Denoting the projection of the Seifert fibration by p_c,, we can write the statement of the theorem in the form of the following diagram:

$$\begin{array}{ccc} \Delta Q_c^3 & \supset & f_c \\ | & & | \\ P_c^2 & \supset & K_c \end{array}$$

where vertical arrows denote the mapping p_c.. Thus, atoms are the images of 3-manifolds ΔQ_c^3 foliated in Seifert's sense.

The definition of Seifert fibrations and the description of their most important properties can be found in [3, 7]. Here we shall confine ourselves only to the following three graphic images, viz., Seifert fibrations corresponding to three simplest atoms A, B, and A^*.

Realization of atom A^*

Figure 45 shows the solid torus, viz., the direct product of the disk D^2 by a circle. The solid torus here plays the role of the 3-manifold ΔQ_c^3. It is foliated into concentric Liouville tori whose common axis is the circle, viz., the axis of the solid torus. The construction of Seifert foliation is very simple. Its fibers are parallel on Liouville tori, viz., the cycles parallel to the axis of the solid torus. Seifert foliation has no singular fibers and is obviously a direct product. Projection p_c projects the solid torus onto a disk. Clearly, the projection of the axial circle of the solid torus is the center of the disk. We consider atom A. Such a manifold ΔQ_c^3 always emerges in physical systems when we consider a tubular neighborhood of the minimax critical circle (for the integral f).

Realization of atom B

Figure 46 shows one of typical 3-manifolds ΔQ_c^3 which are also frequently encountered in real physical systems. This is the direct product of a disk with two holes and a circle. Figure 46 shows the foliation on this 3-manifold into Liouville tori. One fiber is singular. It is homeomorphic to two tori glued along the parallel. Obviously, the manifold ΔQ_c^3 can be presented in the form of a solid torus from which two thin "parallel" solid tori are removed. The normal section of this manifold by a plane gives a disk with two holes. This manifold ΔQ_c^3 emerges practically in all cases of integrability of a heavy rigid body.

Seifert fibration in this case has a very simple structure: all fibers are parallel to one another and to the axis of the solid torus, viz., the cycle which is the axis of the singular fiber f_c. The projection p_c is also shown in Fig. 46 as the projection on the tubular neighborhood of figure of eight embedded into a plane. Obviously, this picture corresponds to the atom B.

Realization of atom A^*

The third example is not less "physical", although it is more sofisticated. It emerges, for example, in the Kovalevskaya integrable case. To begin with, let us consider Klein bottle immersed in a 3-dimensional Euclidean space as shown in Fig. 47. We advise to reader who is not familiar with such an immersion to stop here for a few minutes to make sure that he understand indeed its construction. For better understanding, you must take a circle and draw a small figure of eight orthogonal to it at an arbitrary point of the circle. Now move the figure of eight along the circle so that its center always slides along the circle and the figure of eight rotates uniformly in a plane orthogonal to the circle. The speed of rotation is chosen so that the figure of eight returns at its initial point having completed an 180^0 turn. The 2-dimensional surface (with self- intersection) swept by the figure of eight during its motion is the immersed Klein bottle.

Let us now consider a normal tubular neighborhood of this surface. We obtain 3-manifold shown in Fig. 47. It can be described as follows. We must remove from the solid torus a "thin" solid torus passing twice along the axis of the large torus.

This 3-manifold ΔQ_c^3 is foliated into Liouville tori and has a single singular fiber, viz., an immersed Klein bottle to which nearest tori are contracted. On the other hand, the manifold ΔQ_c^3 is a Seifert fibration

whose fibers are circles. A nonsingular fiber is shown in Fig. 48 in the form of a circle passing twice along the axis of the large solid torus.

The normal section of the manifold ΔQ_c^3 by a plane (orthogonal to the axis of the solid torus) is a disk with two holes. A nonsingular circular fiber pierces this disk exactly at two points. If, however, the point of intersection of the circle with the normal section tends to the center of the figure of eight, the nonsingular circular fiber tends to the singular fiber. The singular 1-dimensional fiber of Seifert fibration is a circle along which the center of the figure of eight slides (see above). Clearly, close nonsingular fibers pass along it twice. In the limiting transition, the nonsingular circle is wound twice on the singular circle.

What is the structure of the base of this Seifert fibration? It is clear from Fig. 48 that each fiber can be put in correspondence with the point of its meeting with half the normal section lying, for example, on the left of the vertical segment mn. This is "nearly always" a one-to-one correspondence. The one-to-one correspondence is violated only for circles meeting the half at points of the segment mn. In order to restore the complete one-ti-one correspondence, we must identify the segments sm and sn as shown in Fig. 48. As a result, we obtain a ring with a circle having a vertex s lying on its axis. This vertex must be marked by asterisk. Thus, we have obtained the 2-surface P_c^2 which is the base of the Seifert fibration.

We have described in detail the realization of three simplest atoms. The structure of other atoms is more complex. For example, Fig. 49 shows the atom F_1 describing the transformation of the Liouville torus into two tori (and vice versa). The trace cut on the atom by the "lower" Liouville torus (i.e., the one ascending from the domain $f < c$) is depicted here in the form of two dashed circles. Liouville torus is transformed into two tori (with numbers 2 and 3 in the Fig. 49) when crossing the critical level. The traces of these two tori (on the 2-atom) are shown by dotted curves in the Fig. 49. The bifurcation pattern is completely reflected by the evolution of level curves on a 2-torus.

The more complex the atom, the more muddled the bifurcation pattern for Liouville tori. This pattern cannot be simplified (in the general case) by a small perturbation of the system. It was noted above that the integrals of a Hamiltonian system generally cannot be "disturbed". Herein lies the significant difference between the present theory and the Morse theory in which we can always "move" the critical points to different levels. If we could do something of this kind in the theory of Hamiltonian systems, we would not require any complex atoms and could do with the

three simplest atoms A, B, and A^*. However, as was mentioned above, the Hamilton function in the Hamiltonian mechanics (as well as the corresponding symplectic structure) is assumed to be defined and fixed. The expression "a Hamiltonian system is defined" implies just this fact. A change (perturbation) of a Hamilton function involves a transition to another system (and to different physical problem). Therefore, we cannot perturbate the Hamiltonian (and the symplectic structure on the phase space) and have to dwell deeply into a nontrivial classification of an unperturbed Hamiltonians.

It is expedient to introduce here the concept of simple and complex Hamiltonian.

Definition 11. *A Hamiltonian H (or the flow corresponding to it) will be referred to as simple if there is exactly one connected critical manifold on each critical level $f = c$. All the remaining Hamiltonians will be called complex.*

The molecule of a simple Hamiltonian (dynamical system) clearly may include only the simplest atoms A, B, and A^*. If a molecule contains at least one atom differing from them, its means that the Hamiltonian is complex.

In spite of the physical impossibility to perturbate many of physical Hamiltonians (of integrable dynamical systems), it would be interesting to answer the following interesting theoretical question: can a small perturbation convert a complex Hamiltonian into a simple one? In other words, is the set of simple nondegenerate Hamiltonians (i.e., with a Bott integrals) dense everywhere in the set of all nondegenerate Hamiltonians? (It should be recalled that here we assume the Hamiltonians to be of nonresonance type everywhere).

Proposition 1. *(T. Nguen, [12]). Any complex nondegenerate Hamiltonian can always be approximated as closely as desired (even through a smooth homotopy) by a simple nondegenerate Hamiltonian in the class of perturbations "involving" both H and ω (where ω is a symplectic structure).*

In this case, we naturally "spoil" the initial integrable Hamiltonian system sgrad$H$ by replacing it by some other (however close) new integrable Hamiltonian system sgrad H'. This new system generally will have a different physical meaning.

The interpretation of atoms presented above can be refined. Let us consider the Seifert fibration $p_c : \Delta Q_c^3 - - - > P_c^2$.

Proposition 2. *A two-dimensional base P_c^2 can always be embedded into the space of fibration ΔQ_c^3 as a section of Seifert fibration (in general, ambiguously).*

In other words, there always exists a map $t : P_c^2 - - - > \Delta Q_c^3$ such that the composition $p_c t$ is the identical mapping of the base into itself (Fig. 50).

Therefore, we can arrange within ΔQ_c^3 both the atoms P_c^2, and Liouville tori transversal to it. These tori cut on it level curves whose evolution indicates the bifurcations of the tori themselves within the isoenergy 3-surface.

Thus, atoms can be given the following interpretation: an atom is a 2-dimensional surface with a boundary, which is embedded into ΔQ_c^3 transversally to the tori of the Liouville foliation. Considering the intersection of the tori with the surface P_c^2, we obtain a family of curves, viz., foliation on the atom. These curves are actually the level curves of the function (integral) f.

Let us now go over to the interpretation of molecules.

We shall consider all critical levels on the manifold Q_c^3 and all its pieces ΔQ_c^3(Fig. 51). It is quite clear that the entire manifold Q_h^3 can be represented in the form of the union of its pieces, namely

$$Q_h^3 = \Sigma \Delta Q_c^3,$$

where the sum (union) is taken over all critical levels. Let ΔQ_c^3 and $\Delta Q_{c'}^3$ be two adjacent pieces. Then they are glued (within Q_h^3) along some of their boundary tori. Let T^2 be any of them. There are obviously many possible versions of identification (gluing) of the two boundary tori. Consequently, we must indicate the gluing which is realized in the given case (for the given system). If a basis consisting of two cycles is chosen (and fixed) on each of the tori, then all possible diffeomorphisms of one torus of the other are uniquely defined relative to each other (to within isotopy) by the set of integral-valued matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha\delta$-$\beta\gamma$=1. Thus, in order to specify the diffeomorphism gluing the two boundary tori (Fig. 51), it is sufficient to define the integral-valued matrix. However, the basis is chosen on the tori ambiguously, and hence the matrix determining the gluing is defined ambiguously. Nevertheless, it turns out (we shall not dwell into details and refer the reader to more

detailed works contained in the References) that we can always choose unambiguously on each torus "half the basis", namely, one cycle (one circle). Consequently, the arbitrariness in the choice of the gluing matrix is reduced (although it does not disappear completely). Further, it was found that the rational number $r = (\alpha/\beta) \bmod 1$ is actually defined uniquely. It indicates the "extent of twisting" characterizing the gluing of two boundary tori of the adjacent pieces. The number r can vary between 0 and 1 in the case when $\beta \neq 0$. If, however, $\beta = 0$, we put (by definition)$r = \infty$.

Thus, we have generally explained the meaning of the rational parameter r which participates in the formal definition of molecule. According to this definition, the number r should be placed on the cylinder $T^2 \times D^1$ connecting two adjacent pieces, i.e., two adjacent atoms. In the molecular form of notation, this cylinder is depicted by an edge connecting two adjacent atoms. Therefore, each edge of a molecule is assigned a rational mark.

In the definition of molecule, there appear two more numerical parameters, viz., marks ϵ_i and marks n_k.. We shall omit their description in order to simplify the analysis.

Important remark. In spite of the fact that each 3- dimensional piece is foliated (in Seifert sense) over the base P^2_c, the entire 3-manifold Q^3_h need not necessarily foliate over the 2-surface. The obstacle to such foliation is of topological nature and is "measured" just by numerical marks introduced above (primarily, by the rational marks r_i).

Let us sum up the results obtained.

Substantial interpretation of molecules in terms of a Hamiltonian system

a) *A molecule W^* depicts the Liouville foliation on a constant-energy 3-manifold. Numerical marks characterize the extent of "twisting" of the foliation between its adjacent singular fibers.*

b) *Moving along the edges of a molecule, we see in reality the evolution of the Liouville tori within the isoenergy 3-manifold. In this sense, the molecule gives a clear visualization of global topology of the given system (on the entire Q).*

6. Method for computing topological invariants for specific physical integrable Hamiltonians

Let us consider the natural question concerning the computation of this invariant W*.. It may turn out be so complicated that its compu-

tation is not possible in practice. However, the very first calculations revealed that the invariant can be computed easily, although certain efforts are required in each particular case. We shall now describe an informal algorithm which was used to evaluate the invariants of many specific physical integrable Hamiltonians.

Step 1.

