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A nonlocal problem with integral conditions for one-dimensional biwave equation

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The main aim of this work is to consider classical solution of the nonlocal problem for a bi-wave equation with integral conditions of the first kind. The main goal is to show the method which allows to prove solvability of a nonlocal problem with integral conditions of the first kind. Under smoothness and matching conditions of the given functions, existence and uniqueness of the solution of the problem are proved. Moreover, making use of characteristics method, the analytical solution of the problem is provided as well.

1. Introduction. The bi-wave equation has been studied in some models related to the mathematical theory of elasticity. For example, the mathematical formulation for the displacement equation of a homogeneous isotropic elastic body was obtained by using bi-wave equation (see [10-12]). In some recent researches, the symmetry analysis of bi-wave equations is considered and Fushchych, Roman and Zhdanov in [3] got exact solutions. The existence and uniqueness of the solution to Cauchy initial value problem, bounded valued problem are given by Korzyuk (e.g., see [4-6]). In [1,2], Feng and Neilan developed the finite element methods for approximations of bi-wave equation. The authors have studied mainly one-dimensional bi-wave equation and classical solutions of the equation (see, for example, [7-9]). In our present work, the main result is to show the classical solution to mixed problems under integral conditions for one-dimensional bi-wave equation by using method characteristic. The uniqueness of the solution of the problem is proved. A classical solution is understood as a function that is defined everywhere in the closure of the domain and has all classical derivatives occurring in the equation and conditions of the problem.

Recently, nonlocal boundary value problems for parabolic and hyperbolic equations with integral conditions have been actively studied. Such problems arise in mathematical physics in the study of heat-transfer processes, plasma phenomena, certain technological processes, vibration of a media.

2. Statement of the problem. Fix the domain $Q = (0,\infty) \times (0,l)$ of two independent variables. Let $f : \bar{Q} \ni (t,x) \to f(t,x) \in \mathbb{R}$ be a given function on $\bar{Q}$ and $\varphi_j : [0,l] \ni x \to \varphi_j(x) \in \mathbb{R}$, $j = \overline{0,3}$, be functions on $[0,l]$.

Denote by $\partial_t^2 = \frac{\partial^2}{\partial t^2}$, $\partial_x^2 = \frac{\partial^2}{\partial x^2}$ the second order partial derivatives with respect to t and x, respectively.

Let a, b be two positive real numbers satisfying $a > b$. Consider the bi-wave equation as follows

$$\mathbb{L}u(t,x) = \left(\partial_t^2 - a^2\partial_x^2\right)\left(\partial_t^2 - b^2\partial_x^2\right)u(t,x) = f(t,x), \quad (t,x) \in \overline{Q}, \tag{18}$$

with initial conditions

$$\partial_t^j u(0,x) = \varphi_j(x), \quad j = 0,1,2,3, \quad x \in [0,l], \tag{19}$$

and nonlocal integral conditions

$$\int\limits_0^l \partial_x^i u(t,x)\,dx = \mu_i(t), \;\; i = \overline{0,3}, t \geq 0. \tag{20}$$

1 General solution

In this section, we investigate the general solution of Eq.(18). The formula of the solution is given in the following theorem.

Theorem 1. *The general solution of Eq.(18) is given by the sum*

$$u(t,x) = u_0(t,x) + v_p(t,x), \tag{21}$$

where $u_0(t,x)$ *is the homogeneous solution of Eq.(18) which has the form* $g_1(x-at) + g_2(x+at) + g_3(x-bt) + g_4(x+bt)$ *and* $v_p(t,x)$ *is a particular solution of Eq.(18).*

Proof. Using the following notation

$$\left(\partial_t^2 - b^2\partial_x^2\right) u_0(t,x) = w(t,x), \quad (t,x) \in \overline{Q}, \tag{22}$$

the equation $\mathbb{L}u_0(t,x) = 0$ can be written by the form

$$\left(\partial_t^2 - a^2\partial_x^2\right) w(t,x) = 0, \quad (t,x) \in \overline{Q}. \tag{23}$$

By making the transformation of the independent variables from (t,x) to (y_0, y_1) given by $x - at = y_0,\ \ x + at = y_1$, we obtain the canonical form of Eq.(23) as follows

$$\partial_{y_0}\partial_{y_1}\tilde{w}(y_0,y_1) = 0,$$

where $\tilde{w}(y_0,y_1) = w(t,x)$. Thus, it is easy to see that

$$\tilde{\omega}(y_0,y_1) = h^{(1)}(y_1) + h^{(0)}(y_0),$$

which implies

$$\omega(t,x) = h^{(1)}(x+at) + h^{(0)}(x-at).$$

Hence, we can rewrite Eq.(22) as follows

$$\left(\partial_t^2 - b^2\partial_x^2\right) u_0(t,x) = h^{(0)}(x-at) + \ h^{(1)}(x+at). \tag{24}$$

Once again, using characteristics method, by replacing $x - bt = z_0,\ x + bt = z_1$, Eq.(24) is reduced to the canonical form

$$\partial_{z_0}\partial_{z_1}\tilde{u}_0(z_0,z_1) = h^{(0)}\left(\frac{z_1+z_0}{2} - \frac{a(z_1-z_0)}{2b}\right) + \ h^{(1)}\left(\frac{z_1+z_0}{2} + \frac{a(z_1-z_0)}{2b}\right). \tag{25}$$

In order to get the solution of Eq.(25), firstly, we integrate both sides of Eq.(25) with respect to variable z_0 getting that

$$\begin{aligned}
\partial_{z_1}\tilde{u}_0(z_0,z_1) = h^{(2)}(z_1) + \int_{z_0}^{z_1} h^{(0)}\left(\frac{z_1+\eta}{2} - \frac{a(z_1-\eta)}{2b}\right) d\eta + \\
+ \int_{z_0}^{z_1} h^{(1)}\left(\frac{z_1+\eta}{2} + \frac{a(z_1-\eta)}{2b}\right) d\eta = \\
= \ h^{(2)}(z_1) + \int_{z_1}^{\frac{z_1+z_0}{2} - \frac{a(z_1-z_1)}{2b}} h^{(0)}(\eta)\, d\eta + \int_{z_1}^{\frac{z_1+z_0}{2} + \frac{a(z_1-z_1)}{2b}} h^{(1)}(\eta)\, d\eta = \\
= h^{(3)}(z_1) + h^{(4)}\left(\frac{z_1+z_0}{2} - \frac{a(z_1-z_1)}{2b}\right) + h^{(5)}\left(\frac{z_1+z_0}{2} + \frac{a(z_1-z_1)}{2b}\right).
\end{aligned} \tag{26}$$

Secondly, integrating both sides of Eq.(26) with respect to z_1 , we obtain

$$\tilde{u}_0(z_0, z_1) = h^{(6)}\left(\frac{z_1+z_0}{2} - \frac{a(z_1-z_1)}{2b}\right) + h^{(7)}\left(\frac{z_1+z_0}{2} + \frac{a(z_1-z_1)}{2b}\right) + \\ + h^{(8)}(z_1) + h^{(9)}(z_0). \tag{27}$$

From the last equation, substituting z_0, z_1 for t and x , we obtain the homogeneous solution of Eq.(18) given as follows

$$u_0(t,x) = g_1(x-at) + g_2(x+at) + g_3(x-bt) + g_4(x+bt).$$

The proof of the theorem thus is complete.

In the next theorem, we shall give the sufficient and necessary condition for the general solution belonging the class $C^4\left(\overline{Q}\right)$.

