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Numerical Study of a Linear Differential Game with Two Pursuers and One Evader*

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Abstract. A linear pursuit-evasion differential game with two pursuers and one evader is considered. The pursuers try to minimize the final miss (an ideal situation is to get exact capture), the evader counteracts them. Two case are investigated. In the first case, each one pursuer is dynamically stronger than the evader, in the second one, they are weaker. Results of numerical study of value function level sets (Lebesgue sets) for these cases are given. A method for constructing optimal feedback controls is suggested on the basis of switching lines. Results of numerical simulation are shown.

Keywords: pursuit-evasion differential game, linear dynamics, value function, optimal feedback control.

1. Introduction and Problem Formulation

1. In the paper, a model differential game with two pursuers and one evader is studied. Three inertial objects moves in the straight line. The dynamics descriptions for pursuers P_1 and P_2 are

$$\begin{array}{ll} \ddot{z}_{P_1} = a_{P_1}, & \ddot{z}_{P_2} = a_{P_2}, \\ \dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, & \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, \\ |u_1| \le \mu_1, & |u_2| \le \mu_2, \\ a_{P_1}(t_0) = 0, & a_{P_2}(t_0) = 0. \end{array} \tag{1}$$

Here, z_{P_1} and z_{P_2} are the geometric coordinates of the pursuers, a_{P_1} and a_{P_2} are their accelerations generated by the controls u_1 and u_2. The time constants l_{P_1} and l_{P_2} define how fast the controls affect the systems.

The dynamics of the evader E is similar:

$$\begin{array}{l} \ddot{z}_E = a_E, \;\; \dot{a}_E = (v - a_E)/l_E, \\ |v| \le \nu, \quad a_E(t_0) = 0. \end{array} \tag{2}$$

Let us fix some instants T_1 and T_2. At the instant T_1, the miss of the first pursuer with the respect to the evader is computed, and at the instant T_2, the miss of the

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second one is computed:

$$r_{P_1,E}(T_1) = |z_E(T_1) - z_{P_1,E}(T_1)|, \quad r_{P_2,E}(T_2) = |z_E(T_2) - z_{P_2,E}(T_2)|. \tag{3}$$

Assume that the pursuers act in coordination. This means that we can join them into one player P (which will be called the *first player*). This player governs the vector control $u = (u_1, u_2)$. The evader is counted as the *second player*. The result miss is the following value:

$$\varphi = \min\{r_{P_1,E}(T_1),\ r_{P_2,E}(T_2)\}. \tag{4}$$

At any instant t, all players know exact values of all state coordinates z_{P_1}, $\dot{z}_{P_1}$, a_{P_1}, z_{P_2}, $\dot{z}_{P_2}$, a_{P_2}, z_E, $\dot{z}_E$, a_E. The first player choosing its feedback control minimizes the miss φ, the second one maximizes it.

Relations (1)–(4) define a standard antagonistic differential game. One needs to construct the value function of this game and optimal strategies of the players.

2. Nowadays, there are a lot of publications dealing with differential games where one group of objects pursues another group; see, for example, the following works (in some order): (Stipanovic et al., 2009), (Blagodatskih and Petrov, 2009), (Chikrii, 1997), (Levchenkov and Pashkov, 1990), (Abramyantz and Maslov, 2004), (Pshenichnii, 1976), (Grigorenko, 1991), (Breakwell, 1976). The problem under consideration has two pursuers and one evader. So, from the point of view of number of objects, it is the simplest one. On the other hand, strict mathematical studies of problems "group-on-group" usually include quite strong assumptions for the dynamics of objects, dimension of the state vector and conditions of termination. Conversely, this paper considers the problem without any assumptions of this type. Solution of the problem can be interesting for the group differential games.

3. Now, let us describe a practical problem, whose reasonable simplification gives model game (1)–(4). Suppose that two pursuing objects attacks the evading one on collision courses. They can be rockets or aircrafts in the horizontal plane. A nominal motion of the first pursuer is chosen such that at the instant T_1 the exact capture occurs. In the same way, a nominal motion of the second pursuer is chosen (the capture is at the instant T_2). But indeed, the real positions of the objects differ from the nominal ones. Moreover, the evader using its control can change its trajectory in comparison with the nominal one (but not principally, without sharp turns). Correcting coordinated efforts of the pursuers are computed during the process by the feedback method to minimize the result miss, which is the minimum of absolute values of deviations at the instants T_1 and T_2 from the first and second pursuers, respectively, to the evader.

The passage from the original non-linear dynamics to a dynamics, which is linearized with the respect to the nominal motions, gives (Shima and Shinar, 2002), (Shinar and Shima, 2002) the problem under considerations.

4. The paper includes results of numerical study of game (1)–(4) for two marginal cases: 1) both pursuers P_1 and P_2 are dynamically stronger than the evader E; 2) both pursuers are dynamically weaker. Results for intermediate situations will be published in another work.

Difficulty of the solution is stipulated by the fact that the payoff function φ is not convex (even for the case $T_1 = T_2$). In the paper (Le Ménec, 2011), a case of "stronger" pursuers is considered and analytically methods are applied to the problem of solvability set construction in the game with zero result miss. For $T_1 = T_2$, an exact solution is obtained; if $T_1 \neq T_2$, then some upper approximation for the set is given. In general case, the exact analytical solution cannot be got, in the authors opinion.

The numerical study is based on algorithms and programs for solving linear differential games worked out in the Institute of Mathematics and Mechanics (Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russia). The central procedure is the backward constructing level sets (Lebesgue sets) of the value function. Optimal strategies of the players are constructed by some processing of the level sets.

2. Passage to Two-Dimensional Differential Game

At first, let us pass to relative geometric coordinates $y_1 = z_E - z_{P_1}$, $y_2 = z_E - z_{P_2}$ in dynamics (1), (2) and payoff function (4). After this, we have the following notations:

$$\begin{array}{ll} \ddot{y}_1 = a_E - a_{P_1} & \ddot{y}_2 = a_E - a_{P_2} \\ \dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1} & \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2} \\ \dot{a}_E = (v - a_E)/l_{P_1} & |u_2| \leq \mu_2 \\ |u_1| \leq \mu_1,\ |v| \leq \nu & \varphi = \min\{|y_1(T_1)|,\ |y_2(T_2)|\}. \end{array} \tag{5}$$

State variables of system (5) are y_1, $\dot{y}_1$, a_{P_1}, y_2, $\dot{y}_2$, a_{P_2}, a_E; u_1 and u_2 are controls of the first player; v is the control of the second one. The payoff function φ depends on the coordinate y_1 at the instant T_1 and on the coordinate y_2 at the instant T_2. From general point of view (existence of the value function, positional type of the optimal strategies), differential game (5) is a particular case of a differential game with a positional functional (Krasovskii and Krasovskii).