Computations can be started right away if the Hamiltonian H and some of its supplementary (additional) integral f are defined in an explicit form (i.e., by simple analytical formulas) in some suitable coordinates. It should be recalled that f is defined ambiguously, but the final result is independent of the arbitrariness in its choice. Thus, let us fix a specific integral f. We write down the image of the *momentum mapping*

$$\mu : M^* - - - > R^2$$

defined by the formula $\mu(x) = (H(x), f(x))$. In other words, we put each point x in the phase space in correspondence with two numbers, viz., the values of the Hamiltonian and the integral at this point. It can be assumed that two Cartesian coordinates are introduced on the Euclidean space R^2 and can be denoted by the same letters H and f (Fig. 52). The image of the manifold M^4 obtained as a result of the mapping μ is a certain closed set (compact if M^4 is compact). The function H is found to be quadratic in many cases.

Step 2.

Definition 12. *Consider the differential $d\mu$ of the momentum mapping. A point x_0 in M^4 is called singular (critical) for the mapping μ if the differential at this point has a rank smaller than two. The image of such a point, i.e., $\mu(x_0)$, is called the critical (singular) value of the momentum mapping. The set of all critical values is denoted by Σ and is called the bifurcation diagram of the momentum mapping.*

This set consists of a certain number of piecewise smooth curves and, perhaps, isolated points (Fig. 52). The boundary of the image of the momentum mapping $\mu(M^4)$ always appears in the bifurcation diagram (all boundary points must be critical values). Fig. 52 shows the specific bifurcation diagram of the integrable Euler Hamiltonian from the rigid body dynamics. Isolated points in the set Σ are encountered quite rarely in actual practice.

The evaluation of Σ is quite formal and the only difficulties that are encountered can be associated just with the complexity of the analytic

formulas defining H and f. A large portion of computations (in this step) can be algorithmized and carried out on a computer. It should be remembered that the sets $\mu(M^4)$ and Σ are defined ambiguously since they obviously depend on the choice of the second integral f.

Step 3.

The equation $H = h(= \text{const})$ (see Fig. 52) defines on the (H, f)-plane a straight line orthogonal to the H-axis. It intersects the set $\mu(M^4)$ along a certain system of segments. Such a segment is denoted by AB in Fig. 52. Its complete preimage $\mu^{-1}(AB)$ obviously coincides with the isoenergy 3-surface Q^3_h. By displacing the straight line $H = h$ to the left and to the right, we obtain a set of all possible isoenergy 3-surfaces of the given integrable system. As this straight line is displaced, it intersects the singularities of the bifurcation diagram. Obviously, Q^3_h and the dynamical system may undergo a topological rearrangement at these moments. For any point y inside $\mu(M^4)$ not lying on the bifurcation diagram, its complete preimage (for the momentum mapping) consists of a certain number of Liouville 2- tori. The diagram Σ splits the image $\mu(M^4)$ into connectivity components which can be called *chambers* for the sake of convenience. If the point y moves continuously within one chamber, the number of Liouville tori in its complete preimage $\mu^{-1}(y)$ remains unchanged. Changes can occur only if the point y moves from one chamber to another, i.e., if it intersects the bifurcation diagram. In this step, we must try to determine the number of Liouville tori "suspended" over each chamber of the image $\mu(M^4)$. This problem usually boils down to finding the number of solutions of a certain algebraic equation. The difficulties encountered in this case are well known and can be overcome by using methods which are operative in almost all the physical cases of integrability investigated by us.

Step 4.

We must now consider the points c at which the vertical straight line $H = h$ intersects the curves of the diagram Σ(Fig. 53). Let γ be a segment of the straight line $H = h$ passing from a regular point a to another regular point b (these points may lie in the same or different chambers). A certain number of Liouville tori "hangs" over the regular point a. Let us denote these tori conditionally by points. As the point $\gamma(t)$ moves along γ towards the point c, these tori are smoothly deformed inside Q (through isotopy). The points representing them describe certain smooth arcs which form the edges of the eventual graph-molecule W^*. The main events occur at the instant when the point $\gamma(t)$ merges with the point

c, i.e., with the critical value of the momentum mapping. At this instant, the Liouville tori undergo some topological transformation, which is described with the help of an atom (or, may be, several atoms). This atom "releases" new Liouville tori on the other side of the point c, which continue their motion as the point $\gamma(t)$ slides towards the point b (see Fig. 54). There may be different numbers of Liouville tori to the right and left of the point c. The main task at this stage of investigations is to find the atoms that emerge at the instant of transformation over the point c. The solution of this problem requires a certain skill since we do not possess at this instant a universal algorithm for recognizing the atoms. Specific arguments have to be invoked for each particular dynamical system.

Step 5.

Thus, we were able to find at the previous stage the atoms "suspended" over all critical values (points) c lying on the segment $H = h$. Joining the atoms by edges in accordance with the evolution of Liouville tori, we obtain the molecule W. It now remains for us to find the numerical marks. Although a complete algorithm has not been found this stage also, complete solution have been obtained for most of the investigated physical systems. The theory for the computation of marks was worked out recently by A.V.Bolsinov and P.Topalov. In order to find the marks r_i, we must know how the critical periodic solutions of the system, which are the "axes" of the adjacent atoms, are "turned" relative to one another. The computation of the marks n_k is a quite delicate procedure. However, this problem has been reduced in recent times to a counting of the homology groups of the 3-manifold Q for most cases. Since the topology of actually encountered Q is quite simple (all their homology groups are known), we directly obtain the marks n_k. Hence we obtain a marked molecule at this latest step. In spite of the noninvariance of the momentum mapping and the bifurcation diagram (relative to the choice of the integral f), the final result, viz., the object W^*, is free from the above arbitrariness. This completes the investigation.

Knowing the invariant for a given Hamilton function, we can successfully answer the questions concerning the other Hamiltonians to which it is topologically equivalent (or, conversely, not equivalent), its complexity, its arrangement (position) in the table of all integrable systems, etc.

7. A BRIFE HISTORICAL COMMENTARY

Since this paper aims at being just a brief review of the topological classification of integrable systems (Hamiltonians), the historical review presented here is quite cursory and lays no claim to being comprehensive (like any historical survey, it is also inevitably subjective).

The first investigations on the topology of integrable systems were carried out by Poincare' and are based on the problems of celestial mechanics. A review of the main stages of this development can be found, for example, in the well-known work "Topology and Mechanics" by S.Smale.

Smale's work, which was published in Invent.Math. in 1970, marked a new stage in the development of the symplectic topology. Smale proposed new methods for studying integral manifolds of Hamiltonian systems and used them, among other things, to the n-body problem. Next, he chalked out the investigation plan which determined for a long time the trend of development in this field. A large number of publications devoted to the topological analysis of Hamiltonian equations began to appear since 1970. In particular, Smale's investigations were continued by S.Katok [18]. The noticeable transition from an analytical approach for integrable Hamiltonians to the geometrical and topological approach is due to several reasons which were described in detail by M.Kharlamov in his book "Topological Analysis of Integrable Problems of the Rigid Body Dynamics" [19]. We shall reproduce some fragments from this book.

"The traditional approach towards the solution of problem in mechanics, which consisted if finding the cases of integrability and indicating the methods of reducing the problem to quadratures, essentially led to a stalemate in the rigid body dynamics. In the first place, this is due to the basic nonintegrability of the general problem (Poincare'-Hughesson theorem about the nonexistence of algebraic integrals, Kozlov's theorem on the nonexistence of an integral that is analytic in small parameter in the vicinity of the Euler case). Secondly, the construction of the solution involves a large number of technical difficulties. The method of algebraic relations is no longer acceptable: a complete investigation of even the fourth- order relations requires an enormous amount of work, while analytical computations for higher degrees (orders) are practically impossible. Considerable promise is offered by the methods of global analysis of mechanical systems employing the differential geometry apparatus, theories of smooth manifolds and smooth mappings, KAM-theory, and the Morse theory. The emergence of these techniques is due primarily to the problems of celestical mechanics. The program of topological

investigations of classical mechanical systems outlined by S.Smale and the ways of its realization in natural systems with a symmetry indicated by him also lent an impetus to the phase topology studies in the rigid body dynamics."([19], *pp.* 3 – –4).

Liouville's bifurcations of tori for the cases of Euler and Lagrange were the first to be investigated. See, for example, the works of A. Iacob [20], Ya. Tatarinov [22], R. Cushman and H.Knorrer [21] etc. Much progress was made in a series of works by M.Kharlamov and T.Pogosyan (see also review in [19]). Kharlamov analyzed the phase topology of many mechanical systems with first integrals that are nonlinear in momenta.

All these investigations led to a rich body of experimental material showing a vast variety of topological transformations of common integral levels and clearly pointing towards the existence of deep-rooted latent regularities which govern these bifurcations. The inadequacy of the earlier methods was also revealed. According to Kharlamov (1987), *"the main difficulty lies in that the additional integrals of the Euler-Poisson equations in the Euler–Zhukovskii, Kovalevskaya, and Chaplygin–Sretenskii solutions are nonlinear in angular velocity components in mobile axes, and Smale's technique is not applicable for studying such manifolds... The regular level surface of the first integral of a completely integrable Hamiltonian system is a union of tori filled with quasi-periodic trajectories. The problem about the structure of critical integral surfaces dividing the regions of independence of integrals is more complicated and more interesting from a practical point of view. These surfaces are the carriers of all singular motions and completely determine the nature of transformations occurring in integral manifolds upon a change in the constants of first integrals... The presence of explicit regularities in many, at times quite nontrivial examples of bifurcations raises hopes about the emergence of a "general Morse theory of integrable systems" in the nearest future..."*([19], *pp.*4 – –5).

The idea about the creation of such a general theory of Morse type for integrable Hamiltonians kept "hanging in the air", and such a theory was indeed constructed. In 1986 (one year before the publication of the above remarks by Kharlamov), A.Fomenko published in Izv.Akad.Nauk SSSR an article entitled "Topology of constant energy surfaces of integrable Hamiltonian systems and obstacles to integrability", in which the author formulated and proved the basic structural theorem that describes the bifurcation mechanism in a nondegenerate case. In this paper, a new topological invariant, viz., the molecule W, which provides a rough topological classification of integrable nondegenerate Hamiltonians, was

introduced.

In 1988, the molecule W acquired numerical marks and was transformed into the marked molecule W^* in a joint paper published by Fomenko and Zieschang [7]. This invariant made it possible to distinguish between integrable Hamiltonians up to fine topological equivalence.

The final theorem about the fine classification of integrable Hamiltonians (including the inverse theorem about the realization of a marked molecule) was proved by Bolsinov, Matveev, and Fomenko in their paper "Topological classification of integrable Hamiltonian systems with two degrees of freedom. A list of low-complexity systems" [6]. This fine classification was found to be discrete, which means that the set of classes of finely nonequivalent integrable systems does not contain continuous parameters (this fact could not be predicted earlier). The theory underwent a very rapid growth from 1986 onwards, and the main stages of its development are partially reflected in [1 – 17, 24, 25].

Among other things, the invariants of many physical integrable Hamiltonians were computed. A.Oshemkov [13] studied the dynamics of a heavy rigid body, while L.Polyakova investigated the Toda lattices. Selivanova [15] considered the geodesic flows on a 2-torus, and Nguen and Polyakova [23], on a 2-sphere. This was followed by a series of works devoted to the fine topological classification of other physical integrable systems by Okuneva, Kruglikov, Matveev (Jr.), Topalov, Kalashnikov (Jr.), Orel and others.

8. Class (H) of isoenergy three-dimensional integrable manifolds. "Five faces" of this class

We shall now describe five classes of compact 3-manifolds whose external forms differ considerably from one another. It is remarkable that all these classes are actually identical and can be treated as "five faces" of some important class of three- dimensional manifolds.

8.1. Class (H) of the isoenergy 3-surfaces. We denote by (M) the class (set) of all smooth connected orientable compact 3-manifolds.