Theorem 2. *The general solution of Eq.(18) belongs to the class* $C^4\left(\overline{Q}\right)$ *if and only if*

$$g_1, g_3 \in C^4(-\infty, l], g_2, g_4 \in C^4[0,\infty), v_p(t,x) \in C^4\left(\overline{Q}\right). \tag{28}$$

Proof. It is obvious that if the conditions (28) are satisfied, then the homogeneous solution $u_0(t,x) = g_1(x-at) + g_2(x+at) + g_3(x-bt) + g_4(x+bt)$ of Eq.(18) belongs to the class $C^4\left(\overline{Q}\right)$.

Conversely, let $u_0(t,x) \in C^4\left(\overline{Q}\right)$ be the solution of the homogeneous equation (18). Then the derivatives $\partial_t^j \partial_x^k u_0(t,x)$ belong to $C^0\left(\overline{Q}\right)$ where $j,k \in \{0,1,2,3,4\}$ satisfying $j+k=4$. The derivatives are the following

$$\partial_t^j \partial_x^k u_0(t,x) = (-a)^j d^4 g_1(x-at) + (a)^j d^4 g_2(x+at) + \\ + (-b)^j d^4 g_3(x-bt) + (b)^j d^4 g_4(x+bt), \tag{29}$$

where d^4 - the fourth order ordinary derivative. Equalities (29) are considered as a system of five equations with respect to the derivatives $d^4 g_i,\ i=\overline{1,4}$. It is easy to verify that the determinants of the fourth order of the matrix of these equations are not equal to zero. Hence, the derivatives $d^4 g_i$ are expressed in terms of the derivatives $\partial_t^j \partial_x^k u_0(t,x)$ of solution $u_0(t,x)$ of homogeneous equation (18). This completes the proof.

2 Particular solution

In this section, we shall discuss how to construct the particular solution on domain Q of Eq.(18). The method is that the problem is considered on the subdomain of Q . The construction follows from two steps.

First step. Firstly, we divide the domain Q into subdomains $Q^{(m)} = \left(\frac{(m-1)l}{a}, \frac{ml}{a}\right) \times (0,l)$ by the lines $t = \dfrac{ml}{a},\quad m \in \mathbb{N}$. Now, we consider the following problem in the subdomain $Q^{(m)}$

$$\left(\partial_t^2 - a^2 \partial_x^2\right) w_p(t,x) = f(t,x). \tag{30}$$

Integrating both sides of last equation yields

$$w_p^{(m)} = f^{(1,m)}(x-at) + f^{(2,m)}(x+at) - \frac{1}{4a^2}\int_{ml}^{x+at} dz \int_{l-ml}^{x-at} f\left(\frac{z-y}{2a}, \frac{z+y}{2}\right) dy, \tag{31}$$

for $(t,x) \in \overline{Q^{(m)}}$. If the function f belongs $C^2\left(\overline{Q}\right)$, $f^{(1,m)} \in C^3([-ml, 2l-ml])$, and $f^{(2,m)} \in C^3([(m-1)l, (m+1)l])$, $m \in \mathbb{N}$, then the function $w_p^{(m)}$ belongs to the class $C^3(\overline{Q^{(m)}})$ satisfying Eq. (30).

In order to get the particular solution w_p of equation (30) on the set $\overline{Q}$, we define a new function as follows

$$w_p(t,x) = w_p^{(m)}(t,x), \quad (t,x) \in \overline{Q^{(m)}}, \quad m \in \mathbb{N}. \tag{32}$$

Note that w_p is required to be smooth on $\overline{Q}$. By selecting functions $f^{(j,m)}$ and f from the class $C^2\left(\overline{Q}\right)$, the particular solution (32) should be in class $C^3\left(\overline{Q}\right)$. From this observation, firstly, we can choose the functions $f^{(j,1)}$, $j=1,2$, as follow

$$w_p^{(1)}(0,x)=\partial_t w_p^{(1)}(0,x)=0,\ \partial_t^2 w_p^{(1)}(0,x)=f(0,x),\ \partial_t^3 w_p^{(1)}(0,x)=\partial_t f(0,x). \tag{33}$$

Secondly, the functions $f^{(j,2)}$, $j=1,2$ are chosen such that the function $w_p^{(2)}$ satisfies Cauchy conditions on the line $t=\frac{l}{a}$, i.e.

$$w_p^{(2)}\left(\frac{l}{a},x\right)=w_p^{(1)}\left(\frac{l}{a},x\right), \partial_t w_p^{(2)}\left(\frac{l}{a},x\right)=\partial_t w_p^{(1)}\left(\frac{l}{a},x\right). \tag{34}$$

From (34) and Eq.(30), we obtain the following equalities for the second and third derivatives of $w_p^{(2)}$

$$\begin{gathered}\partial_t^2 w_p^{(2)}\left(\frac{l}{a},x\right)=f\left(\frac{l}{a},x\right)+a^2\partial_x^2 w_p^{(2)}\left(\frac{l}{a},x\right)=\\ =f\left(\frac{l}{a},x\right)+a^2\partial_x^2 w_p^{(1)}\left(\frac{l}{a},x\right)=\partial_t^2 w_p^{(1)}\left(\frac{l}{a},x\right),\\ \partial_t^3 w_p^{(2)}\left(\frac{l}{a},x\right)=\partial_t f\left(\frac{l}{a},x\right)+a^2\partial_t\partial_x^2 w_p^{(2)}\left(\frac{l}{a},x\right)=\\ =\partial_t f\left(\frac{l}{a},x\right)+a^2\partial_t\partial_x^2 w_p^{(1)}\left(\frac{l}{a},x\right)=\partial_t^3 w_p^{(2)}\left(\frac{l}{a},x\right).\end{gathered} \tag{35}$$

From (34), (35) together with assumption $f\in C^2\left(\overline{Q}\right)$ implies that $w_p(t,x)\in C^3\left(\overline{Q^{(1)}}\cup\overline{Q^{(2)}}\right)$. Similarly, we can choose the functions $f^{(j,k)}$, $j=1,2,k=3,4,...,m$ and can be proved that $w_p(t,x)\in C^3\left(\bigcup\limits_{k=1}^{m}\overline{Q^{(k)}}\right)$. Once again we assume that $w_p^{(m+1)}$ satisfies the conditions on the line $t=\frac{ml}{a}$. It means that

$$w_p^{(m+1)}\left(\frac{ml}{a},x\right)=w_p^{(m)}\left(\frac{ml}{a},x\right), \partial_t w_p^{(m+1)}\left(\frac{ml}{a},x\right)=\partial_t w_p^{(m)}\left(\frac{ml}{a},x\right). \tag{36}$$

It can be proved that $w_p(t,x)\in C^3\left(\bigcup\limits_{k=1}^{m+1}\overline{Q^{(k)}}\right)$ by choosing $f^{(j,m+1)}$, $j=1,2$ and using the relations (36) and (30) together with assumption $f\in C^2\left(\overline{Q}\right)$.

Hence, from the facts that $w_p(t,x)\in C^3\left(\bigcup\limits_{k=1}^{m+1}\overline{Q^{(k)}}\right)$ and $m\in\mathbb{N}$ is any number, we conclude that $w_p(t,x)\in C^3\left(\overline{Q}\right)$.