A standard approach, which is set forth in (Krasovskii and Subbotin, 1974) and (Krasovskii and Subbotin, 1988) for study linear differential games with fixed terminal instant and payoff function depending on some state coordinates at the terminal instant is to pass to new state coordinates. They can be treated as values of the target coordinates forecasted to the terminal instant under zero controls. In our situation, we have two instants T_1 and T_2, but coordinates computed at these instants are independent; namely, at the instant T_1, we should take into account $y_1(T_1)$ only, and at the instant T_2, we use the value $y_2(T_2)$. This fact allows us to use the mentioned approach when solving differential game (5). With that, we pass to new state coordinates x_1 and x_2, where $x_1(t)$ is the value of y_1 forecasted to the instant T_1, and $x_2(t)$ is the value of y_2 forecasted to the instant T_2.

The forecasted values are computed by formula

$$x_i = y_i + \dot{y}_i\tau_i + a_{P_i} l_{P_i}^2 h(\tau_i/l_{P_i}) + a_E l_E^2 h(\tau_i/l_E), \quad i = 1, 2. \tag{6}$$

Here, x_i, y_i, and $\dot{y}_i$ depends on t; $\tau_i = T_i - t$; $h(\alpha) = e^{-\alpha} + \alpha + 1$. Emphasize that the values τ_1 and τ_2 are connected to each other by the relation $\tau_1 - \tau_2 = \text{const} = T_1 - T_2$. One has $x_i(T_i) = y_i(T_i)$.

The dynamics in the new coordinates x_1, x_2 is the following (Le Ménec, 2011):

$$\begin{aligned}
&\dot{x}_1 = -l_{P_1} h(\tau_1/l_{P_1}) u_1 + l_E h(\tau_1/l_E) v,\\
&\dot{x}_2 = -l_{P_2} h(\tau_2/l_{P_2}) u_2 + l_E h(\tau_2/l_E) v,\\
&|u_1| \le \mu_1,\ |u_2| \le \mu_2,\ |v| \le \nu,\\
&\varphi\big(x_1(T_1), x_2(T_2)\big) = \min\{|x_1(T_1)|,\ |x_2(T_2)|\}.
\end{aligned} \tag{7}$$

The first player governs the controls u_1, u_2 and minimizes the payoff φ; the second one has the control v and maximizes φ.

Note that the control u_1 (u_2) affects only the horizontal (vertical) component $\dot{x}_1$ ($\dot{x}_2$) of the velocity vector $\dot{x} = (\dot{x}_1, \dot{x}_2)$. When $T_1 = T_2$, the second summand in dynamics (7) is the same for $\dot{x}_1$ and $\dot{x}_2$. Thus, the component of the velocity vector $\dot{x}$ depending on the second player control is directed at any instant t along the bisectrix of the first and third quadrants of the plane x_1, x_2. When $v = +\nu$, the angle between the axis x_1 and the velocity vector of the second player is 45°; when $v = -\nu$, the angle is 225°. This property simplifies the dynamics in comparison with the case $T_1 \ne T_2$.

Let $x = (x_1, x_2)$ and $V(t, x)$ be the value function at the position (t, x). For any $c \ge 0$, the value function level set

$$W_c = \big\{(t, x) : V(t, x) \le c\big\}$$

coincides with the maximal stable bridge (see (Krasovskii and Subbotin, 1974) and (Krasovskii and Subbotin, 1988)) built from the terminal set

$$M_c = \big\{(t, x) : t = T_1,\ |x_1| \le c;\ t = T_2,\ |x_2| \le c\big\}.$$

The set W_c can be treated as the solvability set for the considered game with the result not greater than c. When $c = 0$, one has the situation of the exact capture. The exact capture means equality to zero, at least, one of $x_1(T_1)$ and $x_2(T_2)$.

Comparing dynamics capabilities of each of pursuers P_1 and P_2 and the evader E, one can introduce parameters (Le Ménec, 2011) $\eta_i = \mu_i/\nu_i$ and $\varepsilon_i = l_E/l_{P_i}$, $i = 1, 2$. They define the shape of the maximal stable bridges in the individual games P_1 against E and P_2 against E.

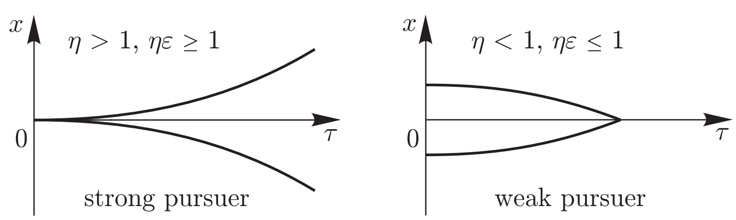

Fig. 1. Different variants of the stable bridges evolution in an individual game

Consider two cases: 1) $\eta_i > 1$, $\eta_i\varepsilon_i \ge 1$, $i = 1, 2$; 2) $\eta_i < 1$, $\eta_i\varepsilon_i \le 1$, $i = 1, 2$. In the first case, each of pursuers P_1 and P_2 is stronger than the evader E; in the second one, both pursuers are weaker. The maximal stable bridges in the individual games in the first case look as it is shown in Fig. 1 (at the left); the right subfigure in Fig. 1 gives the outline for the second case. The horizontal axis is the backward time τ, the vertical axis is the one-dimensional state variable x.

3. Level Sets of the Value Function

As it was mentioned above, a level set W_c of the value function is the maximal stable bridge for dynamics (7) built in the space t, x from the target set M_c. A time section (t-section) $W_c(t)$ of the bridge W_c at the instant t is a set in the plane of two-dimensional variable x.