Theorem 10. *(S. V. Matveev and A. T. Fomenko). Any 3-manifold X^3 from the class (M) is an isoenergy 3-surface of a certain Hamiltonian system on 4-dimensional symplectic manifold $M^4 = X^3 \times D^1$, where D^1 is a segment.*

In order to prove this theorem, it is sufficient to establish the symplecticity of the manifold $M^4 = X^3 \times D^1$. If the symplecticity is proved, we

can choose for the Hamiltonian H he natural function $H(x,t)=t$, where x and t are the real coordinates on the direct product of X^3 and D^1. In this case, X^3 is a 3-surface of the level of the function H, or an isoenergy manifold. The symplecticity of the 4-manifolds M^4 follows from the theorem on the existence of immersion of any oriented compact 3-manifold X^3 into a 4-dimensional Euclidean space R^4. Taking the small tubular neighborhood U of the immersed 3-manifold (Fig. 55), we obtain an immersed 4-manifold. Since a canonical symplectic structure of the type $dp \wedge dq$ exists on R^4, the restriction of this structure to the immersed image of the 4-manifold M^4 leads to a symplectic 2-form on M^4.

Definition 13. *We denote by (H) the class of all orientable compact closed 3-manifolds that are isoenergy surfaces of integrable nondegenerate Hamiltonian systems (i.e., systems that are integrable with the help of the Bott integrals). For the sake of brevity, we call such 3-manifold isointegrable 3-manifold (or integrable isoenergy 3-manifolds).*

The class (H) forms a certain subset in the class (M). This at once raises the following question: *does (H) coincide with (M) or not?* This is an interesting question, since if the class (H) turns out to be smaller than the class (M), we immediately obtain new topological obstacles for integrability of Hamiltonian systems. Indeed, let us suppose that we are interested in the integrability of a certain specific Hamiltonian system on a specific isoenergy 3-surface Q_0. Suppose that we have succeeded (somehow) in studying the topology of Q_0 and found that this manifold does not appear in the class (H). But this directly provides an answer to our original question, i.e., we can state that this system is not integrable on the given Q_0. In other words, any Hamiltonian system on any manifold not belonging to the class (H) is nonintegrable (in the class of Bott integrals). It will be shown below that the class (M) is indeed smaller that the class (M).

8.2. Class (Q) of three-dimensional manifolds glued from two types of blocks. Let us consider two quite simple 3-dimensional manifolds A^3 and B^3 with a boundary. These manifolds are described as follows.

Manifold A^3

This manifold is diffeomorphic to the direct product of a 2-disk by a circle, i.e., $A^3 = D^2 \times S^1$(Fig. 56). Its boundary is diffeomorphic to the 2-torus T^2.

Manifold B^3

It is diffeomorphic to the direct product of the disk N^2 with two holes and a circle, i.e., $B^3 = N^2 \times S^1$. Its boundary consists of three tori.

Definition 14. *We denote by (Q) the class of all orientable closed compact 3-manifolds that can be presented in the form*

$$Q^3 = aA^3 + bB^3,$$

where a and b are nonnegative integers, and the plus sign indicates the gluing of manifolds along the diffeomorphisms of boundary tori. In other words, Q^3 is obtained from gluing of a copies of manifold A^3 to b copies of manifold B^3 along certain identities of their boundary tori (so that a manifold without boundary is obtained).

Obviously, the numbers a and b cannot be arbitrary (they are connected through some simple relations).

8.3. Class (W) of waldhausen manifolds (graph-manifolds).

Definition 15. *We denote by (W) the class of all orientable compact closed 3-manifolds W such that*

(a) *W contains a certain finite set of nonintersecting 2-tori, and*

(b) *after the removal of these tori from W, we obtain an open 3-manifold whose each connected component is a locally trivial fibration with a circle as a fiber over a certain 2-dimensional manifold (perhaps with a boundary).*

This class of manifolds was introduced into the 3- dimensional topology by F.Waldhausen. It was called Graphenmannigfaltigkeiten and appeared in the works of Waldhausen and his followers from within the internal problems of 3-dimensional topology without any connection with Hamiltonian mechanics or symplectic geometry. Manifolds of this kind will be called *graph-manifolds* sometimes. Waldhausen classified all such manifolds and Fomenko found that this classification is closely related with the classification of integrable systems. It can be stated that this class (W) had a rebirth.

8.4. The class (S).

Definition 16. *A closed 3-dimensional compact orientable manifold belongs to the class (S) if and only if it can be endowed by a Bott function all whose critical manifolds are smooth circles and all nonsingular 2-dimensional level surfaces consist of an unconnected union of several 2-tori.*

It is important to note that such a function need not be an integral of any Hamiltonian system. Thus, like the classes (Q) and (W), the class (S) is also not connected externally with the Hamiltonian mechanics in any way.

On the other hand, it was mentioned above that the data accumulated so far from the investigations of integrable systems convincingly prove the predominance in physics and mechanics of integrals whose critical manifolds are circles (in other words, integrals of Hamiltonian systems with critical tori and Klein bottles are rarely encountered). Hence it seems natural to consider the class of smooth functions (which need not be integrable) mentioned in the definition 16.

8.5. The class (T) of isointegrable manifolds corresponding to hamiltonians with tame integrals. Although the integral f is a Bott integral at nearly all the energy levels $Q^3 = (H = \text{const})$ in most of the physical systems investigated so far, this integral may not be of the Bott type for some special energy values (that usually fill a set of measure zero) and it may acquire singularities that are more complicated than what is envisaged by the definition of the Bott nature of integrals. One can naturally question the extent to which the classification of isoenergy 3-surfaces of integrable systems is stable relative to the extension of the class of integrals under consideration. The structure of the class of isoenergy 3-surfaces for Hamiltonians H having an additional, though not necessarily Bott integral, must also be studied. Naturally, we are primarily interested in Hamiltonians whose integrals are not of the Bott type, but do not have a "strong pathology", i.e., do not possess a "too complicated structure". By doing so, we certainly expand the class of the Hamiltonians under investigation. The investigations of physical systems have revealed that the class of Hamiltonians H admitting tame integrals (see definition below) is the most natural one.

Definition 17. *A smooth integral f will be called a tame integral (on a given isoenergy 3-manifold Q) if for any critical value c of the function f the corresponding level surface $f^{-1}(c)$ is tame. This means that there exists a homeomorphism of the entire manifold Q on itself, which transforms the set $f^{-1}(c)$ into a polyhedron.*

Thus, although a tame integral may not be of the Bott type, it is still not "so terrible" and its level surfaces are actually polyhedra in Q.

Definition 18. *We denote by (T) the class of all 3-dimensional orientable compact closed manifolds that are isoenergy 3-surfaces of Hamiltonian systems integrable with the help of tame integrals.*

Obviously, any Bott integral is a tame integral. The converse, however, is not true. Hence we have a trivial inclusion: the class $(H)$is contained in the class (T). Thus, by expanding the class of integrable Hamiltonians, we can a priori extend the class of isointegrable manifolds as well. Is it actually the case?

8.6. The class (R) of manifolds glued from round handles. This class is being actively investigated at present in many works devoted to the theory of dynamical systems. However, it was not associated with the Hamiltonians physics until the creation of the theory of fine topological classification for integrable Hamiltonians.

Let us recall the definition of the Morse circular function.

Definition 19. *A Bott function on a smooth manifold M^n is called a Morse circular function if all its critical manifolds are circles.*

Naturally, such systems need not necessarily be the integrals of any Hamiltonian systems of equations. It can be seen from the definition that the functions described above as those characterizing the class (S) are Morse circular functions. The converse, of course, is not true. Morse circular functions emerge in the Morse-Smale flow theory and have been studied in great details, for example, by Thurston, Asimov, Miyoshi and Sharko. It was shown by Miyoshi that the manifold M^n(of arbitrary dimension) admits a Morse circular function if and only if it can be decomposed into a sum of *circular (round) handles.* We shall not do into the definition of round handles since we shall not require them here.

Definition 20. *We denote by (R) the class of all orientable compact closed 3-manifolds admitting a Morse circular function (i.e., decomposing into a sum of round handles).*

Thus, we have described *six* classes of 3-manifolds: (H), (Q), (W), (S), (T), and (R).

In each of these, we isolate a subclass consisting of *irreducible manifolds.* This subclass is denoted by the same letter but is assigned the index "0". Thus, we arrive at the following subclasses: $(H)_0, (Q)_0, (W)_0, (S)_0, (T)_0$, and $(R)_o$.

8.7. Theorem on the coincidence if five classes. The following theorem is the outcome of joint efforts of several authors, including A. V. Brailov, S. V. Matveev, A. T. Fomenko, and H. Zieschang.

Theorem 11.

a) *Five (out of six) classes of 3-manifolds described above actually coincide, i.e.* $(H) = (Q) = (W) = (S) = (T)$.

The class (H) is contained in the class (R), the latter being "strictly larger" than the class (H).

b) *The class (H) is strictly smaller than (i.e., does not exhaust) the class (M) of all 3-manifolds.*

c) *All six classes of irreducible 3-manifolds coincide, i.e.,* $(H)_0 = (Q)_0 = (W)_0 = (S)_0 = (T)_0 = (R)_0$.

d) *If* Q_1 *and* Q_2 *are two manifolds from the class (H), their connected sum* $Q = Q_1 \# Q_2$ *also belongs to the class (H).*

e) *If a manifold Q from the class (H) is reducible, i.e., if it can be presented as the connected sum of some manifolds* Q_1 *and* Q_2, *(i.e.,* $Q = Q_1 \# Q_2$*), both the manifolds* Q_1 *and* Q_2 *also belong to the class (H).*

Among other things, the following corollary is obtained from this important theorem.

Proposition 3. *Not every orientable compact closed 3-manifold can serve as an isoenergy 3-surface of a Hamiltonian system that can be integrated with the help of a Bott integral.*

In other words, not every manifold from the class (M) is an isointegrable 3-surface of a nondegenerate system. According to what has been stated above, this means that new topological obstacles appear in the path of integrability, both in the class of Bott integrals and tame integrals. In the following theorem, we shall give an example of a very effective criterion allowing us to establish the nonintegrability of Hamiltonian systems. It should be recalled that a manifold is called *hyperbolic* if it can be endowed by a complete Riemannian metric of a constant negative sectional curvature.

Theorem 12. *(S. Matveev, A. Fomenko).*

a) *The class (H) does not contain hyperbolic manifolds. Consequently, all hyperbolic 3-manifolds lie outside the class (H) and any Hamiltonian system having a compact closed hyperbolic manifold as an isoenergy 3-surface is nonintegrable (on a given surface) in the class of Bott integrals (and even in the class of tame integrals).*

b) *The number N(d) of closed compact orientable irreducible 3-manifolds (from the class (M)) of complexity d is defined for* $d \leqslant 6$ *by the table shown in Fig. 57 (results of the computer experiment).*

c) *All closed orientable 3-manifolds of complexity below 8 lie in the class (H).*

d) *If al irreducible 3-manifolds are arranged in the ascending order of their complexity (see Fig. 57), the first hyperbolic manifolds appear with a complexity $d = 9$, i.e., as soon as the manifolds not belonging to the class (H) appear.*

The first hyperbolic manifolds appearing with a complexity 9 are also known to possess some other remarkable properties (see [2]).

Thus, Theorem 12 points towards a close connection between the Hamiltonian physics, hyperbolic geometry, and the theory of Matveev's complexity of 3-manifolds.

9. Application of the fine topological classification theory of integrable systems to geodesic flows on a 2-sphere and 2-torus

9.1. Hypothesis on geodesic flows.

Any theory is deemed to be quite acceptable if it can be used to predict some properties of hitherto undiscovered objects. Hence it is natural to try to realize this criterion on the theory of classification of integrable systems.

It turns out that the new theory allows us to predict important geometrical properties of integrable Hamiltonians (Hamiltonian dynamical systems) that have not been discovered so far. Let us consider an example of such a prediction.

We begin this section by discussing a geometrical problem raised by A. Fomenko.

A) Is it true that any Riemannian metric on a 2-torus with an integrable (in Liouville sense) nondegenerate geodesic flow is "similar" to one of the well-known Riemannian metrics with a linear or quadratic integral? Such metrics will be called *linearly-quadratically integrable metrics*, or 1-metrics and 2- metrics.