Second step. We divide the domain Q into subdomains $\Omega^{(k)}=\left(\frac{(k-1)l}{b},\frac{kl}{b}\right)\times(0,l)$ by the lines $t=\frac{kl}{b}, k\in\mathbb{N}$. We consider, in the subdomain $\Omega^{(k)}$, the following problem

$$\left(\partial_t^2-b^2\partial_x^2\right)v_p(t,x)=w_p(t,x). \tag{37}$$

By integrating both sides of the last equation, we obtain the solution of Eq.(37)

$$v_p^{(k)}=f^{(3,k)}(x-bt)+f^{(4,k)}(x+bt)-\frac{1}{4b^2}\int\limits_{kl}^{x+bt}dz\int\limits_{l-kl}^{x-bt}w_p\left(\frac{z-y}{2a},\frac{z+y}{2}\right)dy, \tag{38}$$

for $(t,x)\in\overline{\Omega^{(k)}}$. If function $w_p\in C^3\left(\overline{Q}\right)$ ($f\in C^2\left(\overline{Q}\right)$), $f^{(3,k)}\in C^4([-kl,2l-kl])$ and $f^{(4,k)}\in C^4([(k-1)l,(k+1)l])$, $k\in\mathbb{N}$, then function $v_p^{(k)}$ belongs to the class $C^4(\overline{\Omega^{(k)}})$ satisfying Eq.(18) .

In order to get the particular solution v_p of equation (18) on the set $\overline{Q}$, we define a new function as follows

$$v_p(t,x)=v_p^{(k)}(t,x),\ \ (t,x)\in\overline{\Omega^{(k)}},\ \ k\in\mathbb{N}. \tag{39}$$

Note that v_p is required to be smooth on $\overline{Q}$. By selecting functions $f^{(i,k)}$ and f from the class $C^2\left(\overline{Q}\right)$, the particular solution of Eq.(18) should be in the class $C^4\left(\overline{Q}\right)$. From this observation, we can choose the functions $f^{(i,1)}$, $i=3,4$, as follow

$$v_p^{(1)}(0,x)=\partial_t v_p^{(1)}(0,x)=\partial_t^2 v_p^{(1)}(0,x)=\partial_t^3 v_p^{(1)}(0,x)=0,\ \partial_t^4 v_p^{(1)}(0,x)=f(0,x). \tag{40}$$

The functions $f^{(i,2)}$, $i=3,4$ are chosen such that the function $v_p^{(2)}$ satisfies Cauchy conditions on the line $t=\frac{l}{b}$, i.e.

$$v_p^{(2)}\left(\frac{l}{b},x\right)=v_p^{(1)}\left(\frac{l}{b},x\right),\partial_t v_p^{(2)}\left(\frac{l}{b},x\right)=\partial_t v_p^{(1)}\left(\frac{l}{b},x\right). \tag{41}$$

From Eq.(37) and Eq.(30), we obtain the following equalities for the second, third and fourth derivatives of $v_p^{(2)}$

$$\begin{gathered}\partial_t^2 v_p^{(2)}\left(\frac{l}{b},x\right)=w_p\left(\frac{l}{b},x\right)+b^2\partial_x^2 v_p^{(2)}\left(\frac{l}{b},x\right)=\\ =w_p\left(\frac{l}{b},x\right)+b^2\partial_x^2 v_p^{(1)}\left(\frac{l}{b},x\right)=\partial_t^2 v_p^{(1)}\left(\frac{l}{b},x\right),\\ \partial_t^3 v_p^{(2)}\left(\frac{l}{b},x\right)=\partial_t w_p\left(\frac{l}{b},x\right)+b^2\partial_t\partial_x^2 v_p^{(2)}\left(\frac{l}{b},x\right)=\\ =\partial_t w_p\left(\frac{l}{b},x\right)+b^2\partial_t\partial_x^2 v_p^{(1)}\left(\frac{l}{b},x\right)=\partial_t^3 v_p^{(1)}\left(\frac{l}{b},x\right),\\ \partial_t^4 v_p^{(2)}\left(\frac{l}{b},x\right)=\partial_t^2 w_p\left(\frac{l}{b},x\right)+b^2\partial_t^2\partial_x^2 v_p^{(2)}\left(\frac{l}{b},x\right)=\\ =\partial_t^2 w_p\left(\frac{l}{b},x\right)+b^2\partial_t^2\partial_x^2 v_p^{(1)}\left(\frac{l}{b},x\right)=\partial_t^4 v_p^{(1)}\left(\frac{l}{b},x\right).\end{gathered} \tag{42}$$

From (41), (42) together with assumption $f\in C^2\left(\overline{Q}\right)$ implies that $v_p(t,x)\in C^4\left(\overline{\Omega^{(1)}\cup\Omega^{(2)}}\right)$. Next, we can choose, similarly, the functions $f^{(i,h)}$, $i=3,4,h=3,4,...,k$, and can be proved that $v_p(t,x)\in C^4\left(\bigcup\limits_{h=1}^{k}\overline{\Omega^{(h)}}\right)$. Once again, we assume that $v_p^{(k+1)}$ satisfies the conditions on the line $t=\frac{kl}{b}$. It means that

$$v_p^{(k+1)}\left(\frac{kl}{b},x\right)=v_p^{(k)}\left(\frac{kl}{b},x\right),\partial_t w_p^{(k+1)}\left(\frac{kl}{b},x\right)=\partial_t v_p^{(k)}\left(\frac{kl}{b},x\right). \tag{43}$$

It can be proved that $v_p(t,x)\in C^4\left(\bigcup\limits_{h=1}^{k+1}\overline{\Omega^{(h)}}\right)$ by choosing $f^{(i,k+1)}$, $i=3,4$ and using the relations (43) and equality (37) together with assumption $f\in C^2\left(\overline{Q}\right)$.

Thus, from the facts that $v_p(t,x)\in C^4\left(\bigcup\limits_{h=1}^{k+1}\overline{\Omega^{(h)}}\right)$ and $k\in\mathbb{N}$ is any number, we conclude that $v_p(t,x)\in C^4\left(\overline{Q}\right)$.

Theorem 3. *If the function f in Eq.(18) belongs to the class $C^2\left(\overline{Q}\right)$, then the function v_p defined by formulas (39),(38),and (31) belongs to the class $C^4\left(\overline{Q}\right)$ and is a solution of equation (18) satisfying the conditions (40).*

Proof. The proof follows from the preceding arguments.

3 Solving problem

Lemma 1. *Let $u(t,x)$ satisfies equation (18), initial data (19) and matching conditions*

$$\int\limits_0^l\varphi_0(x)\,dx=\mu_0(0),\int\limits_0^l\varphi_1(x)\,dx=d\mu_0(0),\int\limits_0^l\varphi_2(x)\,dx=d^2\mu_0(0),\int\limits_0^l\varphi_3(x)\,dx=d^3\mu_0(0),$$

hold then (20) are equivalent to the conditions of type:

$$d^4\mu_0(t) - \left(a^2+b^2\right) d^2\mu_2(t) + a^2b^2\left(\partial_x^3 u(t,l) - \partial_x^3 u(t,0)\right) = \int_0^l f(t,x)\,dx,$$
$$u(t,l) - u(t,0) = \mu_1(t),\, \partial_x u(t,l) - \partial_x u(t,0) = \mu_2(t),$$
$$\partial_x^2 u(t,l) - \partial_x^2 u(t,0) = \mu_3(t),\quad t \geq 0. \tag{44}$$

Proof. Let $u(t,x)$ satisfies equation (18), initial data (19) and conditions (20). Integrating (18) over $[0,l]$ and using (20) we get:

$$\partial_t^4\left(\int_0^l u(t,x)\,dx\right) - \left(a^2+b^2\right)\partial_t^2\left(\int_0^l \partial_x^2 u(t,x)\,dx\right) +$$
$$+a^2b^2\left(\int_0^l \partial_x^4 u(t,x)\,dx\right) = \int_0^l f(t,x)\,dx.$$

Note that

$$\int_0^l u(t,x)\,dx = \mu_0(t),\ \int_0^l \partial_x^2 u(t,x)\,dx = \mu_2(t),$$

then we immediately get (44). Let now (44) holds for the solution $u(t,x)$ of (18). Integrating (18) over $[0,l]$ we get a ODE:

$$\frac{\partial^4}{\partial t^4}\left(\int_0^l u(t,x)\,dx - \mu_0(t)\right) = 0,$$

Consistency matching conditions give initial data

$$\frac{\partial^i}{\partial t^i}\left(\int_0^l u(0,x)\,dx - \mu_0(0)\right) = 0,\ \ i = \overline{0,3}.$$

By virtue of uniqueness of a solution to the Cauchy problem, we obtain

$$\int_0^l u(t,x)\,dx = \mu_0(t).$$

This means that conditions (20) hold.