To be definite, let $T_1 \geq T_2$. Then, for any $t \in (T_2, T_1]$, the set $W_c(t)$ is a vertical stripe around the axis x_2. Its width along the axis x_1 equals the width of the bridge in the individual game P_1–E at the instant $\tau = T_1 - t$ of the backward time. At the instant $t = T_1$, half-width of $W_c(T_1)$ is equal to c.

Denote by $W_c(T_2 + 0)$ the right limit of the set $W_c(t)$ as $t \to T_2 + 0$. Then, the set $W_c(T_2)$ is cross-like, obtained by union of the vertical stripe $W_c(T_2 + 0)$ and a horizontal one around the axis x_1 with the width equal $2c$ along the axis x_2.

When $t \leq T_2$, the backward construction of the sets $W_c(t)$ is made starting from the set $W_c(T_2)$.

The algorithm, which is suggested by the authors for constructing the approximating sets $\widetilde{W}_c(t)$, uses a time grid in the interval $[0, T_1]$: $t_N = T_1$, t_{N-1}, $\ldots$, $t_S = T_1$, t_{S-1}, t_{S-2}, $\ldots$. For any instant t_k from the taken grid, the set $\widetilde{W}_c(t_k)$ is built on the basis of the previous set $\widetilde{W}_c(t_{k+1})$ and a dynamics obtained from (7) by fixing its value at the instant t_{k+1}. So, dynamics (7), which varies in the interval $(t_i, t_{i+1}]$, is changed by a dynamics with simple motions (Isaacs, 1965). The set $\widetilde{W}_c(t_k)$ is treated as a collection of all positions at the instant t_k, where from the first player guarantees guiding the system to the set $\widetilde{W}_c(t_{k+1})$ under "frozen" dynamics (7) and discrimination of the second player, that is, when the second player announces its constant control v, $|v| \leq \nu$, in the interval $[t_i, t_{i+1}]$.

Due to symmetry of dynamics (7) and the sets $W_c(T_1)$, $W_c(T_2)$ with the respect to the origin, one gets that for any $t \leq T_1$ the t-section $W_c(t)$ is symmetric also.

3.1. Maximal Stable Bridges for the Case of Strong Pursuers

Simultaneous dynamic advantage of P_1 and P_2 with the respect to E implies that for any c, $W_c(\overline{t}) \subset W_c(\underline{t})$ if $\underline{t} < \overline{t}$. This means that the bridge W_c expands in the backward time. The latter allows to make independent constructions in all four quadrants. And due to the central symmetry, it is sufficient to make the constructions in the I and II quadrants only.

Let us give results of constructing t-sections $W_c(t)$ for the following values of game parameters:

$$\mu_1 = 2, \quad \mu_2 = 3, \quad \nu = 1,$$
$$l_{P_1} = 1/2, \; l_{P_2} = 1/0.857, \; l_E = 1.$$

Equal terminal instants. Let $T_1 = T_2 = 6$. Fig. 2 shows results of constructing the set W_0 (that is, with $c = 0$). In the figure, one can see several time sections $W_0(t)$ of this set. The bridge has a quite simple structure. At the initial instant $\tau = 0$ of the backward time (when $t = 6$), its section coincides with the target set M_0, which is the union of two coordinate axes. Further, at the instants $t = 4$, 2, 0, the cross thickens, and two triangles are added to it. The widths of the vertical and horizontal parts of the cross correspond to sizes of the maximal stable bridges in the individual games with the first and second pursuers. These triangles are located in the II and IV quadrants (where the signs of x_1 and x_2 are different, in other words, when the

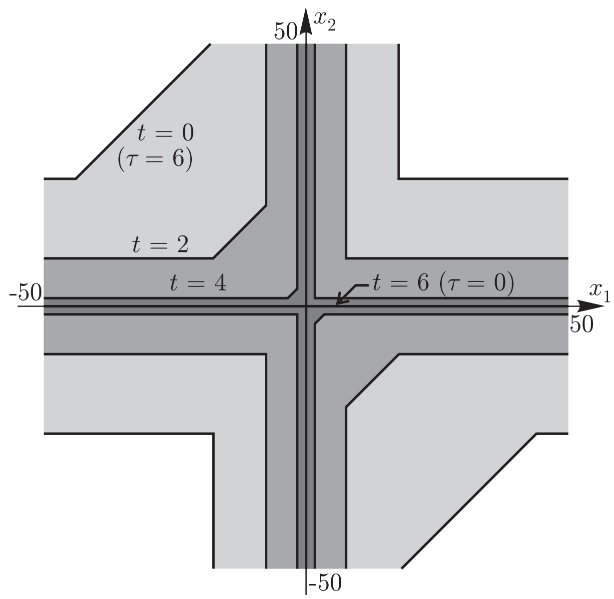

Fig. 2. Two strong pursuers, equal terminal instants: time sections of the bridge W_0

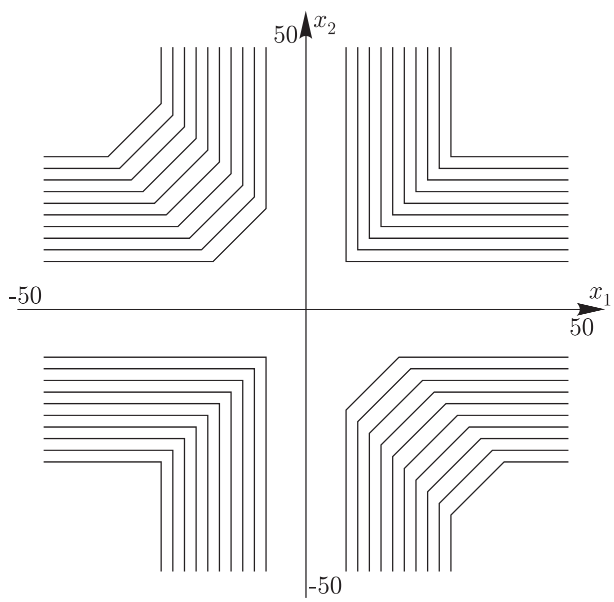

Fig. 3. Two strong pursuers, equal terminal instants: level sets of the value function, $t = 2$