B) Is it true that any Riemannian metric on a 2-sphere with an integrable nondegenerate geodesic flow is "similar" to one of the known Riemannian metrics with an integral of degree 1,2,3 or 4? Such metrics will be called $1-, 2-, 3-$, or 4-metrics.

This assumption is found to be almost true from the point of view of complexity of a such metrics. This fact was proved by T. Z. Nguen, L. S. Polyakova, and V. V. Kalashnikov (Jr.).

9.2. Integrable geodesic flows on a 2-sphere and a 2-torus. Let us consider the familiar cases of completely integrable (in Liouville sense) geodesic flows on 2-dimensional Riemannian manifolds M^2. Suppose that

local regular coordinates q_1 and q_2 are defined on M^2. In this case, the coordinates q_1, q_2, p_1, *and* p_2 will be the local coordinates on the cotangent fibration T^*M. By *geodesic flows* of the given Riemannian metric $ds^2 = \Sigma g_{ij} dq_i dq_j$ we mean a Lagrangian dynamical system on a tangent fibration TM (to the given 2-manifold) with a Lagrangian function $L = \Sigma g_{ij} \dot{q}_i \dot{q}_j$. Identifying TM with T^*M with the help of a Riemannian metric, we obtain the corresponding Hamiltonian system v on T^*M with a Hamiltonian $H = \Sigma a_{ij} p_i p_j$, where the matrix $A = (a_{ij})$ is defined as $A = \frac{1}{4} G^{-1}$ and $G = (g_{ij})$. The integrals that are velocity polynomials on TM have the integrals that are momenta polynomials on T^*M corresponding to them.

It is well known (V.V.Kozlov) that a geodesic flow of smoothness class C^1 on a compact 2-dimensional orientable manifold M of the genus $g > 1$ cannot have n additional integral that is analytic in momenta, i.e., is continuously differentiable on T^*M, and is independent of the Hamiltonian H.

On the other hand, it is well-known (G. D. Birkhoff, V. N. Kolokoltsev, and then I.K.Babenko and N.N.Nechoroshev) that a 2-sphere and a 2-torus have remarkable Riemannian metrics whose geodesic flows have an additional polynomial integral that is quadratic of linear in momenta. For the sake of brevity, we shall say that the metrics themselves have a linear or quadratic integral. The complete description of all such metrics on 2-torus and 2-sphere was recently obtained by I. K. Babenko and N. N. Nechoroshev who filled the gap in the initial classification of V. V. Kolokoltsev. They discovered a new series of integrable 1-2-metrics missed by Kolokoltsev.

Among other things, the geodesic flows corresponding to such metrics are integrable in Liouville sense. We shall call such metrics as *linearly quadratically integrable metrics or 1- 2-metrics.* The properties of these familiar classical metrics have been studied quite extensively and these metrics emerge on different accounts in various problems of Riemannian geometry and topology. Let us consider an infinite-dimensional space of all possible smooth Riemannian metrics g_{ij} on a 2-sphere and 2- torus.

Definition 21. *We call a Riemannian metric nondegenerately integrable if the geodesic flow corresponding to it is integrable in Liouville sense, and the additional integral is a Bott integral on regular isoenergy 3-surfaces. It should be recalled that according to the definition, all its critical submanifolds are nondegenerate.*

Of course, each linearly-quadratically integrable Riemannian metric

is nondegenerately integrable. Is the converse also true in the case of a 2-torus? The class of nondegenerately integrable metrics may turn out to be a priori wider that the well-investigated class of 1-2-integrable metrics. As a matter of fact, a Bott function (integral) is just a general position "smooth function" and need not be linear or quadratic (in canonical coordinates of the cotangent bundle).

Problem: Is it true that any nondegenerately integrable Riemannian metric on a 2-torus is linearly-quadratically integrable and on a 2-sphere is $1-2-3-4$-*metric?*

In other words, do nondegenerately integrable metrics exist on T^2 (and on S^2) which have independent (of H) polynomial integral of a degree greater that two for 2-torus, and a degree greater than 4 on a 2-sphere? We mean here that these metrics cannot be reduced to the integrals of degree 1 and 2. It should be noted that the well-known Kovalevskaya integral of a corresponding geodesic flow on the 2-sphere has the degree *four* and the Goraychev-Chaplygin integral has a degree 3.

It should be noted that the existence of integrable natural mechanical systems on the 2-torus with higher-degree integrals in momenta was partially investigated [26, 27]. In the work by M. Byalyi [26] the Hamiltonian system $v = \mathrm{sgrad} H$ with the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(x, y)$$

is considered, where V is 2π-periodic function in the variables x and y. The system $v = \mathrm{sgrad} H$ defines the motion of a material particle over a 2-torus in a potential field. The general problem can be formulated as follows: find the potential V for which there exists a polynomial integral F_n of degree n in momenta with 2π-periodic coefficients (non-reducible to the integrals of degree 1 and 2). The components of even and odd degrees of F_n are separately first integrals, and hence we can assume that F_n is the sum of homogeneous polynomials of only even or only odd degrees. The cases of linear and quadratic integrals are of the classical type. It is well known that:

a. The integral F_1 linear in momenta exists if and only if

$$V = f(mx + ny),$$

where m and n are integers and f is $a 2\pi$-periodic function; and $F_1 = mp_2 - np_1$.

b. The quadratic in momenta integral F_2 there exists if and only if

$$V = f_1(m_1 x + n_1 y) + f_2(m_2 x + n_2 y),$$

where m_1 and n_1 are integers and $m_1 m_2 - n_1 n_2 = -1$, while f_1 is $a2\pi$-periodic function. Here

$$F_2 = (x_1 + x_2)p_1^2 + 4p_1p_2 - (x_1 + x_2)p_2^2 + 2(x_1 - x_2)(f_1 + f_2)$$

and $\quad x_1 = m_1/n_1.$

For higher degrees, Byalyi obtained the following results.

Proposition 4. *(Byalyi [26]). The integral F_3 cubic in momenta exists for a Hamiltonian H if and only if Case 1 is realized, i.e., when there exists an integral F_1 linear in momenta, and F_3 can be expressed explicitly in terms of F_1 and H.*

Proposition 5. *(Byalyi [26]). The fourth-degree integral F_4 in momenta exists for a Hamiltonian H if and only if Case 2 is realized, i.e., there exists an integral F_2 quadratic in momenta and F_4 can be expressed in terms of F_2 and H.*

The proof of these two theorems is based on a detailed analysis of the equation $\{F_n, H\} = 0$, where $\{,\}$ is a Poisson bracket. It was noted that for $n > 4$ the calculations become too cumbersome, although they apparently lead to the same results.

V. Kozlov and D. Treschev [27] used another approach to this problem: they assumed that the potential $V(x_1, ..., x_n)$ of the natural system with the Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}p_ip_j + V(x_1, ..., x_n)$$

is a trigonometrical polynomial of the coordinates $x_1, \ldots, x_n$ on the torus T^n, but no additional constraints are imposed on the degrees of additional integrals. The main theorem specifying a necessary and sufficient condition for the potential $V(x_1, ..., x_n)$, which ensures the complete integrability (according to Birkhoff) of the Hamiltonian system with a positive-definite quadratic form $\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}p_ip_j$ leads to the following important corollary. If a Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}p_ip_j + V(x_1, ..., x_n)$$

has n polynomial integrals with independent higher-degree homogeneous forms, there exists n independent involutive polynomial integrals of a degree not exceeding 2.

Therefore, the results obtained in [26, 27], are strong arguments in favor of the fact that there are no integrable natural mechanical systems with higher-order nontrivial integrals in momenta on the torus T^2. However, this question still remains open. In this connection V. Kozlov and A. Fomenko formulated the hypothesis consisting in that any nondegenerate integrable Riemannian metric on a 2-torus (i.e., the one having a Bott integral) "resembles" in a certain exact sense a linearly-quadratically integrable metric. Let us consider this hypothesis.

9.3. Complexity of integrable geodesic flows on a 2-sphere and a 2-torus. The phase space of the geodesic flow of a Riemannian metric on a 2-dimensional surface is diffeomorphic to a tangent bundle of this surface. Since the Hamiltonian H is defined by the Riemannian metric, all nondegenerate constant-energy 3-surfaces Q^3_h are diffeomorphic to the fibration with a fiber S^1 and with the initial 2-surface as the base. Each such circle S^1 is embedded into the relevant tangent plane to the surface and consists of the endpoints of the tangent vectors of unit length. For example, all nondegenerate isoenergy 3-surfaces Q^3_h for the case of a geodesic flow on a 2-sphere are diffeomorphic to the projective space RP^3. In the case of 2-torus, we have $Q^3_h = T^3$(i.e., it is 3-dimensional torus).

E. Selivanova [15], T. Nguen and L. Polyakova [23] obtained a fine topological classification of integrable geodesic flows on the 2-torus and on the 2-sphere, which have additional integrals f linear or quadratic in momenta. This allowed the authors to classify the complexities of these integrable geodesic flows on the 2sphere and the 2-torus (see below).

Theorem 13. *(T. Nguen and L. Polyakova). Integrable geodesic flows (of Riemannian metrics) on a 2-sphere, which have an additional integral f linear or quadratic in momenta, fill the following region $A \cup B$ on the molecular complexity table (Fig. 58):*

a. *The set $A = \{(m,n) = (6,4)$ or $m/2+3 \leqslant n \leqslant m-2, m \geqslant 6$, where $m = 4k+2, n = 2l$, and k,l are arbitrary nonnegative integers$\}$. The set A depicting geodesic integrable flows with a quadratic additional integral on the table, is shown in Fig. 58 by small dark circles (large dark dots).*

b. *The set $B = \{(m,n) = (2,1)$ or $m/2+2 \leqslant n \leqslant m-1, m \geqslant 6$, where $m = 4k+2, n = 2l+1$, and k,l are arbitrary nonnegative integers$\}$. The set B, which represents integrable geodesic flows with a linear additional integral in table, is shown in Fig. 58 by small circles (large light dots).*

Theorem 14. *(E. Selivanova). Integrable geodesic flows on a 2-torus fill the following region C in the molecular complexity table (Fig. 59):*

The set $C = \{(m,n) = (0,1)$ *or* $m/2+2 \leqslant n \leqslant m, m = 4k, n = 2(l+1)$, *where* k *and* l *are arbitrary nonnegative integers*$\}$. *The points of this set are depicted in Fig. 59 by small dark disks.*

It should be noted that the isoenergy 3-surfaces $Q^3 = (H = \text{const})$ is diffeomorphic to the 3-torus T^3 in the case of geodesic flows on the 2-torus and to the $Q^3 = RP^3$ in the case of a 2-sphere. All mathematically feasible nondegenerate integrable Hamiltonian systems on an isoenergy 3-sphere were classified by T. Nguen and A. Fomenko [8]. The complexities of "mathematically existing" integrable systems on the 3-torus T^3 and on the projective space RP^3 were described by T. Nguen [16]. He calculated the shape of the zone in the complexity table which is filled with complexities of such integrable systems. These zones were called the S^3-zone, T^3-zone, and RP^3-zone.

According to Figs. 58 and 59, the zones depicting integrable geodesic flows on the 2-sphere (and the 2-torus respectively) form a "net" (net subset) within the RP^3-zone (resp., T^3-zone).

9.4. Hypothesis: linearly-quadratically integrable metrics "approximate" any nondegenerate integrable riemannian metric on a 2-torus. The following hypothesis was formulated by A.Fomenko and is actually a detailed version of the hypothesis mentioned above.

Hypothesis. Let g_{ij} be an arbitrary nondegenerate integrable Riemannian metric on a 2-torus (or 2-sphere) (i.e., the corresponding geodesic flow is integrable in the Liouville sense). Then the complexity of this metric (of its geodesic flow) coincides with the complexity of a certain linearly- quadratically integrable Riemannian metric for the case of 2-torus (and with complexity of a certain $1-2-3-4$-metric for the case of 2-sphere).

In other words, the hypothesis states that *from the point of view of complexity*, the well-known classical integrable geodesic flows exhaust all nondegenerate integrable geodesic flows on a 2-torus and on a 2-sphere. This means that in a certain sense, the integrability of geodesic flows on such 2- surfaces has a "linear and quadratic nature" (2-torus) or "$1-2-3-4$-nature" (2-sphere) from the point of view of their complexity. It should be born in mind that on a 2-sphere there exist integrable geodesic flows with an additional integral of degree 3 and 4, which are not reduced to 1-2-integrals. These flows are produced by Goryachev-Chaplygin case and Kovalevskaya case (from rigid body dynamics). This theorem was proved by Bolsinov and Fomenko.