We know that the classical solution $u \in C^4\left(\overline{Q}\right)$ of Eq.(18) has the following form

$$u(t,x) = g_1(x-at) + g_2(x+at) + g_3(x-bt) + g_4(x+bt) + v_p(t,x), \tag{45}$$

where $g_j, j = \overline{1,4}-$ arbitrary functions belonging to the class C^4. Substituting $u(t,x)$ into the Cauchy conditions (19), we obtain the system of equations for g_j

$$g_1(x) + g_2(x) + g_3(x) + g_4(x) = \varphi_0(x),$$
$$-adg_1(x) + adg_2(x) - bdg_3(x) + bdg_4(x) = \varphi_1(x),$$
$$a^2d^2g_1(x) + a^2d^2g_2(x) + b^2d^2g_3(x) + b^2d^2g_4(x) = \varphi_2(x),$$
$$-a^3d^3g_1(x) + a^3d^3g_2(x) - b^3d^3g_3(x) + b^3d^3g_4(x) = \varphi_3(x), \tag{46}$$

where d^ig_j- the $i^{th}-$order ordinary derivative of the function $g_j, j = \overline{1,4}$. Solving system (46) yields

$$g_1(z) = g_1^{(0)}(z) = \frac{1}{2a\left(a^2-b^2\right)}\left(b^2\int_0^z \varphi_1(\xi)d\xi + a\int_0^z (z-\xi)\varphi_2(\xi)d\xi - \right.$$
$$\left. -\int_0^z \frac{(z-\xi)^2}{2}\varphi_3(\xi)d\xi - ab^2\varphi_0(z) + C_1b^2 + aC_2z + C_3a - C_4z^2 - C_5z - C_6\right), \tag{47}$$

$$g_2(z) = g_2^{(0)}(z) = \frac{1}{2a(a^2-b^2)} \left(-b^2 \int\limits_0^z \varphi_1(\xi)d\xi + a \int\limits_0^z (z-\xi)\varphi_2(\xi)d\xi + \right.$$
$$\left. + \int\limits_0^z \frac{(z-\xi)^2}{2}\varphi_3(\xi)d\xi - ab^2\varphi_0(z) - C_1b^2 + aC_2z + C_3a + C_4z^2 + C_5z + C_6 \right), \tag{48}$$

$$g_3(z) = g_3^{(0)}(z) = \frac{1}{2b(a^2-b^2)} \left(-a^2 \int\limits_0^z \varphi_1(\xi)d\xi - b \int\limits_0^z (z-\xi)\varphi_2(\xi)d\xi + \right.$$
$$\left. + \int\limits_0^z \frac{(z-\xi)^2}{2}\varphi_3(\xi)d\xi + a^2b\varphi_0(z) - C_1a^2 - bC_2z - C_3b + C_4z^2 + C_5z + C_6 \right), \tag{49}$$

$$g_4(z) = g_4^{(0)}(z) = \frac{-1}{2b(a^2-b^2)} \left(-a^2 \int\limits_0^z \varphi_1(\xi)d\xi + b \int\limits_0^z (z-\xi)\varphi_2(\xi)d\xi + \right.$$
$$\left. + \int\limits_0^z \frac{(z-\xi)^2}{2}\varphi_3(\xi)d\xi - a^2b\varphi_0(z) - C_1a^2 + bC_2z + C_3b + C_4z^2 + C_5z + C_6 \right), \tag{50}$$

where $z \in [0, l]$ and $C_1, C_2, C_3, C_4, C_5, C_6 \in \mathbb{R}$ are arbitrary constants.

In a similar way, substituting $u(t, x)$ into the nonlocal boundary conditions (44), we obtain the following system of equations for $g_j, j = 1, 2, 3, 4$

$$\begin{aligned}
g_1(l-at) + g_2(l+at) + g_3(l-bt) + g_4(l+bt) + v_p(t,l) - v_p(t,0) - \\
-g_1(-at) - g_2(at) - g_3(-bt) - g_4(bt) = \mu_1(t), \\
dg_1(l-at) + dg_2(l+at) + dg_3(l-bt) + dg_4(l+bt) + \partial_x v_p(t,l) - \partial_x v_p(t,0) - \\
-dg_1(-at) - dg_2(at) - dg_3(-bt) - dg_4(bt) = \mu_2(t), \\
d^2g_1(l-at) + d^2g_2(l+at) + d^2g_3(l-bt) + d^2g_4(l+bt) + \partial_x^2 v_p(t,l) - \partial_x^2 v_p(t,0) - \\
-d^2g_1(-at) - d^2g_2(at) - d^2g_3(-bt) - d^2g_4(bt) = \mu_3(t), \\
d^3g_1(l-at) + d^3g_2(l+at) + d^3g_3(l-bt) + d^3g_4(l+bt) + \partial_x^3 v_p(t,l) - \\
-d^3g_1(-at) - d^3g_2(at) - d^3g_3(-bt) - d^3g_4(bt) - \partial_x^3 v_p(t,0) - \\
-\frac{1}{a^2b^2}\left(\int\limits_0^l f(t,x) - d^4\mu_0(t) + \left(a^2+b^2\right) d^4\mu_2(t) \right) = 0
\end{aligned} \tag{51}$$

If we integrate (51) with respect to t, we have

$$g_1(-at) + g_3(-bt) - g_2(l+at) - g_4(l+bt) = v_p(t,l) - \\ -v_p(t,0) - \mu_1(t) - g_2(at) - g_4(bt) + g_1(l-at) + g_3(l-bt),$$

$$\frac{g_1(-at)}{-a} + \frac{g_3(-bt)}{-b} - \frac{g_2(l+at)}{a} - \frac{g_4(l+bt)}{b} = \tilde{C}_1 - \frac{g_2(at)}{a} + \\ + \int\limits_0^t \left(\partial_x v_p(\xi,l) - \partial_x v_p(\xi,0) - \mu_2(\xi)\right) d\xi - \frac{g_4(bt)}{b} + \frac{g_1(l-at)}{-a} + \frac{g_3(l-bt)}{-b},$$

$$\frac{g_1(-at)}{(-a)^2} + \frac{g_3(-bt)}{(-b)^2} - \frac{g_2(l+at)}{(a)^2} - \frac{g_4(l+bt)}{(b)^2} = \tilde{C}_2 t + \tilde{C}_3 - \frac{g_2(at)}{(a)^2} + \\ + \int\limits_0^t (t-\xi)\left(\partial_x^2 v_p(\xi,l) - \partial_x^2 v_p(\xi,0) - \mu_3(\xi)\right) d\xi - \frac{g_4(bt)}{(b)^2} + \frac{g_1(l-at)}{(-a)^2} + \frac{g_3(l-bt)}{(-b)^2}, \tag{52}$$