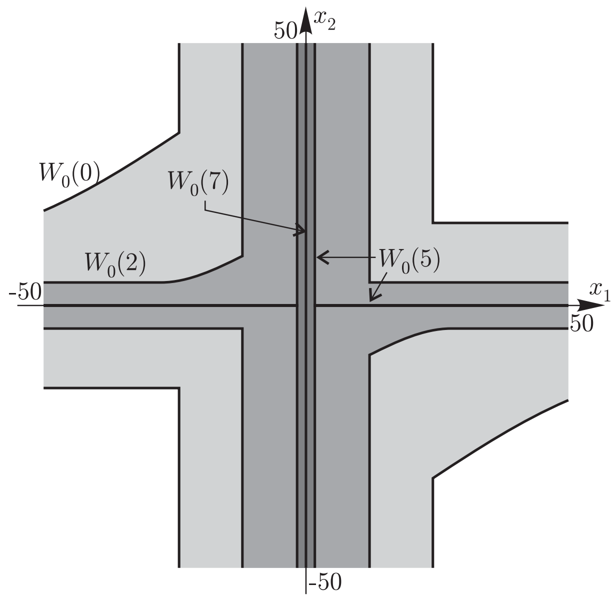

Fig. 4. Two strong pursuers, different terminal instants: time sections of the bridge W_0

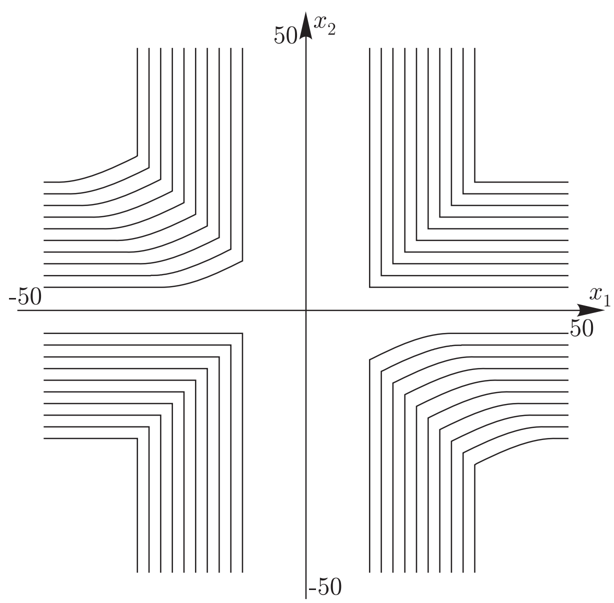

Fig. 5. Two strong pursuers, different terminal instants: level sets of the value function, $t = 2$

evader is between the pursuers) give the zone where the capture is possible only under collective actions of both pursuers.

Time sections $W_c(t)$ of other bridges W_c, $c > 0$, have a shape similar to $W_0(t)$. In Fig. 3, one can see the sections $W_c(t)$ at $t = 2$ ($\tau = 4$) for a collection $\{W_c\}$ corresponding to some series of values of the parameter c. For other instants t, the structure of the sections $W_c(t)$ is similar.

Different terminal instants. Let $T_1 = 7$, $T_2 = 5$. Results of construction of the set W_0 are given in Fig. 4. When $t < 5$, time sections $W_0(t)$ grow both horizontally and vertically; two additional triangles appear, but now they are curvilinear.

Total structure of the sections $W_c(t)$ at $t = 2$ is shown in Fig. 5.

3.2. Maximal Stable Bridges for the Case of Weak Pursuers

Now, we consider a variant of the game when both pursuers are weaker than the evader. Let us take the parameters

$$\mu_1 = 0.9, \quad \mu_2 = 0.8, \quad \nu = 1, \quad l_{P_1} = l_{P_2} = 1/0.7, \quad l_E = 1.$$

Let us show results for the case of different terminal instants only: $T_1 = 7$, $T_2 = 5$.

Since in this variant the evader is more maneuverable than the pursuers, they cannot guarantee the exact capture.

Fix some level of the miss, namely, $\bigl|x_1(T_1)\bigr| \le 2.0$, $\bigl|x_2(T_2)\bigr| \le 2.0$. Time sections $W_{2.0}(t)$ of the corresponding maximal stable bridge are shown in Fig. 6. The upper-left subfigure corresponds to the instant when the first player stops to pursue. The upper-right subfigure shows the picture for the instant, when the second pursuer finishes its pursuit. At this instant, the horizontal strip is added, which is a bit wider than the vertical one contracted during the passed period of the backward time. Then, the bridges contracts both in horizontal and vertical directions, and two additional curvilinear triangles appear (see middle-left subfigure). The middle-right subfigure gives the view of the section when the vertical strip collapses, and the lower-left subfigure shows the configuration just after the collapse of the horizontal strip. At this instant, the section loses connectivity and disjoins into two parts symmetrical with respect to the origin. Further, these parts continue to contract (as it can be seen in the lower-right subfigure) and finally disappear.

Time sections $\bigl\{W_c(t)\bigr\}$ are given in Fig. 7 at the instant $t = 0$ ($\tau_1 = 7$, $\tau_2 = 5$).

4. Optimal Feedback Control

Using knowledge of the value function provided by its level sets W_c, we can construct optimal strategies of the first and second players. Let us do it dividing the plane x_1, x_2 for every instant t to some cells. Inside each cell, the optimal control takes some extremal values.

Rewrite system (7) as

$$\dot{x} = \mathscr{D}_1(t)u_1 + \mathscr{D}_2(t)u_2 + \mathscr{E}(t)v,$$
$$|u_1| \le \mu_1,\ |u_2| \le \mu_2,\ |v| \le \nu.$$

Here, $x = (x_1, x_2)$; vectors $\mathscr{D}_1(t)$, $\mathscr{D}_2(t)$, and $\mathscr{E}(t)$ look like

$$\mathscr{D}_1(t) = \bigl(-l_{P_1}h((T_1 - t)/l_{P_1}),\, 0\bigr), \qquad \mathscr{D}_2(t) = \bigl(0,\, -l_{P_2}h((T_2 - t)/l_{P_2})\bigr),$$
$$\mathscr{E}(t) = \bigl(l_E h((T_1 - t)/l_E),\, l_E h((T_2 - t)/l_E)\bigr).$$