If the hypothesis formulated above is correct, the point depicting any

of such integrable metric in the complexity table must lie in the remarkable region discovered by the authors mentioned above. This zone is marked by small light and dark disks in Figs. 758 and 59.

The integrable Riemannian metric on the 2-dimensional manifold is called *orientable* if all the saddle-type critical circles of the corresponding geodesic flow (viz., Hamiltonian system) have orientable separatrix diagrams.

It turns out that the hypothesis formulated above is "almost correct" in the class of orientable metrics on a 2-sphere. Namely, the following statement is valid.

Proposition 6. *(Nguen, Polyakova and Kalashnikov (Jr.)). The number m of critical circles of an arbitrary nondegenerate orientable Riemannian metric on a 2-sphere has the following form: $m = 4k + 2$ for a certain integer k.*

Corollary 1. *(Nguen and Polyakova). Nondegenerately integrable and orientable geodesic flows on a 2-sphere are contained in the following "net subset" within the RP^3-region depicted by light disks, dark disks and light disks with a point. Here, light and dark disks represent geodesic flows with linear and quadratic integrals respectively. Light disks with a point depict unknown (and probably nonexisting altogether) integrable Riemannian metrics.*

Thus, we can see that the "zone of all orientable integrable metrics" practically coincides with the "zone of linearly-quadratically integrable metrics" with the exception of a straight line lying at the lower boundary of the angle in Fig. 58. It would be extremely interesting to complete the analysis of this special straight line at the boundary (and hence obtain the final answer to the question formulated above).

The statement that $m = 4k$(which is similar to the theorem formulated above) is valid for an arbitrary nondegenerate integrable geodesic flow of a Riemannian metric on a 2-torus. Here n is an even number (see above). Therefore, the linearly- quadratically "integrability zone" (representing the classical integrable metrics on a 2-torus in the complexity table) covers at least "half" the region corresponding to all "mathematically possible" integrable geodesic flows (of nondegenerate orientable metrics) on a 2-torus. In Fig. 59, dark disks correspond to Liouville geodesic flows on a 2-torus, while light disks with a point represent unknown (and perhaps nonexisting altogether) integrable geodesic flows on the 2-torus.

10. Topological classification of classical cases of integrability in the dynamics of a heavy rigid body

We shall briefly consider here the results of detailed investigation carried out by A. A. Oshemkov who calculated the invariant W for integrable cases of the dynamics of a rigid body. It should be recalled that integrable Hamiltonians of this type depend on several parameters. The topology and the properties of the system change upon the variation of these parameters. Therefore, it is important to determine the number of *different topological types* contained in each classical(multiparametric) case of integrability.

Theorem 15. *(Oshemkov, see* [13]*). All integrable Hamiltonians H of classical integrability cases (viz., Euler, Lagrange, Kovalevskaya, and also Goryachev-Chaplygin, Zhukovskii, Sretenskii, Clebsch and Steklov cases), as well as the integrable series of Hamiltonians of a 4-dimensional rigid body (see description in* [5]*), have Bott integrals on nearly all energy levels. Each such (multiparametric) family splits into a few different rough topological types.*

By way of an illustration, we shall give the *complete list of all topological types* into which the Euler, Lagrange, Sretenskii, Zhukovskii, Kovalevskaya, and Clebsch cases split, as well as the integrable series of a 4-dimensional rigid body. The remaining cases and the details of analysis can be found in the work by A.Oshemkov [13]. Oshemkov's theorem (see Theorem 16 below) completes the cycle of investigations carried out by many authors who studied the topology of integral manifolds in the dynamics of a heavy rigid body from various points of view.

In Theorem 16, we shall use the following notations. For each integrable series of Hamiltonians, we indicate all compact isoenergy 3-surfaces encountered in this series. For each such manifold, we indicate the molecule W which defines uniquely the topological type of the system (Liouville foliation) on a given 3-manifold. For the convenience of classification, all the molecules which were encountered in the cases described above are labelled by numbers from 1 to 17. These molecules are shown in Fig. 60.

In Theorem 16, we shall indicate both the 3-manifold Q^3 and the number of the molecule specifying the system on this manifold. For example, the notation *"on S^3 - type 1"* indicates that an isoenergy 3-manifold is diffeomorphic to a 3-sphere, while the Hamiltonian system is defined on it by a molecule of type 1 (see Fig. 60).

We shall denote the direct product of a circle and a 2- sphere, by

$S^1 \times S^2$, the 3-dimensional projective space, by RP^3, the connected sum of two copies $S^1 \times S^2$ by K^3, i.e.,

$$K^3 = (S^1 \times S^2)\#(S^1 \times S^2),$$

and the connected sum of three copies $S^1 \times S^2$ by N^3.

Theorem 16. *(Theorem of rough topological classification). See Oshemkov,* [13].

In the integrability cases listed below, the complete list of all different topological classes (within each series) has the form:

a) *Euler case splits into three classes:*

on S^3 – *type 1,*
on $S^1 \times S^2$ – *type 5,*
on RP^3 – *type 5.*

b) *Zhukovskii case splits into nine classes:*

on S^3 – *type 1,*
on S^3 – *type 2,*
on S^3 – *type 4,*
on $S^1 \times S^2$ – *type 1,*
on $S^1 \times S^2$ – *type 2,*
on $S^1 \times S^2$ – *type 4,*
on RP^3 – *type 1,*
on RP^3 – *type 2,*
on RP^3 – *type 3.*

c) *Kovalevskaya case splits into ten classes:*

on S^3 – *type 1,*
on S^3 – *type 2,*
on S^3 – *type 4,*
on S^3 – *type 6,*
on $S^1 \times S^2$ – *type 2,*
on $S^1 \times S^2$ – *type 6,*
on K^3 – *type 6,*
on RP^3 – *type 6,*
on RP^3 – *type 7,*
on RP^3 – *type 8.*

d) *Sretenskii case splits into eight classes:*

on S^3 – *type 1,*
on S^3 – *type 3,*
on S^3 – *type 10,*
on $S^1 \times S^2$ – *type 2,*

on $S^1 \times S^2$ – *type 11*,
on RP^3 – *type 9*,
on RP^3 – *type 11*,
on K^3 – *type 10*.

e) *Goryachev-Chaplygin case (Sretenskii's particular case) splits into two classes:*

on S^3 – *type 3*,
on RP^3 – *type 9*.

f) *Lagrange case splits into three classes:*

on S^3 – *type 1*,
on RP^3 – *type 1*,
on $S^1 \times S^2$ – *type 1*.

g) *Clebsch case splits into ten classes:*

on S^1 – *type 1*,
on S^3 – *type 2*,
on $S^1 \times S^3$ – *type 1*,
on $S^1 \times S^3$ – *type 2*,
on $S^1 \times S^3$ – *type 5*,
on $S^1 \times S^3$ – *type 12*,
on RP^3 – *type 5*,
on RP^3 – *type 12*,
on N^3 – *type 5*,
on N^3 – *type 12*.

h) *Steklov case splits into six classes with molecules W of the form* 1, 2, 5, 12, 13 *and* 17.

i) *The case of a 4-dimensional rigid body splits into nine classes with molecules W of the form* 1, 2, 5, 9, 12, 13, 14, 15, *and* 16.

It is convenient to present Oshemkov's theorem in the form shown in Fig. 60. The figure not only lists all the molecules W encountered in the integrability series, but also shows graphically the distribution of isoenergy 3-manifolds depending on the type of a molecule. Among other things, the 3-manifolds on which certain molecules are realized can be seen clearly. The obtained pattern gives grounds for formulating new problems since some interesting (and not quite clear so far) regularities are manifested in it.