$$\frac{g_1(-at)}{(-a)^3} + \frac{g_3(-bt)}{(-b)^3} - \frac{g_2(l+at)}{(a)^3} - \frac{g_4(l+bt)}{(b)^3} = \tilde{C}_4 t^2 + \tilde{C}_5 t + \tilde{C}_6 - \frac{g_2(at)}{(a)^3} + \\ + \int\limits_0^t \frac{(t-\xi)^2}{2}\left(\partial_x^3 v_p(\xi,l) - \partial_x^3 v_p(\xi,0)\right) d\xi - \frac{g_4(bt)}{(b)^3} + \frac{g_1(l-at)}{(-a)^3} + \frac{g_3(l-bt)}{(-b)^3} - \\ - \frac{1}{a^2b^2}\left(\int\limits_0^t \frac{(t-\xi)^2}{2}\left(\int\limits_0^l f(\xi,x)\,dx - d^4\mu_0(\xi) + \left(a^2+b^2\right) d^2\mu_2(\xi)\right) d\xi\right),$$

where

$$\tilde{C}_1 = \frac{C_5 l + C_4 l^2 - \left(a^2+b^2\right)\int\limits_0^l \varphi_1(y)\,dy + \frac{1}{2}\int\limits_0^l (l-y)^2 \varphi_3(y)\,dy}{a^2b^2},$$

$$\tilde{C}_2 = \frac{2C_4 l + \left(a^2+b^2\right)\left(\varphi_1(0) - \varphi_1(l)\right) + \int\limits_0^l (l-y)\varphi_3(y)\,dy}{a^2b^2},$$

$$\tilde{C}_3 = \frac{C_2 l + \left(a^2+b^2\right)\left(\varphi_0(0) - \varphi_0(l)\right) + \int\limits_0^l (l-y)\varphi_2(y)\,dy}{a^2b^2},$$

$$\tilde{C}_4 = \frac{\left(a^2+b^2\right)\left(d\varphi_1(0) - d\varphi_1(l)\right) + \int\limits_0^l \varphi_3(y)\,dy}{2a^2b^2},$$

$$\tilde{C}_5 = \frac{\left(a^2+b^2\right)\left(d\varphi_0(0) - d\varphi_0(l)\right) + \int\limits_0^l \varphi_2(y)\,dy}{a^2b^2},$$

$$\tilde{C}_6 = \frac{\left(a^2+b^2\right)\left(C_5 l + C_4 l^2 + \frac{1}{2}\int\limits_0^l (l-y)^2\varphi_3(y)\,dy\right) - \left(a^4+a^2b^2+b^4\right)\int\limits_0^l \varphi_1(y)\,dy}{a^2b^2}.$$

Solving system (52), we obtain

$$
\begin{gathered}
g_1(z)=g_1^{(k+1)}(z)=\frac{1}{4a\left(a^2-b^2\right)}\Big(4a\left(a^2-b^2\right)g_1^{(k)}(l+z)-\\
-2a^4H\left(\frac{-z}{a}\right)-2a^3b^2J\left(\frac{-z}{a}\right)+2a^3K\left(\frac{-z}{a}\right)+2a^4b^2L\left(\frac{-z}{a}\right)-2b^2\int\limits_0^l\varphi_1(\xi)d\xi+\\
+2a^4\int\limits_0^{-z/a}\mu_2(\xi)\,d\xi+2a^3b^2\int\limits_0^{-z/a}\left(\frac{-z}{a}-\xi\right)\mu_3(\xi)\,d\xi-a^4\int\limits_0^{-z/a}\left(\frac{-z}{a}-\xi\right)^2\mu_2''(\xi)\,d\xi-\\
-a^2b^2\int\limits_0^{-z/a}\left(\frac{-z}{a}-\xi\right)^2\mu_2''(\xi)\,d\xi+a^2\int\limits_0^{-z/a}\left(\frac{-z}{a}-\xi\right)^2\mu_0^{(4)}(\xi)\,d\xi+\\
+\int\limits_0^l(z+l-y)\left(z\varphi_3(y)-2a\varphi_2(y)\right)dy+\int\limits_0^l\Big((l-y)^2+z(l-y)\Big)\varphi_3(y)\,dy-\\
-2aC_2l+2C_5l+2C_4l^2+4C_4lz-2\left(a^3+ab^2\right)\left(\varphi_0(0)-\varphi_0(l)\right)+\\
+2\left(a^2z+b^2z\right)\left(\varphi_1(0)-\varphi_1(l)\right)-2\left(a^3z+ab^2z\right)\left(d\varphi_0(0)-d\varphi_0(l)\right)-\\
-\int\limits_0^{-z/a}\int\limits_0^l(z+a\xi)^2f(\xi,x)\,dxd\xi+\left(a^2z^2+b^2z^2\right)\left(d\varphi_1(0)-d\varphi_1(l)\right)\Big),\,z\in\left[-(k+1)\,l,-kl\right],
\end{gathered}
\tag{53}
$$

$$
\begin{gathered}
g_2(z)=g_2^{(k+1)}(z)=\frac{1}{4a\left(a^2-b^2\right)}\Big(4a\left(a^2-b^2\right)g_2^{(k)}(z-l)-\\
-2a^4H\left(\frac{z-l}{a}\right)+2a^3b^2J\left(\frac{z-l}{a}\right)-2a^3K\left(\frac{z-l}{a}\right)+2a^4b^2L\left(\frac{z-l}{a}\right)+\\
+2\int\limits_0^l(z-y)\left((z-l)\,\varphi_3(y)+2a\varphi_2(y)\right)dy+2\int\limits_0^l(l-y)^2\varphi_3(y)\,dy+2aC_2l+\\
+2a^4\int\limits_0^{(z-l)/a}\mu_2(\xi)\,d\xi-\int\limits_0^{(z-l)/a}\int\limits_0^l\left((z-l)/a+a\xi\right)^2f(\xi,x)\,dxd\xi-\\
-2a^3b^2\int\limits_0^{(z-l)/a}\left((z-l)/a-\xi\right)\mu_3(\xi)\,d\xi-a^4\int\limits_0^{(z-l)/a}\left((z-l)/a-\xi\right)^2\mu_2''(\xi)\,d\xi-\\
-a^2b^2\int\limits_0^{(z-l)/a}\left((z-l)/a-\xi\right)^2\mu_2''(\xi)\,d\xi+a^2\int\limits_0^{(z-l)/a}\left((z-l)/a-\xi\right)^2\mu_0^{(4)}(\xi)\,d\xi-\\
-2b^2\int\limits_0^l\varphi_1(\xi)d\xi+2C_5l+2C_4l^2+4C_4l\,(z-l)+2\left(a^3+ab^2\right)\left(\varphi_0(0)-\varphi_0(l)\right)+\\
+2\left(a^2+b^2\right)(z-l)\left(\varphi_1(0)-\varphi_1(l)\right)+2\left(a^3+ab^2\right)(z-l)\left(d\varphi_0(0)-d\varphi_0(l)\right)+\\
+\left(a^2+b^2\right)(z-l)^2\left(d\varphi_1(0)-d\varphi_1(l)\right)\Big),\,z\in\left[(k+1)\,l,(k+2)\,l\right],
\end{gathered}
\tag{54}
$$