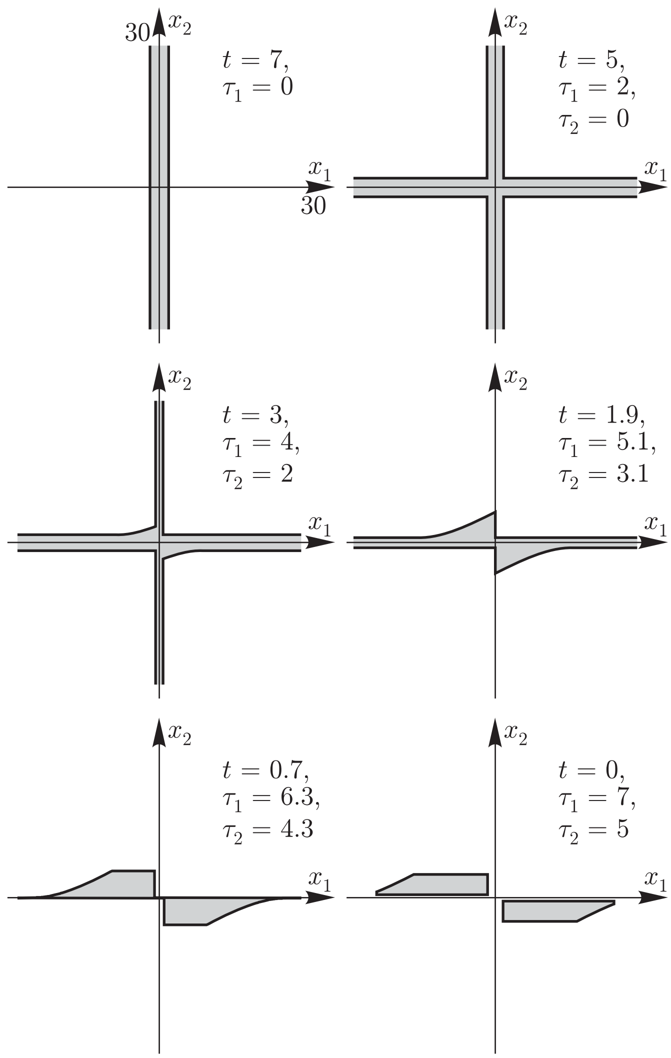

Fig. 6. Two weak pursuers, different termination instants: time sections of the maximal stable bridge $W_{2,0}$

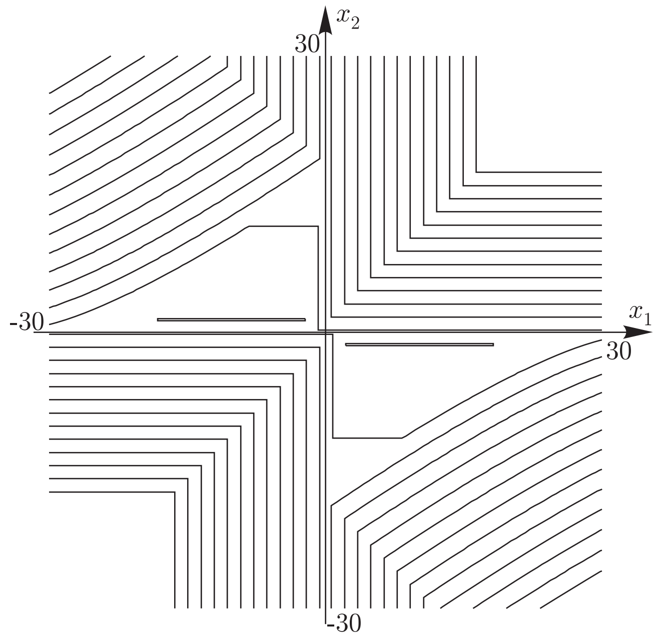

Fig. 7. Two weak pursuers, different terminal instants: level sets of the value function, $t = 0$

We see that the vector $\mathscr{D}_1(t)$ ($\mathscr{D}_2(t)$) is directed along the horizontal (vertical) axis; when $T_1 = T_2$, the angle between the axis x_1 and the vector $\mathscr{E}(t)$ equals 45°; when $T_1 \neq T_2$, the angle changes in time.

4.1. Switching Lines in the Case of Strong Pursuers

Feedback control of the first player. Analyzing the change of the value function along a horizontal line in the plane x_1, x_2 for a fixed instant t, one can conclude that the minimum of the function is reached in the segment of intersection of this line and the set $W_0(t)$. With that, the function is monotonic at both sides of the segment. For points at the right (at the left) from the segment, the control $u_1 = \mu_1$ ($u_1 = -\mu_1$) directs the vector $\mathscr{D}_1(t)u_1$ to the minimum.

Splitting the plane into horizontal lines and extracting for each line the segment of minimum of the value function, one can gather these segments into a set in the plane and draw a switching line through this set, which separates the plane into two parts at the instant t. At the right from this switching line, we choose the control $u_1 = \mu_1$, and at the left the control is $u_1 = -\mu_1$. On the switching line, the control u_1 can be arbitrary obeying the constraint $|u_1| \leq \mu_1$. The easiest way is to take the vertical axis x_2 as the switching line.

In the same way, using the vector $\mathscr{D}_2(t)$, we can conclude that the horizontal axis x_1 can be taken as the switching line for the control u_2.

Thus,

$$u_i^*(t,x) = \begin{cases} \mu, & \text{if } x_i > 0, \\ -\mu, & \text{if } x_i < 0, \\ \text{any } u_i \in [-\mu, \mu] & \text{if } x_i = 0. \end{cases} \tag{8}$$

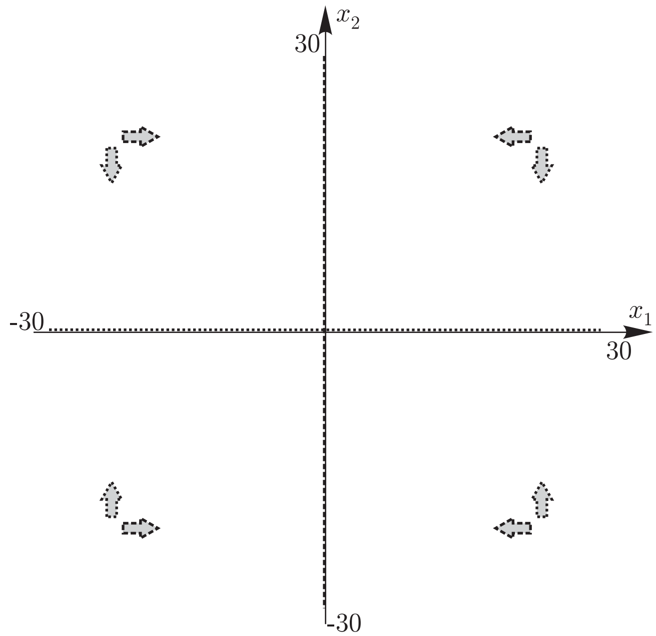

Fig. 8. Two strong pursuers, equal terminal instants: switching lines for the first player

The switching lines (the coordinate axes) at any t divide the plane x_1, x_2 into 4 cells. In each of these cells, the optimal control of the first player is constant. The synthesis of the first player optimal control is the same for all time instants and is shown in Fig. 8. Arrows denote the direction of the vectors $\mathscr{D}_i(t)u_i^*$, $i = 1, 2$.