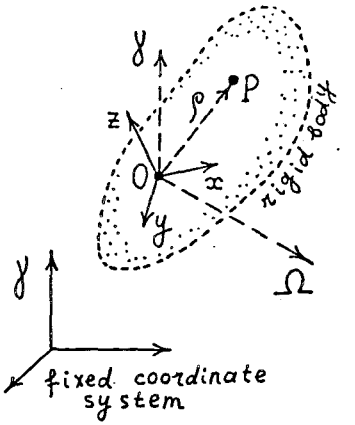


Fig. 1

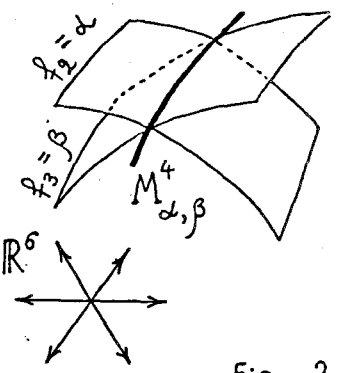


Fig. 2

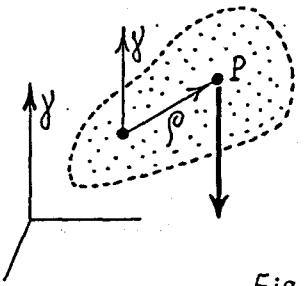


Fig. 3

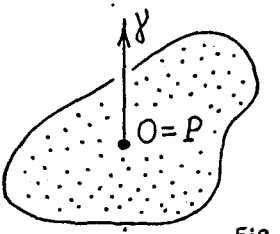


Fig. 4

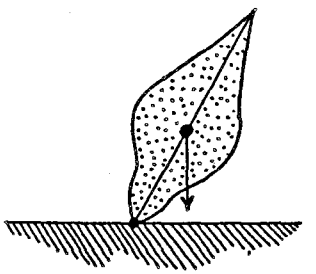


Fig. 5

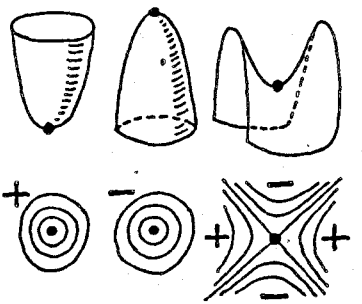


Fig. 6

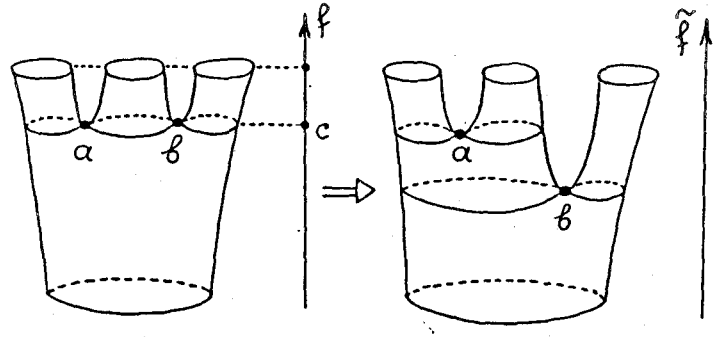


Fig. 7

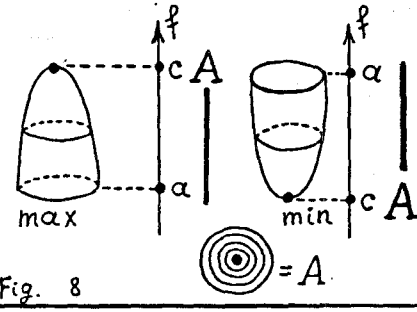


Fig. 8

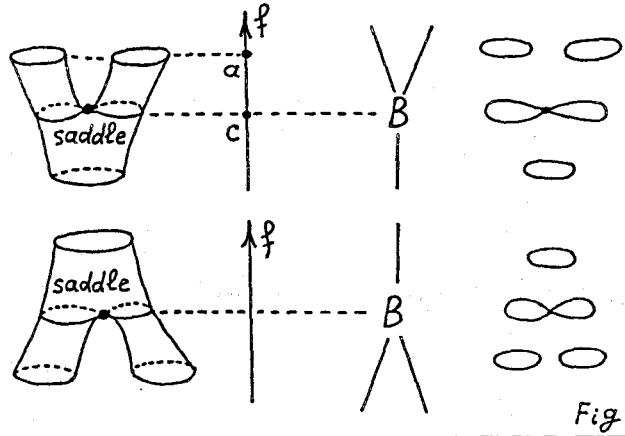


Fig. 9-a

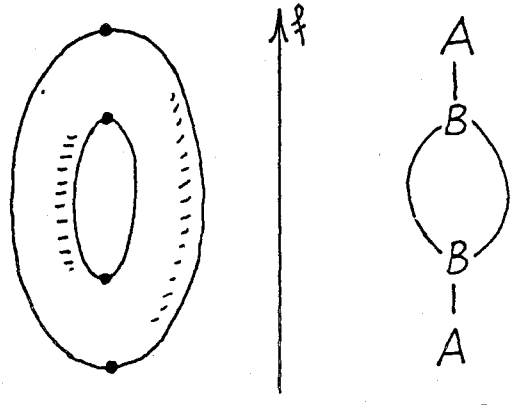


Fig. 9-8

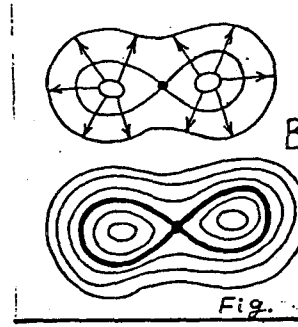


Fig. 10

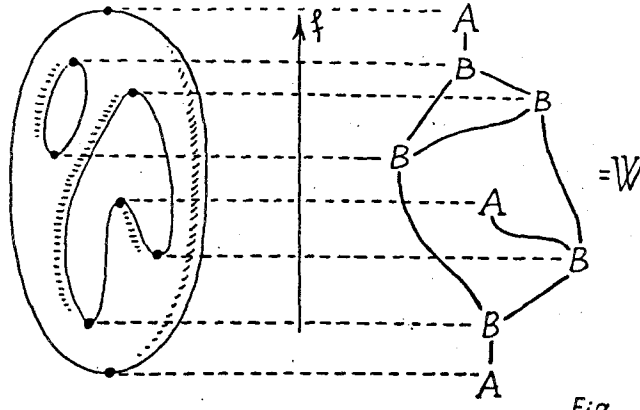


Fig. 11

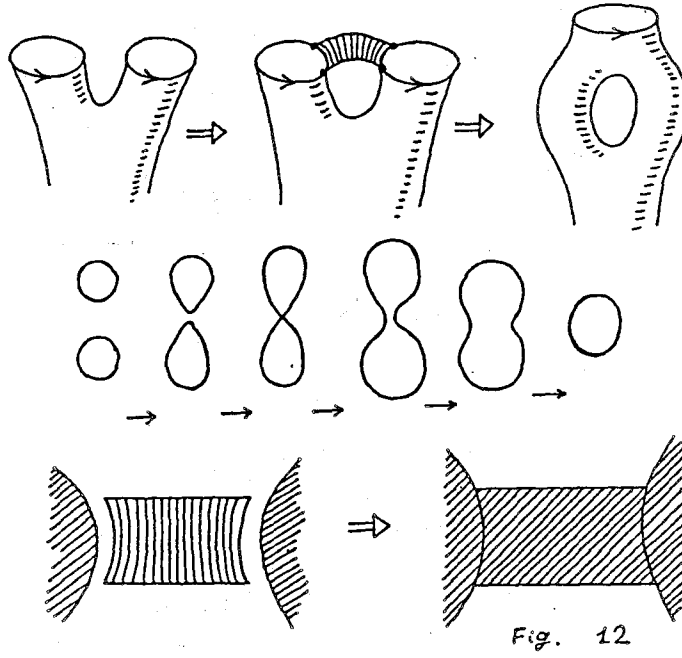


Fig. 12

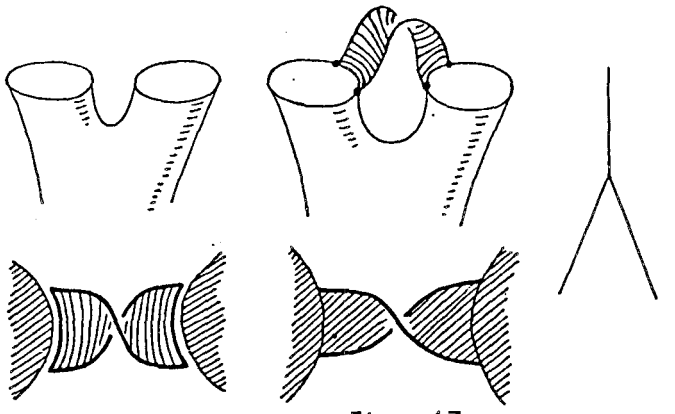


Fig. 13

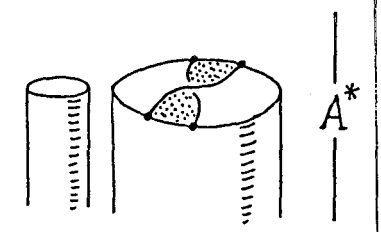


Fig. 14

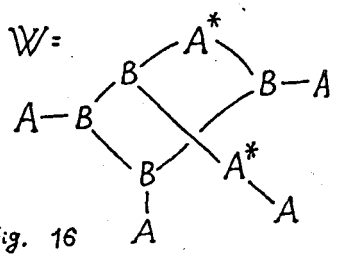


Fig. 16

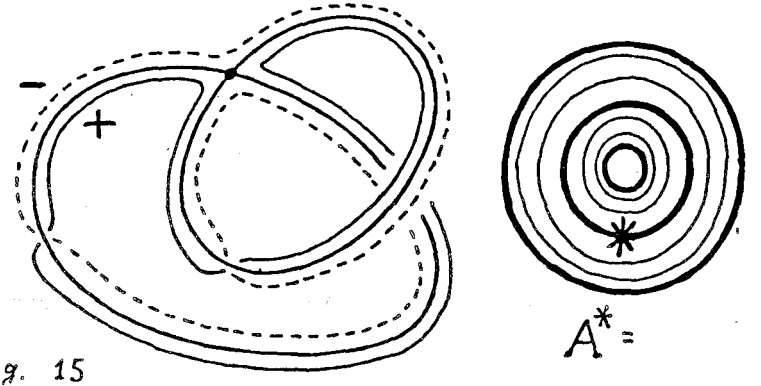
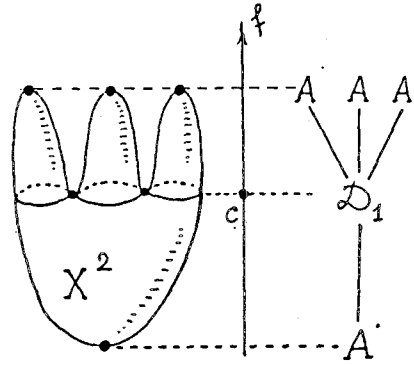


Fig. 15



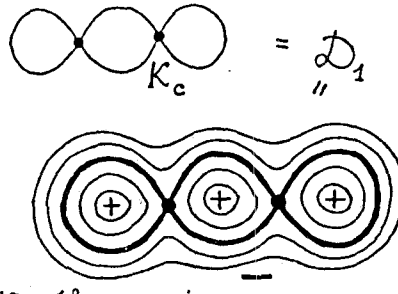


Fig. 18

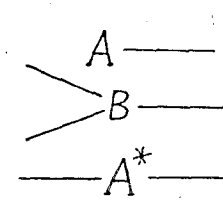
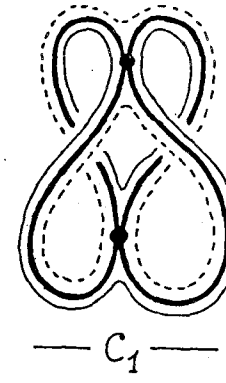


Fig. 17

Fig. 19



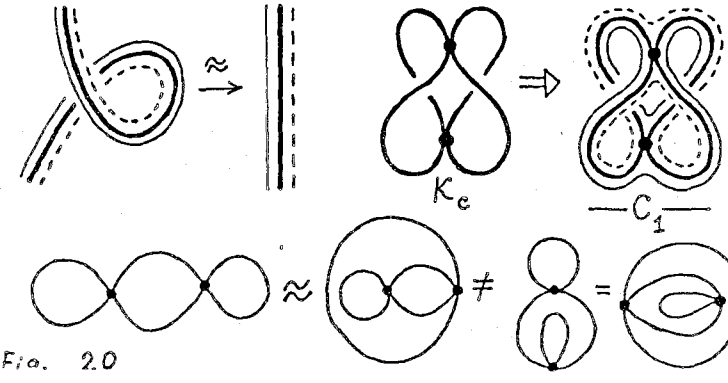
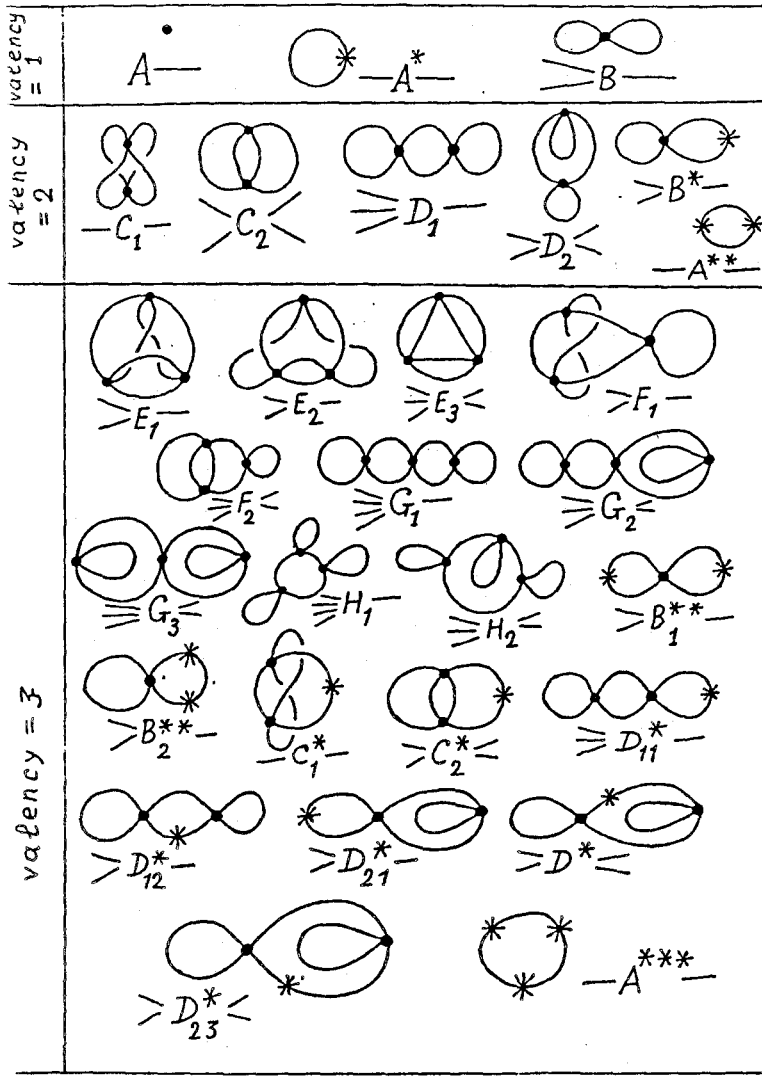


Fig. 20

Fig. 21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A
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B
C1
C2
D1
D2
B*
A**
E1
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E3
F1
F2
G1
G2
G3
H1
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B1**
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C1*
C2*
D11*
D12*
D21*
D*
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A***

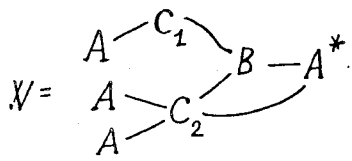
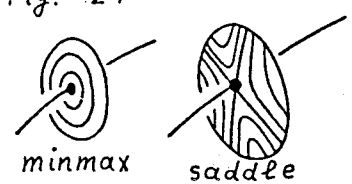