$$g_3(z) = g_3^{(k+1)}(z) = \frac{1}{4(a^2b-b^3)}\Big(4(a^2b-b^3)\, g_3^{(k)}(l+z) +$$
$$+2b^4 H\left(\frac{-z}{b}\right) + 2b^3a^2 J\left(\frac{-z}{b}\right) - 2b^3 K\left(\frac{-z}{b}\right) - 2b^4a^2 L\left(\frac{-z}{b}\right) + 2a^2\int\limits_0^l \varphi_1(\xi)d\xi -$$
$$-2b^4\int\limits_0^{-z/b}\mu_2(\xi)\,d\xi - 2a^2b^3\int\limits_0^{-z/b}\left(\frac{-z}{b}-\xi\right)\mu_3(\xi)\,d\xi + b^4\int\limits_0^{-z/b}\left(\frac{-z}{b}-\xi\right)^2\mu_2''(\xi)\,d\xi +$$
$$+a^2b^2\int\limits_0^{-z/b}\left(\frac{-z}{b}-\xi\right)^2\mu_2''(\xi)\,d\xi - b^2\int\limits_0^{-z/b}\left(\frac{-z}{b}-\xi\right)^2\mu_0^{(4)}(\xi)\,d\xi + \tag{55}$$
$$+2\int\limits_0^l (z+l-y)(-z\varphi_3(y)+b\varphi_2(y))\,dy - \int\limits_0^l (l-y)^2\varphi_3(y)\,dy +$$
$$+2bC_2l - 2C_5l - 2C_4l^2 - 4C_4lz + 2(a^2b+b^3)(\varphi_0(0)-\varphi_0(l)) -$$
$$-2(a^2z+b^2z)(\varphi_1(0)-\varphi_1(l)) + 2(a^2bz+b^3z)(d\varphi_0(0)-d\varphi_0(l)) +$$
$$+\int\limits_0^{-z/b}\int\limits_0^l (z+b\xi)^2 f(\xi,x)\,dxd\xi - (a^2z^2+b^2z^2)(d\varphi_1(0)-d\varphi_1(l))\Big),\ z\in[-(k+1)l,-kl],$$

$$g_4(z) = g_4^{(k+1)}(z) = \frac{1}{4(b^3-a^2b)}\Big(4(b^3-a^2b)\, g_4^{(k)}(z-l) -$$
$$-2b^4H\left(\frac{z-l}{b}\right) + 2b^3a^2J\left(\frac{z-l}{b}\right) - 2b^3K\left(\frac{z-l}{b}\right) + 2b^4a^2L\left(\frac{z-l}{b}\right) +$$
$$+2\int\limits_0^l (z-y)((z-l)\varphi_3(y)+b\varphi_2(y))\,dy + \int\limits_0^l (l-y)^2\varphi_3(y)\,dy + 2bC_2l -$$
$$+2b^4\int\limits_0^{(z-l)/b}\mu_2(\xi)\,d\xi - 2a^2b^3\int\limits_0^{(z-l)/b}((z-l)/b-\xi)\,\mu_3(\xi)\,d\xi -$$
$$-b^4\int\limits_0^{(z-l)/b}((z-l)/b-\xi)^2\mu_2''(\xi)\,d\xi + a^2b^2\int\limits_0^{(z-l)/b}((z-l)/b-\xi)^2\mu_2''(\xi)\,d\xi + \tag{56}$$
$$+b^2\int\limits_0^{(z-l)/b}((z-l)/b-\xi)^2\mu_0^{(4)}(\xi)\,d\xi - \int\limits_0^{(z-l)/b}\int\limits_0^l ((z-l)/b+a\xi)^2 f(\xi,x)\,dxd\xi$$
$$-2a^2\int\limits_0^l\varphi_1(\xi)d\xi + C_5l + C_4l^2 + 2C_4l(z-l) + 2(a^2b+b^3)(\varphi_0(0)-\varphi_0(l)) +$$
$$+2(a^2+b^2)(z-l)(\varphi_1(0)-\varphi_1(l)) + 2(a^2b+b^3)(z-l)(d\varphi_0(0)-d\varphi_0(l)) +$$
$$+(a^2+b^2)(z-l)^2(d\varphi_1(0)-d\varphi_1(l))\Big),\ z\in[(k+1)l,(k+2)l],$$

where

$$K(t) = v_p(t,l) - v_p(t,0),$$

$$H(t) = \int\limits_0^t (\partial_x v_p(\xi,l) - \partial_x v_p(\xi,0))\,d\xi,$$

$$J(t)=\int_0^t\left(\partial_x^2 v_p(\xi,l)-\partial_x^2 v_p(\xi,0)\right)d\xi,$$

$$L(t)=\int_0^t\left(\partial_x^3 v_p(\xi,l)-\partial_x^3 v_p(\xi,0)\right)d\xi.$$

With the functions $g_1, g_3 \in C^4(-\infty, l]$, $g_2, g_4 \in C^4[0,\infty)$ and the requirements of smoothness of the given functions in the problem (18-20), one should require that the values of the functions $g_i^{(k)}, i=\overline{1,4}; k=0,1,2,3,...$, and the values of their first, second, third and fourth derivatives should coincide at their common points, i.e., the following relations should be held

$$\begin{aligned}
d^p g_1^{(k+1)}(-kl)&=d^p g_1^{(k)}(-kl), \quad p=\overline{0,4},\\
d^p g_2^{(k+1)}(l+kl)&=d^p g_2^{(k)}(l+kl), \quad p=\overline{0,4},\\
d^p g_3^{(k+1)}(-kl)&=d^p g_3^{(k)}(-kl), \quad p=\overline{0,4},\\
d^p g_4^{(k+1)}(l+kl)&=d^p g_4^{(k)}(l+kl), \quad p=\overline{0,4}.
\end{aligned}\tag{57}$$

Lemma 1. *The necessary and sufficient conditions for the relations (57) hold for all* $k\in\mathbb{N}$ *is that they hold just for only case* $k=0$.

Proof. From formulas (53)-(56), we get

$$\begin{aligned}
&d^p g_1^{(k+1)}(-kl)-d^p g_1^{(k)}(-kl)=d^p g_1^{(k)}(l-kl)-d^p g_1^{(k-1)}(l-kl)=\\
&=d^p g_1^{(k-1)}(2l-kl)-d^p g_1^{(k-2)}(2l-kl)=...=d^p g_1^{(1)}(0)-d^p g_1^{(0)}(0),\ p=\overline{0,4},
\end{aligned}$$

and

$$\begin{aligned}
&d^p g_2^{(k+1)}(l+kl)-d^p g_2^{(k)}(l+kl)=d^p g_2^{(k)}(kl)-d^p g_2^{(k-1)}(kl)=\\
&=d^p g_2^{(k-1)}(kl-l)-d^p g_2^{(k-2)}(kl-l)=...=d^p g_2^{(1)}(l)-d^p g_2^{(0)}(l), \quad p=\overline{0,4}.
\end{aligned}$$

Similarly for $g_3^{(k)}, g_4^{(k)}$.