Feedback control of the second player. For a fixed instant t, consider a split of the plane x_1, x_2 into lines parallel to the vector $\mathscr{E}(t)$. Take segments of local minimum and local maximum of the value function on all lines. One can easily see that for any line (except lines passing near the origin), there are two segments of local minimum and one of local maximum located between them. The segments of minimum appear by intersection of the line with the set $W_0(t)$. The segment of maximum for the case $T_1 = T_2$ coincides with the rectilinear part of the boundary of some set $W_c(t)$ and has slope angle equal to 45°. If $T_1 \neq T_2$, then the segment of maximum degenerates to a point coinciding with the corner point of a curvilinear triangle. For any point in the line outside all the segments, the control v is chosen in such a way that the vector $\mathscr{E}(t)v$ is oriented to the direction of growth of the value function. So, there are two parts of the line, where $v = \nu$, and two parts, where $v = -\nu$.

For a fixed instant t, the switching lines for the second player comprise of the coordinate axes and some line $\Pi(t)$, which passes through the middles of the segments of local minimum, if $T_1 = T_2$, and through the corner points of curvilinear triangles, if $T_1 \neq T_2$. An unpleasant peculiarity is that if $T_1 \neq T_2$, then one should take $v = \pm\nu$ in the switching line $\Pi(t)$; choices $|v| < \nu$ are not optimal.

Inside each of 6 cells, to which the plane is separated by the switching lines of the second player, the control is taken either $v = \nu$ or $v = -\nu$.

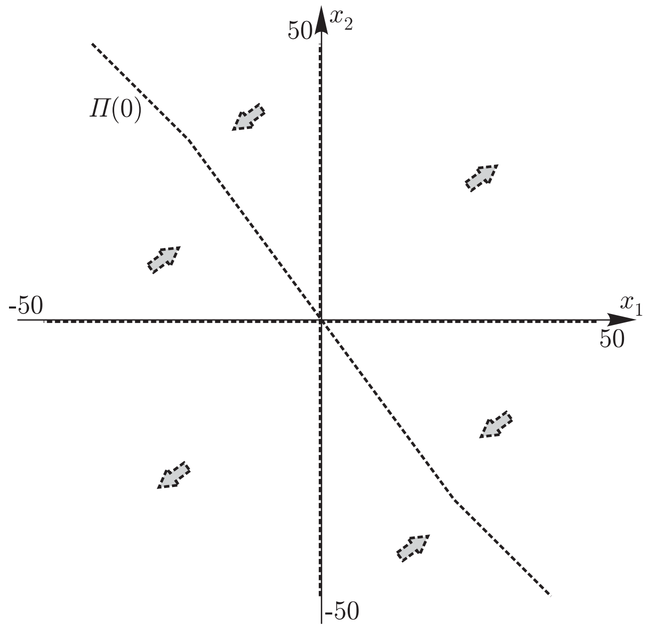

Fig. 9. Two strong pursuers, equal terminal instants: switching lines for the second player, $t = 0$

The second player optimal synthesis for the case $T_1 = 7$, $T_2 = 5$ is shown in Fig. 9 for $t = 0$. Arrows denote direction of the vectors $\mathscr{E}(t)v^*$.

4.2. Switching Lines in the Case of Weak Pursuers

In the case of pursuers weaker than the evader, the structure of the sets W_c is more complex in some neighborhood of the origin. This leads to more complicated shape of the switching lines both for the first and second players.

Switching lines of the first player are given in Fig. 10 at the instant $t = 0$ ($\tau_1 = 7$, $\tau_2 = 5$). The dashed line is the switching line for the component u_1; the dotted one is for the component u_2. The switching lines are obtained as a result of the analysis of the function $x \to V(t, x)$ in horizontal (for u_1 in accordance with the direction of the vector $\mathscr{D}_1(t)$) and vertical (for u_2 in accordance with the direction of the vector $\mathscr{D}_2(t)$) lines. If in the considered horizontal (vertical) line the minimum of the value function is attained in a segment, then the middle of such a segment is taken as a point for the switching line. Arrows show the directions of the vectors $\mathscr{D}_1(t)u_1^*$ and $\mathscr{D}_2(t)u_2^*$ in 4 cells.

In Fig. 11 switching lines and the directions of the vectors $\mathscr{E}(t)v^*$ are shown for $t = 0$. In this picture, we have 4 cells with constant values of the second player control.

4.3. Generating Feedback Controls. Discrete Scheme of Control

Switching lines are built as a result of processing the boundary of the sets $W_c(t)$. With that, some grid of instants t_k, where the t-sections $W_{c_j}(t_k)$ of the maximal stable bridges W_{c_j} are constructed by the backward procedure. The values c_j are also taken in some grid. For any instant t_k, approximating switching lines are stored as polygonal lines in the memory of a computer.