Fig. 22

Fig. 23

Fig. 24



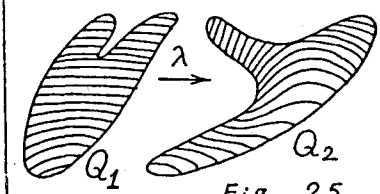


Fig. 25

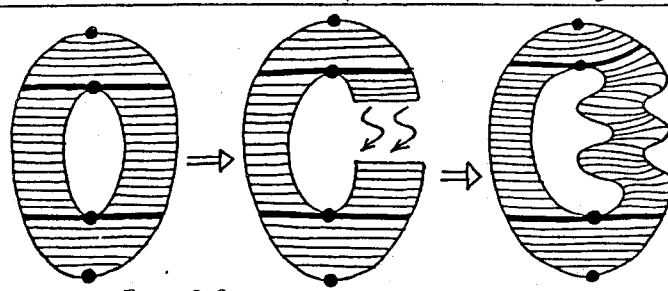


Fig. 26

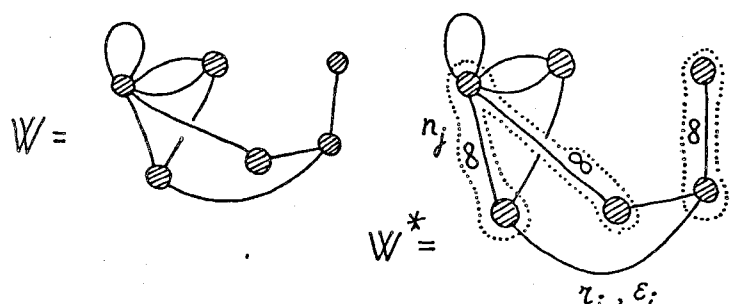


Fig. 27

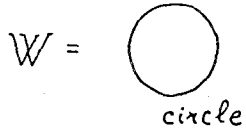


Fig. 28

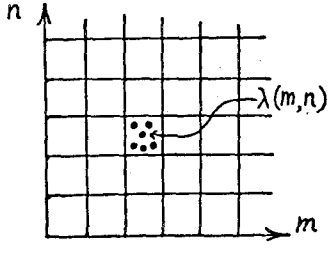


Fig. 30

0	0	0	54	
0	0	0	247	
0	0	11	530	
0	5	24	561	
0	10	24	128	
1	3	2	8	

n

m

Fig. 32

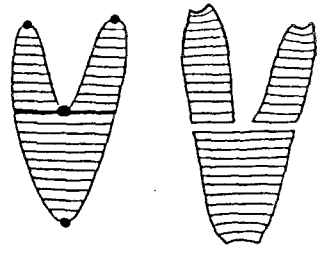


Fig. 29

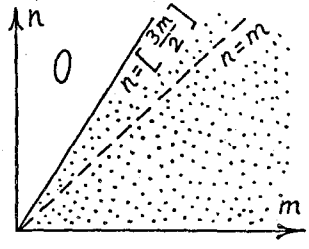


Fig. 31

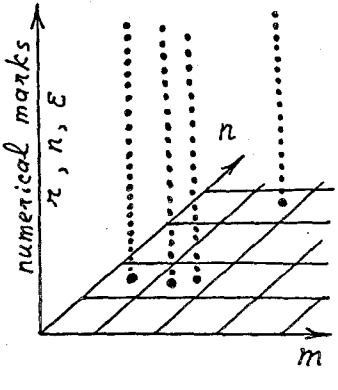


Fig. 33

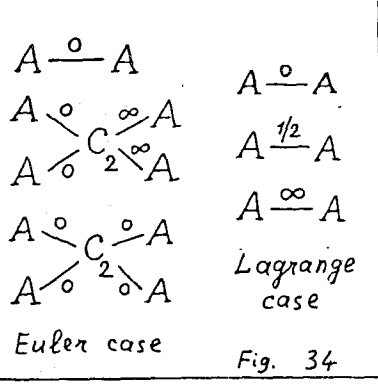


Fig. 34

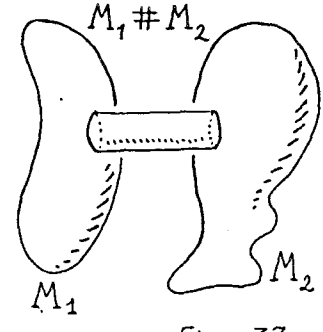


Fig. 37

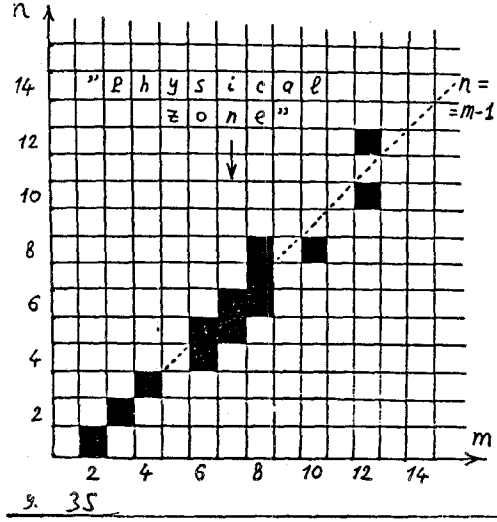


Fig. 35

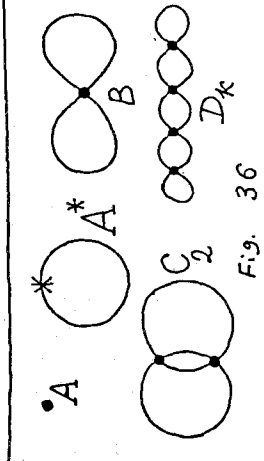


Fig. 36

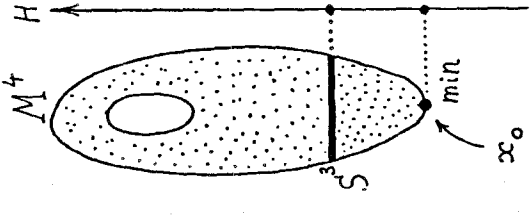
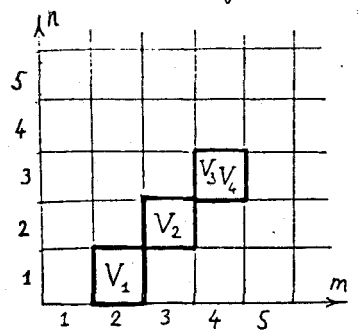
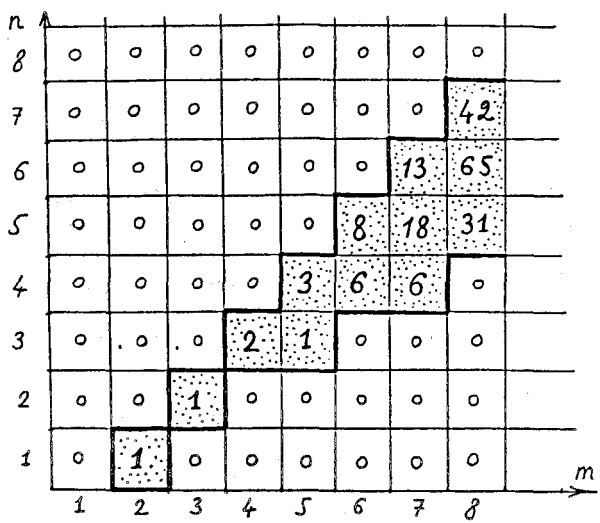


Fig. 38

Fig 39



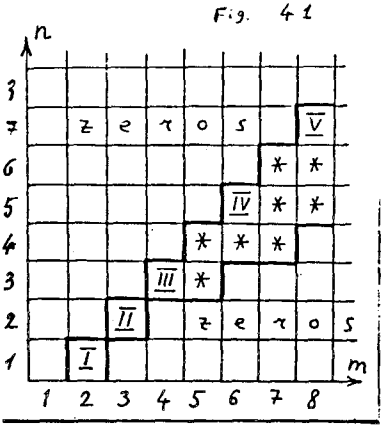


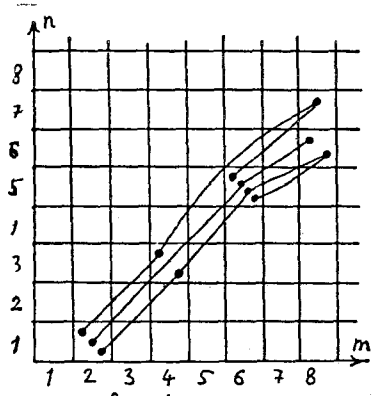
$\sum_n$ = 1 1 2 4 14 37 138

The number of molecules $\mathcal{W}$ on S^3

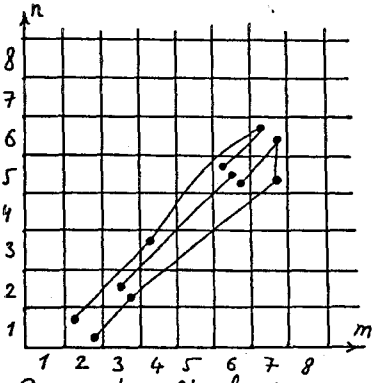
Fig. 40

Fig. 41





Kovalevskaya case



Goryachev-Chaplygin-Sretenskiĭ case

Fig. 42

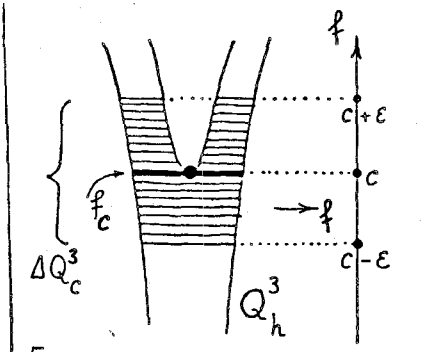


Fig. 43

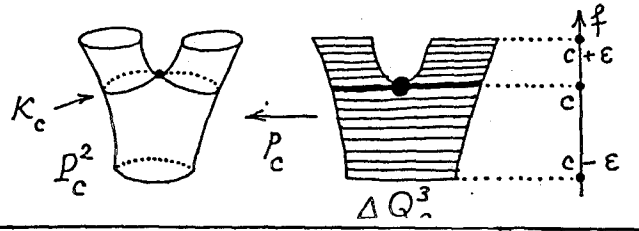


Fig. 44

Fig. 45

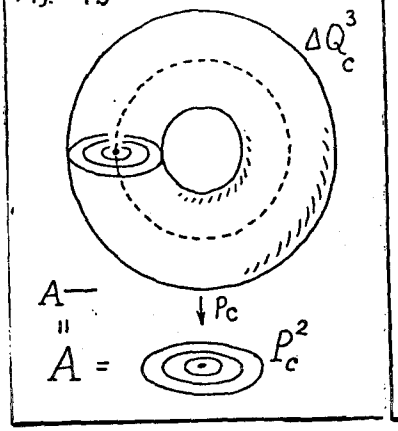
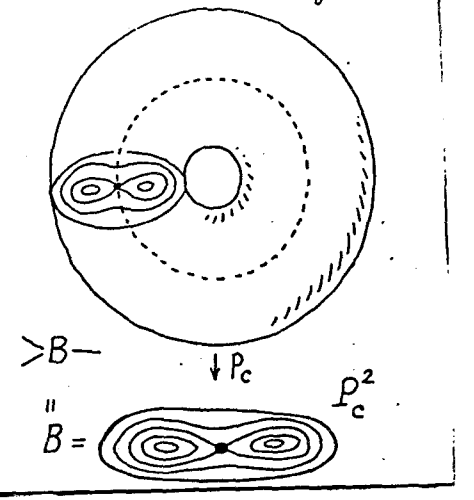