Lemma 2. *The relations (57) hold for case* $k=0$ *if and only if the following matching conditions hold*

$$\begin{aligned}
&\int_0^l \varphi_j(y)\,dy=\mu_0^{(j)}(0),\ j=\overline{0,3},\\
&\left(a^2+b^2\right)\int_0^l d^2\varphi_2(y)\,dy-a^2b^2\int_0^l d^4\varphi_0(y)\,dy-d^4\mu_0(0)=-\int_0^l f(0,\xi)\,d\xi,\\
&\left(a^2+b^2\right)\int_0^l d^2\varphi_3(y)\,dy-a^2b^2\int_0^l d^4\varphi_1(y)\,dy-d^5\mu_0(0)=-\int_0^l \partial_t f(0,\xi)\,d\xi,\\
&d^i\varphi_j(l)-d^i\varphi_j(0)=d^j\mu_{i+1}(0),\ \ i=\overline{0,2}, j=\overline{0,3}, f(0,0)=f(0,l).
\end{aligned}\tag{58}$$

Proof. For the case $k=0$, the equalities (57) are rewritten as follow

$$g_1^{(1)}(0)-g_1^{(0)}(0)=\frac{a^3\left(v_p(0,l)-v_p(0,0)-\mu_1(0)+\varphi_0(l)-\varphi_0(0)\right)}{2\left(a^3-ab^2\right)}=0,$$

$$\begin{aligned}
dg_1^{(1)}(0)-dg_1^{(0)}(0)=&\frac{a^2\left(\varphi_1(0)-\varphi_1(l)+d\mu_1(0)-a\left(d\varphi_0(0)-d\varphi_0(l)+\mu_2(0)\right)\right)}{2\left(a^3-ab^2\right)}+\\
&+\frac{a^2\left(\partial_t v_p(0,0)-\partial_t v_p(0,0)+a\left(\partial_x v_p(0,l)-\partial_x v_p(0,0)\right)\right)}{2\left(a^3-ab^2\right)}=0,
\end{aligned}$$

$$d^2g_1^{(1)}(0)-d^2g_1^{(0)}(0)=\frac{\left(a\left(\varphi_2(l)-\varphi_2(0)-d^2\mu_1(0)\right)-\left(2a^2+b^2\right)\left(d\varphi_1(l)-d\varphi_1(0)-d\mu_2(0)\right)\right)}{2\left(a^3-ab^2\right)}+$$
$$+\frac{\left(ab^2\left(d^2\varphi_0(0)-d^2\varphi_0(l)+\mu_3(0)\right)+a\left(\partial_t^2v_p(0,l)-\partial_t^2v_p(0,0)\right)\right)}{2\left(a^3-ab^2\right)}+$$
$$+\frac{ab^2\left(\partial_x^2v_p(0,0)-\partial_x^2v_p(0,l)\right)}{2\left(a^3-ab^2\right)}=0,$$

$$d^3g_1^{(1)}(0)-d^3g_1^{(0)}(0)=\frac{\left(\left(\varphi_3(0)-\varphi_3(l)+d^3\mu_1(0)\right)-a\left(d\varphi_2(0)-d\varphi_2(l)\right)-d^2\mu_2(0)\right)}{2\left(a^3-ab^2\right)}+$$

$$+\frac{\left(a^2+b^2\right)\int\limits_0^l d^2\varphi_2(y)\,dy-a^2b^2\int\limits_0^l d^4\varphi_0(y)\,dy-d^4\mu_0(0)+\int\limits_0^l f(0,\xi)\,d\xi}{2\left(a^3-ab^2\right)}+$$
$$+\frac{\left(a\left(\partial_t^2\partial_xv_p(0,l)-\partial_t^2\partial_xv_p(0,0)\right)-b^2\left(d^2\varphi_1(0)-d^2\varphi_1(l)\right)\right)}{2\left(a^3-ab^2\right)}+$$
$$+\frac{\left(ab^2\left(d^3\varphi_0(0)-d^3\varphi_0(l)\right)+\left(\partial_t^3v_p(0,0)-\partial_t^3v_p(0,l)\right)\right)}{2\left(a^3-ab^2\right)}+$$
$$+\frac{ab^2\left(\partial_x^3v_p(0,0)-\partial_x^3v_p(0,l)\right)+b^2\left(\partial_x^2\partial_tv_p(0,l)-\partial_x^2\partial_tv_p(0,0)\right)}{2\left(a^3-ab^2\right)}=0,$$

$$d^4g_1^{(1)}(0)-d^4g_1^{(0)}(0)=\frac{\left(a\left(d\varphi_3(0)-d\varphi_3(l)+d^3\mu_2(0)\right)-a^2\left(d^2\varphi_2(0)-d^2\varphi_2(l)\right)\right)}{2\left(a^4-a^2b^2\right)}+$$

$$+\frac{\left(a^2+b^2\right)\int\limits_0^l d^2\varphi_3(y)\,dy-a^2b^2\int\limits_0^l d^4\varphi_1(y)\,dy-d^5\mu_0(0)+\int\limits_0^l\partial_tf(0,\xi)\,d\xi}{2\left(a^4-a^2b^2\right)}+$$
$$+\frac{\left(a^2b^2\left(d^4\varphi_0(0)-d^4\varphi_0(l)\right)+ab^2\left(d^3\varphi_1(l)-d^3\varphi_1(0)\right)\right)}{2\left(a^4-a^2b^2\right)}+$$
$$+\frac{\left(\left(\partial_t^4v_p(0,l)-\partial_t^4v_p(0,0)\right)+ab^2\left(\partial_x^3\partial_tv_p(0,l)-\partial_x^3\partial_tv_p(0,0)\right)\right)}{2\left(a^4-a^2b^2\right)}+$$
$$+\frac{a\left(\partial_x\partial_t^3v_p(0,0)-\partial_x\partial_t^3v_p(0,l)\right)-b^2\left(\partial_x^2\partial_t^2v_p(0,l)-\partial_x^2\partial_t^2v_p(0,0)\right)}{2\left(a^4-a^2b^2\right)}=0.$$

Similarly, we have $d^pg_2^{(1)}(l)-d^pg_2^{(0)}(l)=0, d^pg_4^{(1)}(l)-d^pg_4^{(0)}(l)=0$ and $d^pg_3^{(1)}(0)-d^pg_3^{(0)}(0)=0$ with $p=\overline{0,4}$.

Notice that from the Cauchy conditions for particular solution v_p, we obtain

$$\partial_t^4v_p(0,l)=f(0,l),\partial_t^4v_p(0,0)=f(0,0),$$
$$\partial_x^i\partial_t^jv_p(0,l)=\partial_x^i\partial_t^jv_p(0,0)=0,0\le i,j\le 3.$$

Last equalities are satisfied if and only if the homogeneous matching conditions (58) are satisfied.

Lemma 3. *For any number* $k\in\{0,1,2,...\}$ *functions* $g_i^{(k)}(z), i=\overline{1,4}$, can be represented in the forms

$$g_1^{(k)}(z)=\psi_1^{(k)}(z,a,b)+\frac{C_1b^2+aC_2z+C_3a-C_4z^2-C_5z-C_6}{2a\left(a^2-b^2\right)};$$
$$g_2^{(k)}(z)=\psi_2^{(k)}(z,a,b)+\frac{-C_1b^2+aC_2z+C_3a+C_4z^2+C_5z+C_6}{2a\left(a^2-b^2\right)};$$
$$g_3^{(k)}(z)=\psi_3^{(k)}(z,a,b)+\frac{-C_1a^2-bC_2z-C_3b+C_4z^2+C_5z+C_6}{2b\left(a^2-b^2\right)};$$

$$g_4^{(k)}(z) = \psi_4^{(k)}(z,a,b) + \frac{C_1a^2 - bC_2 z - C_3 b - C_4 z^2 - C_5 z - C_6}{2b\left(a^2-b^2\right)},$$

where the functions $\psi_i^{(k)},\ i=\overline{1,4}$ *, do not depend on the constants* C_1,C_2,C_3,C_4,C_5,C_6 .

Proof. The assertion of the lemma is proved for the function $g_i^{(k)}, i=\overline{1,4}$ by using method of mathematical induction. For the case $k=0$, this assertion follows from formula ((47)-(50)).