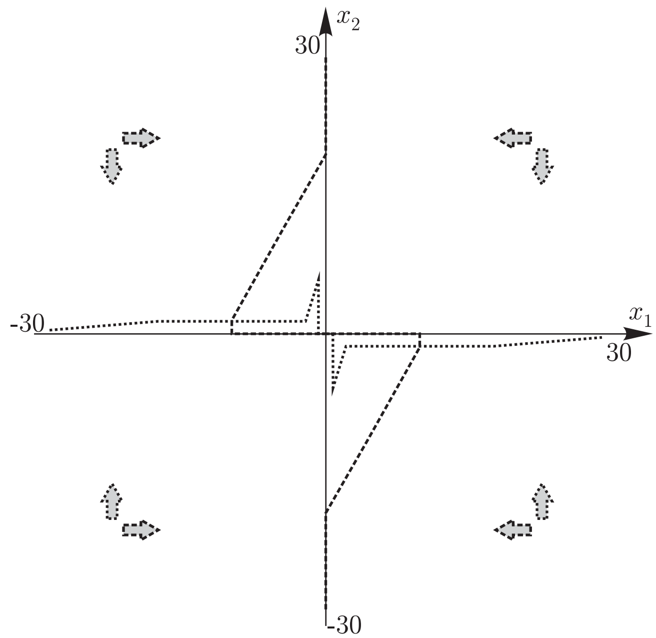

Fig. 10. Two weak pursuers, equal terminal instants: switching lines for the first player, $t = 0$

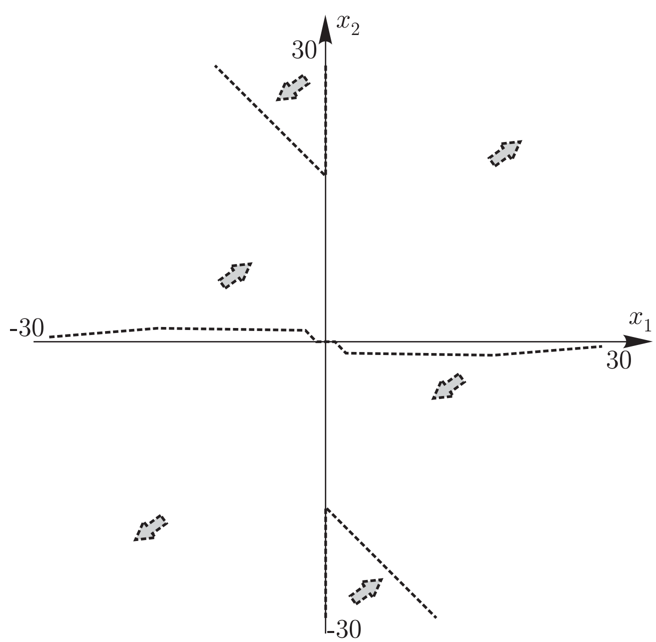

Fig. 11. Two weak pursuers, equal terminal instants: switching lines for the second player, $t = 0$

Having a position $x(t_k)$ at the instant t_k, it is possible to compute the controls $u_1^*\big(t_k, x(t_k)\big)$ and $u_2^*\big(t_k, x(t_k)\big)$ analyzing location of the point $x(t_k)$ with the respect to the switching lines for u_1 and u_2. The vectors $\mathscr{D}_1(t_k)$ and $\mathscr{D}_2(t_k)$ are used for this. In the case of strong pursuers, the axis x_2 is the switching line for the control u_1, and the axis x_1 is the switching line for the control u_2. The values of u_1^* and u_2^* are defined by formula (8). In the case of weak pursuers, the switching line is unique for each component u_i of the control too. Drawing a ray from the point $x(t_k)$ with the directing vector $\mathscr{D}_i(t_k)$, one can decide whether it crosses a switching line corresponding to the index i. If it does not, then $u_i^*\big(t_k, x(t_k)\big) = -\mu_i$; if it crosses, then $u_i^*\big(t_k, x(t_k)\big) = \mu_i$.

The first player control chosen at the instant t_k is kept until the instant t_{k+1}. At the position $\big(t_{k+1}, x(t_{k+1})\big)$, a new control value is chosen, etc. So, the feedback control generated by the switching lines is applied in a discrete control scheme (Krasovskii and Subbotin, 1974, Krasovskii and Subbotin, 1988).

To construct $v^*\big(t_k, x(t_k)\big)$ we use the vector $\mathscr{E}(t_k)$. Compute how many times (even or odd) a ray with the beginning at the point $x(t_k)$ and the directing vector $\mathscr{E}(t_k)$ crosses the second player switching lines. If the number of crosses is even (absence of crosses means that the number equals zero and is even), then we take $v^*\big(t_k, x(t_k)\big) = +\nu$; otherwise, $v^*\big(t_k, x(t_k)\big) = -\nu$. The chosen control is kept until the next instant t_{k+1}. In the position $\big(t_{k+1}, x(t_{k+1})\big)$, a new control is built, etc.

This synthesis for the first (second) player is suboptimal. Analysis of its closeness to an optimal one needs an additional study. Namely, it is necessary to show that under a coordinated choice of diameters Δt and Δc of grids in t and c, the feedback control of the first (second) player built on the basis of switching lines guarantees the limit of result as $\Delta t \to 0$ and $\Delta c \to 0$, which is not greater (not less) than $V(t_0, x_0)$ for any initial position (t_0, x_0). Such a study for linear differential games with convex t-sections $W_c(t)$ of maximal stable bridges is made in the works (Botkin and Patsko, 1982, Zarkh, 1990, Patsko, 2006). In the problem under consideration the sections $W_c(t)$ are not convex, and this fact preconditions the difficulty of this problem.

5. Simulation Results

Let the pursuers P_1, P_2, and the evader E move in the plane. At the initial instant $t_0 = 0$, velocities of all objects are parallel (Fig. 12) and sufficiently greater than the possible changes of the lateral velocity components. The instant of longitudinal coincidence of objects P_1 and E is T_1; the instant of coincidence of the objects P_2 and E is T_2. The dynamics of lateral motion is described by relations (1), (2); the resulting miss is given by formula (4).