Fig. 46



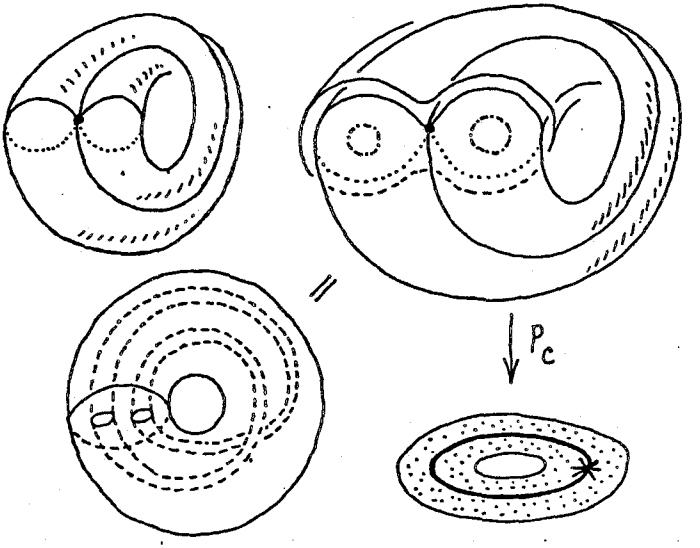


Fig. 47

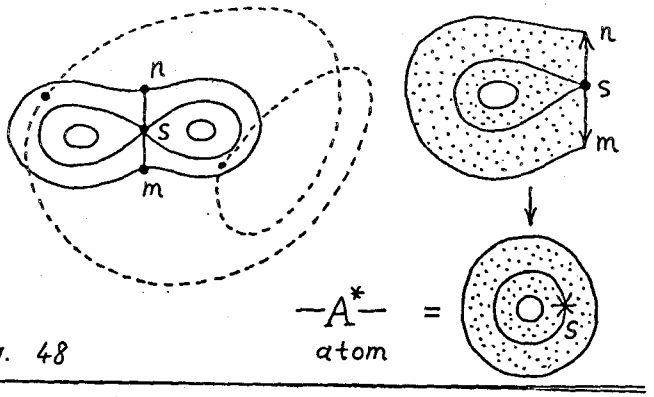


Fig. 48

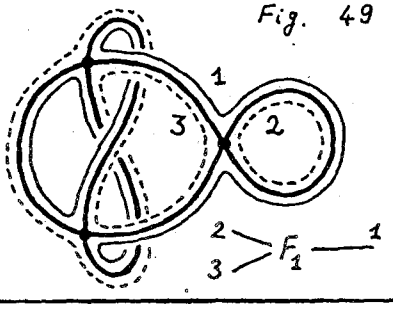


Fig. 49

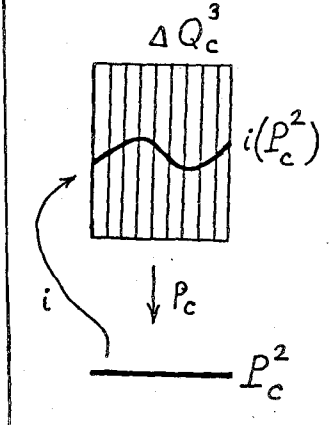
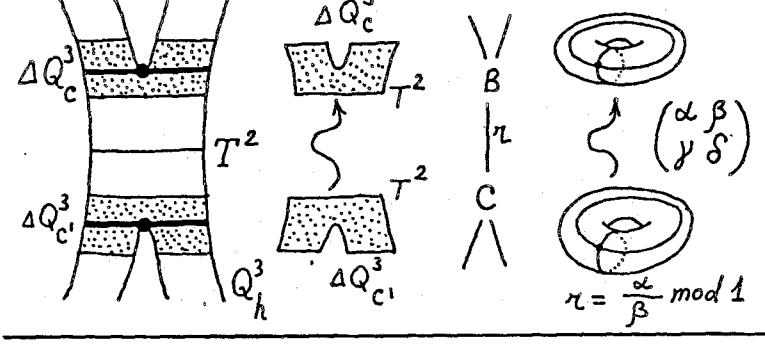


Fig. 50

Fig. 51



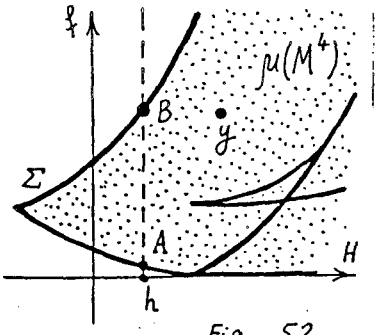


Fig. 52

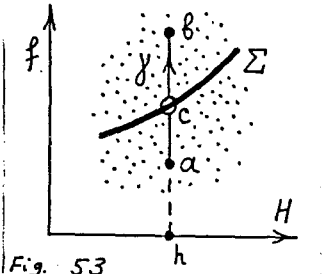


Fig. 53

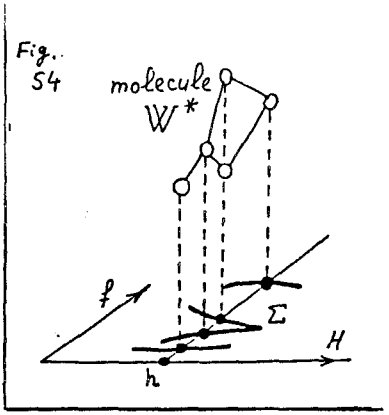


Fig. 54

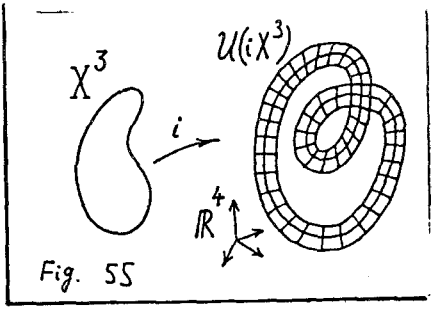


Fig. 55

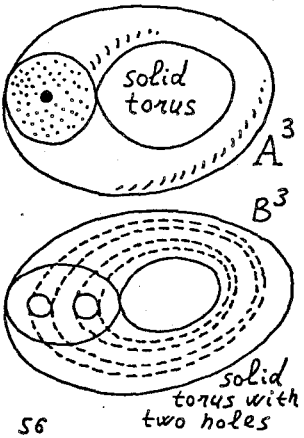


Fig. 56

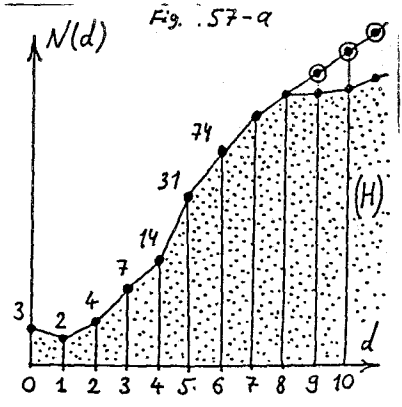


Fig. 57-a

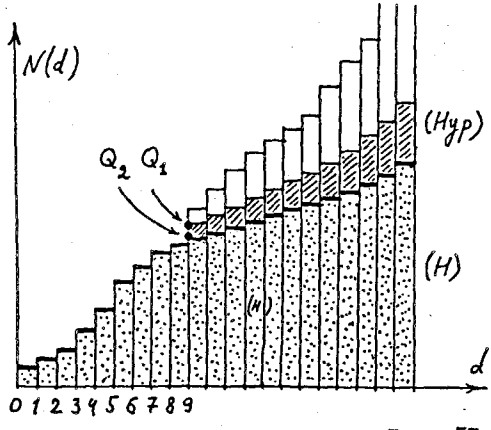


Fig. 57-b

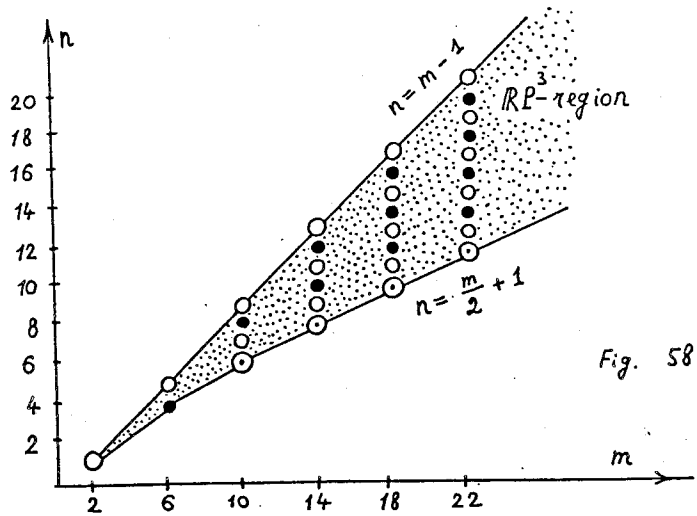

n
m
n = m − 4
RP^3-region
$n = \frac{m}{2} + 1$
20
18
16
14
12
10
8
6
4
2
2
6
10
14
18
22

Fig. 58

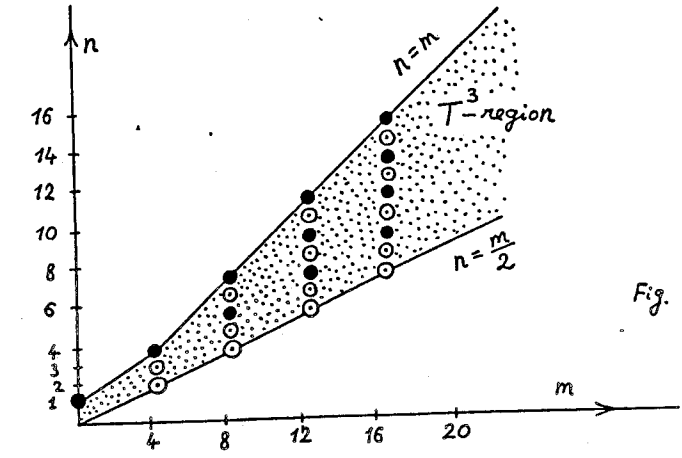

n
m
n = m
T^3-region
$n = \frac{m}{2}$
16
14
12
10
8
6
4
3
2
1
4
8
12
16
20

Fig. 59

Fig. 60-a

No	molecule W	Lagrange	Euler	Zhukovskii	Kovuleov skuga	Steten skii	Gor. Chap.	Clebsch	Stekov	dim 4
1	A—A	S^3 $\mathbb{R}P^3$ $S^1\times S^2$	S^3	S^3 $\mathbb{R}P^3$ $S^1\times S^2$	S^3	S^3		S^3 $S^1\times S^2$	*	*
2	A—B—A | A			S^3 $\mathbb{R}P^3$ $S^1\times S^2$	S^3 $S^1\times S^2$	$S^1\times S^2$		S^3 $S^1\times S^2$	*	*
3	A—A*—A			$\mathbb{R}P^3$		S^3	S^3			
4	A—B—A A—B—A			S^3 $S^1\times S^2$	S^3					
5	A | A—C_2—A | A		$\mathbb{R}P^3$ $S^1\times S^2$					N^3 $\mathbb{R}P^3$ $S^1\times S^2$	*	*
6	A A A | | | B—B—B | | A A				S^3, K^3 $\mathbb{R}P^3$ $S^1\times S^2$					
7	A—A*—B—A*—A | A				$\mathbb{R}P^3$					
8	A | A—A*—C_2—A*—A | A				$\mathbb{R}P^3$					
9	A—B—B—A | | A A					$\mathbb{R}P^3$	$\mathbb{R}P^3$			*
10	A—A*—D<(A, A, A)					S^3 K^3				
11	A—A*—B—B<(A, A, A)					$\mathbb{R}P^3$ $S^1\times S^2$				
12	A A A | | | B—C_2—B | | | A A A							$\mathbb{R}P^3, N^3$ $S^1\times S^2$	*	*

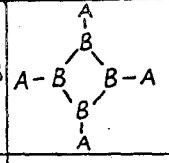
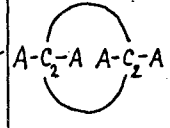
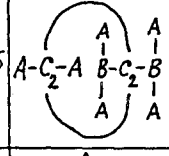
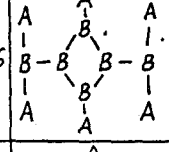
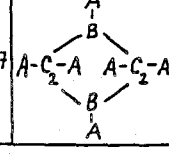
No.	molecule W	Lagrange	Euler	Zhukovskiĭ	Kovalevskaya	Sretenskiĭ	Goryachev-Chaplygin	Clebsch	Steklov	4-dimensional rigid body
13									*	*
14										*
15										*
16										*
17									*	

Fig. 60-B

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