Suppose that the lemma is true for all $k=0,1,...,n-1$, we need prove its assertion for the function $g_i^{(n)}, i=\overline{1,4}$. According to formula (53), we obtain

$$\begin{gathered}
g_1^{(n)}(z) = \frac{1}{4a\left(a^2-b^2\right)}\Big(4a\left(a^2-b^2\right)\psi_1^{(n-1)}(l+z) + \\
+2C_1b^2 + 2aC_2(z+l) + 2C_3a - 2C_4(z+l)^2 - 2C_5(z+l) - 2C_6 - \\
-2a^4H\left(\frac{-z}{a}\right) - 2a^3b^2J\left(\frac{-z}{a}\right) + 2a^3K\left(\frac{-z}{a}\right) + 2a^4b^2L\left(\frac{-z}{a}\right) - 2b^2\int_0^l \varphi_1(\xi)d\xi + \\
+2a^4\int_0^{-z/a}\mu_2(\xi)\,d\xi + 2a^3b^2\int_0^{-z/a}\left(\frac{-z}{a}-\xi\right)\mu_3(\xi)\,d\xi - a^4\int_0^{-z/a}\left(\frac{-z}{a}-\xi\right)^2\mu_2''(\xi)\,d\xi - \\
-a^2b^2\int_0^{-z/a}\left(\frac{-z}{a}-\xi\right)^2\mu_2''(\xi)\,d\xi + a^2\int_0^{-z/a}\left(\frac{-z}{a}-\xi\right)^2\mu_0^{(4)}(\xi)\,d\xi + \\
+\int_0^l (z+l-y)\left(z\varphi_3(y) - 2a\varphi_2(y)\right)dy + \int_0^l \left((l-y)^2 + z(l-y)\right)\varphi_3(y)\,dy - \\
-2aC_2l + 2C_5l + 2C_4l^2 + 4C_4lz - 2\left(a^3+ab^2\right)\left(\varphi_0(0)-\varphi_0(l)\right) + \\
+2\left(a^2z+b^2z\right)\left(\varphi_1(0)-\varphi_1(l)\right) - 2\left(a^3z+ab^2z\right)\left(d\varphi_0(0)-d\varphi_0(l)\right) - \\
-\int_0^{-z/a}\int_0^l (z+a\xi)^2 f(\xi,x)\,dxd\xi + \left(a^2z^2+b^2z^2\right)\left(d\varphi_1(0)-d\varphi_1(l)\right)\Big) = \\
= \psi_1^{(n)}(z,a,b) + \frac{C_1b^2 + aC_2z + C_3a - C_4z^2 - C_5z - C_6}{2a\left(a^2-b^2\right)}.
\end{gathered}$$

Similarly for $g_i^{(n)}, i=2,3,4$.

Lemma 4. *For any number* $k,j,m,n\in\{0,1,2,...\}$ *, the sum* $g_1^{(k)}(x-at) + g_2^{(j)}(x+at) + \ + g_3^{(m)}(x-bt) + g_4^{(n)}(x+bt)$ *does not depend on* $C_i, i=\overline{1,6}$.

Proof. From Lemma 3, we obtain

$$\begin{gathered}
g_1^{(k)}(x-at) + g_2^{(j)}(x+at) + g_3^{(m)}(x-bt) + g_4^{(n)}(x+bt) = \\
= \psi_1^{(k)}(x-at,a,b) + \psi_2^{(j)}(x+at,a,b) + \psi_3^{(m)}(x-bt,a,b) + \psi_4^{(n)}(x+bt,a,b) + \\
+\frac{C_1b^2 + aC_2(x-at) + C_3a - C_4(x-at)^2 - C_5(x-at) - C_6}{2a\left(a^2-b^2\right)} + \\
+\frac{-C_1b^2 + aC_2(x+at) + C_3a + C_4(x+at)^2 + C_5(x+at) + C_6}{2a\left(a^2-b^2\right)} + \\
+\frac{-C_1a^2 - bC_2(x-bt) - C_3b + C_4(x-bt)^2 + C_5(x-bt) + C_6}{2b\left(a^2-b^2\right)} + \\
+\frac{C_1a^2 - bC_2(x+bt) - C_3b - C_4(x+bt)^2 - C_5(x+bt) - C_6}{2b\left(a^2-b^2\right)} = \\
= \psi_1^{(k)}(x-at,a,b) + \psi_2^{(j)}(x+at,a,b) + \psi_3^{(m)}(x-bt,a,b) + \psi_4^{(n)}(x+bt,a,b).
\end{gathered}$$

Theorem 4. *Assume that* $\varphi_j \in C^{5-j}([0,l])$, $j=\overline{0,3}$, $\mu_0 \in C^5([0,\infty))$, $\mu_i \in C^4([0,\infty))$, $i=1,2,3$, *and* $f \equiv 0$, *then the problem (18)-(20) has unique solution belonging to the class* $C^4(Q)$ *if and only if the following matching conditions hold:*

$$\int_0^l \varphi_j(y)\,dy = \mu_0^{(j)}(0), j=\overline{0,3},$$

$$\left(a^2+b^2\right)\int_0^l d^2\varphi_2(y)\,dy - a^2b^2\int_0^l d^4\varphi_0(y)\,dy - d^4\mu_0(0) = 0,$$

$$\left(a^2+b^2\right)\int_0^l d^2\varphi_3(y)\,dy - a^2b^2\int_0^l d^4\varphi_1(y)\,dy - d^5\mu_0(0) = 0,$$

$$d^i\varphi_j(l) - d^i\varphi_j(0) = d^j\mu_{i+1}(0),\ \ i=\overline{0,2}, j=\overline{0,3}.$$

Proof. The proof follows from the preceding arguments, Lemmas 2 and 3.

Theorem 5. *Assume that* $\varphi_j \in C^{5-j}([0,l])$, $j=\overline{0,3}$, $\mu_0 \in C^5([0,\infty))$, $\mu_i \in C^4([0,\infty))$, $i=1,2,3$ *and* $f \in C^2\left(\overline{Q}\right)$, *then the problem (18)-(20) has unique solution belonging to the class* $C^4(Q)$ *if and only if the following matching conditions hold:*

$$\int_0^l \varphi_j(y)\,dy = d^j\mu_0(0), j=\overline{0,3},$$

$$\left(a^2+b^2\right)\int_0^l d^2\varphi_2(y)\,dy - a^2b^2\int_0^l d^4\varphi_0(y)\,dy - d^4\mu_0(0) = -\int_0^l f(0,\xi)\,d\xi,$$

$$\left(a^2+b^2\right)\int_0^l d^2\varphi_3(y)\,dy - a^2b^2\int_0^l d^4\varphi_1(y)\,dy - d^5\mu_0(0) = -\int_0^l \partial_t f(0,\xi)\,d\xi,$$

$$d^i\varphi_j(l) - d^i\varphi_j(0) = d^j\mu_{i+1}(0),\ \ i=\overline{0,2}, j=\overline{0,3}, f(0,0)=f(0,l).$$

Proof. The proof follows from the preceding arguments, Lemmas 2 and 3, Theorem 3.

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A nonlocal problem with integral conditions for one-dimensional biwave equation

Summary

The main aim of this work is to consider classical solution of the nonlocal problem for a bi-wave equation with integral conditions of the first kind. The main goal is to show the method which allows to prove solvability of a nonlocal problem with integral conditions of the first kind. Under smoothness and matching conditions of the given functions, existence and uniqueness of the solution of the problem are proved. Moreover, making use of characteristics method, the analytical solution of the problem is provided as well.