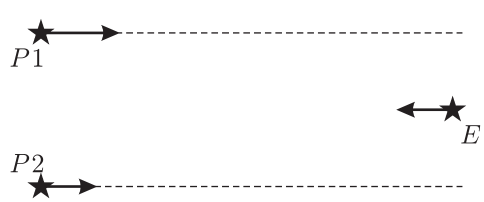

Fig. 12. Schematic initial positions of the pursuers and evader

In all following results, the initial lateral velocities and accelerations are assumed to be zero:

$$\dot{z}^0_{P_1} = z^0_{P_2} = \dot{z}^0_E = 0; \quad a^0_{P_1} = a^0_{P_2} = a^0_E = 0.$$

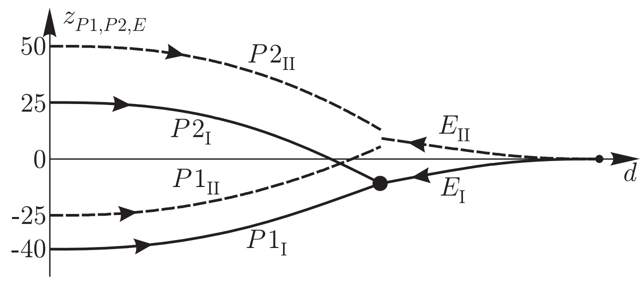

Fig. 13. Two strong pursuers, equal termination instants: trajectories in the original space

In Fig. 13, one can see the trajectories of the objects in the original space for the case of strong pursuers and equal terminal instants for the following game parameters:

$$\mu_1 = 2,\ \mu_2 = 3,\ \nu = 1,\ l_{P_1} = 1/2,\ l_{P_2} = 1/0.857,\ l_E = 1,\ T_1 = T_2 = 6.$$

The pursuers P_1, P_2, and the evader E act optimally. The trajectories drawn by solid lines correspond to the following initial data:

$$z^0_{P_1} = -40,\ z^0_{P_2} = 25,\ z^0_E = 0.$$

The dashed lines denote the trajectories for the following initial lateral parameters:

$$z^0_{P_1} = -25,\ z^0_{P_2} = 50,\ z^0_E = 0.$$

In the first case, the evader is successfully captured (at the terminal instant, the positions of both pursuers are the same as the position of the evader). In the second variant of initial positions, the evader escapes: at the terminal instant no one of the pursuers superposes with the evader. In this case, one can see as the evader aims itself to the middle between the terminal positions of the pursuers (this guarantees to him the maximum of the payoff function φ).

Figs. 14, 15, and 16 correspond to the case of weak pursuers and different terminal instants:

$$\mu_1 = 0.9,\ \mu_2 = 0.8,\ \nu = 1,\ l_{P_1} = l_{P_2} = 1/0.7,\ l_E = 1,\ T_1 = 7,\ T_2 = 5.$$

The initial positions are taken as follows:

$$z^0_{P_1} = -12,\ z^0_{P_2} = 12,\ z^0_E = 0.$$

Trajectories in Fig. 14 are built for the optimal controls of all objects. At the beginning of the pursuit, the evader closes to the first (lower) pursuer. It is done to increase the miss from the second (upper) pursuer at the instant T_2. Further closing is not reasonable, and the evader switches its control to increase the miss from the first pursuer at the instant T_1.

Fig. 15 gives the trajectories, when the pursuers use their optimal feedback controls generated by switching lines, but the evader applies a constant control $v \equiv \nu$ escaping from P_1 and ignoring P_2. In Fig. 16, the situation is given, when the evader, vice versa, keeps control $v \equiv -\nu$ escaping from P_2 and ignoring P_1. In both these situations, the payoff is less than in the case when the second player uses optimal control. When a constant control $v = +\nu$ is applied, the miss to the second pursuer at the instant T_2 is less; when the second player keeps $v = -\nu$, the miss to the first pursuer at the instant T_1 decreases.

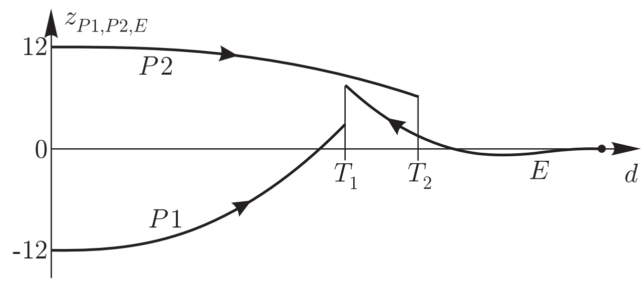

Fig. 14. Two weak pursuers, different termination instants: trajectories of the objects in the original space, optimal control of the second player

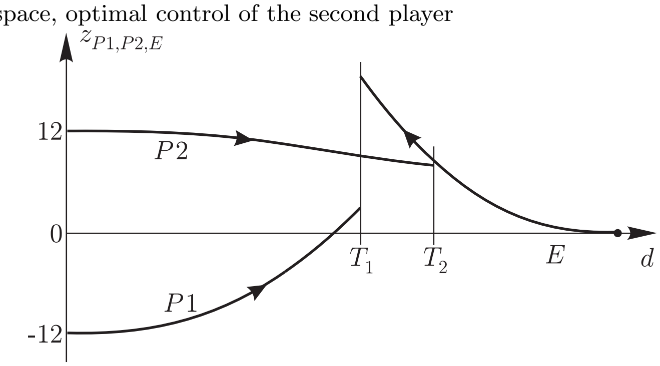

Fig. 15. Two weak pursuers, different termination instants: trajectories of the objects in the original space, constant control of the second player $v = +\nu$

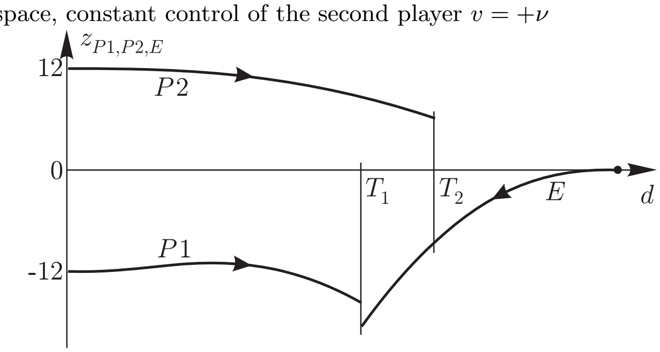

Fig. 16. Two weak pursuers, different termination instants: trajectories of the objects in the original space, constant control of the second player $v = -\nu$

6. Conclusion

A problem of pursuit-evasion with two pursuing objects and one evading object is considered as a two-dimensional antagonistic differential game. Difficulty of numerical solution of this problem is conditioned by the fact that the t-sections of the value function level sets are not convex. For two qualitatively different types of parameters ("strong" pursuers, "weak" pursuers), an analysis of the value function level sets is worked out in the paper. On the basis of this analysis, optimal strategies of players are built.

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