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A NEW GAP GROUP LIBRARY FOR IRREDUCIBLE MAXIMAL SOLVABLE SUBGROUPS OF PRIME DEGREE CLASSICAL GROUPS

ABSTRACT. These notes are concerned with the problem of creating new group libraries for the group theory computer system GAP (Groups, Algorithms, and Programming [1]). Our main objective is to develop efficient algorithms that produce a list of maximal solvable subgroups of special linear groups of prime degree over a finite field and to implement it as part of the GAP.

INTRODUCTION

Research into linear solvable groups was one of the first problems in the group theory. The early results in this field are due to K. Jordan. Subsequently various aspects of linear solvable group theory were studied by a number of mathematicians. Since the 40-s solvable matrix groups as well as closely related to them solvable permutation groups has been investigated by D. A. Suprunenko (see, for example, [2]). As a result of these studies, a detailed description of maximal solvable subgroups of $GL(n, P)$ over an arbitrary field P has been worked out. In particular, for the case of prime degree q, the complete classification of maximal solvable subgroups of $GL(n, P)$ was obtained up to conjugacy, where P is an algebraically closed field or a finite field ([2]).

Although the problem of linear solvable group classification has been a subject of research over a number of years, at present the complete classification is available only for some small degree classical groups, except for the mentioned above case of $GL(q, P)$. In particular, in 1992 M. Short suggested classification of irreducible solvable subgroups of $GL(n, p)$, where $p^n < 256$ ([3]). This classification is included in GAP as the group library *IrredSol* (such functions as AllIrreducibleSolvable-Groups (<fun1>, <val1>, ...) etc.) M. Short has developed algorithms

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that take as input a positive integer n and a prime p ($p^n < 256$) and produce a list of irreducible solvable subgroups of $GL(n, p)$.

In [4, 5] a complete classification of irreducible maximal solvable subgroups of prime degree classical groups was obtained. It seems to be helpful to provide electronic access to this classification.

The main purpose of the present study is to discuss an approach to the solution of this problem. The essence of the approach is to give a detailed description of matrices generating each of conjugacy classes representatives similar to what is done in Short's classification.

Only the simplest case of special linear group $SL(q, k)$ will be considered here. Note that although the case of isometry groups is more complicated, the main features of the problem can be illustrated in the case of $SL(q, k)$.

Let $G = SL(q, k)$, q be a prime, $k = GF(p^l)$, $\alpha \in k^*$ be an element of order $p^l - 1$. Denote by M_q the set of all irreducible maximal solvable subgroups of G. Clearly, the set M_q is the union of three subsets M_{q1}, M_{q2}, M_{q3}, where M_{q1} contains all imprimitive (i.e., monomial) subgroups of M_q and M_{q2}, M_{q3} consist of primitive subgroups with irreducible and scalar maximal abelian normal subgroup respectively.

Theorem ([4, 5]). *Every irreducible maximal solvable subgroups of G is conjugate in G to one and only one of the following groups:*

	Group	Order	Comment
$q = 2$	H_{21}	$2(p^l - 1)$	$p^l \neq 2, 3, 5, 7, 9$
	H_{22}	$2(p^l + 1)$	$p^l \neq 3, 7$
	H_{23}	$2^3 \cdot 3$	$p^l \equiv \pm 3 \pmod 8$
	$s_m H_{23} s_m^{-1}$	$2^3 \cdot 6$	$p^l \equiv \pm 1 \pmod 8$, $m = 0, 1$
$q = 3$	H_{31}	$6(p^l - 1)^2$	$p^l \neq 2, 4$
	H_{32}	$3(p^{2l} + p^l + 1)$	
	H_{33}	$3^3 \cdot 2^3$	$p^l \equiv 1 \pmod 3$, $p^l \not\equiv 1 \pmod 9$
	$s_m H_{33} s_m^{-1}$	$3^3 \cdot 24$	$p^l \equiv 1 \pmod 9$, $m = \overline{0, 2}$

$q>3$	H_{q1}	$q(q-1)(p^l-1)^{q-1}$	$p^l \neq 2$
	H_{q2}	$q(p^{ql}-1)(p^l-1)^{-1}$	
	$s_m H_{q3} s_m^{-1}$	$2q^3(q-1)$	$p^l \equiv 1 \pmod q$, $m=\overline{0,q-1}, q \neq 5,7$
	$s_m H_{q4} s_m^{-1}$	$2q^3(q+1)$	$p^l \equiv 1 \pmod q$, $m=\overline{0,q-1}, q \neq 7$
	$s_m H_{q5} s_m^{-1}$	$q^3 \cdot 24 if$ $q \equiv \pm 3 \pmod 8$; $q^3 \cdot 48 if$ $q \equiv \pm 1 \pmod 8$	$p^l \equiv 1 \pmod q$, $m=\overline{0,q-1}$;
	$s_m H_{q6} s_m^{-1}$	$q^3 \cdot 48$	$q \equiv \pm 1 \pmod 8$, $p^l \equiv 1 \pmod q$, $m=\overline{0,q-1}$.

Here $H_{q1} \in M_{q1}$ is a monomial subgroup, $H_{q2} \in M_{q2}$ is the normalizer of the unique up to conjugacy irreducible maximal abelian subgroup of G and $H_{qi}, s_m H_{qi} s_m^{-1}$ are subgroups of M_{q3}, $i=\overline{3,6}$ ($s_m = diag(1,\ldots 1,\alpha^m)$). Detailed description of the groups H_{qi} will be given below. All subgroups listed in the table are indeed maximal solvable except for the cases mentioned in the column 'comment'. Note that although classification of the set M_q subgroups is similar to the well-known classification of irreducible maximal solvable subgroups of $GL(q,k)$ (see [2]), it does not follow in a simple way from the case $GL(q,k)$.

In these notes subgroups H_{qi} will be described using a consistent polycyclic presentation. A polycyclic presentation for a group H is a presentation $\{X,R\}$, where $X=\{a_1,\ldots a_n\}$ is a generating set for H, and the relations in R are of the following kind:

$$a_i^{r(i)} = \Pi_{k=i+1}^n a_k^{\alpha(i,i,k)} \quad 1 \leq i \leq n; \tag{1}$$

$$a_i^{-1} a_j a_i = \Pi_{k=i+1}^n a_k^{\alpha(i,j,k)} \quad 1 \leq i < j \leq n. \tag{2}$$

The set X is called a polycyclic generating sequence for H. We will denote $a^{-1}ba$ by b^a for each $a,b \in G$. The presentation $\{X,R\}$ is called consistent if $|H| = \Pi_{i=1}^n r(i)$ ([3]). Let us consider each of the subgroups H_{qi}.

1. Imprimitive maximal solvable subgroups

Let L_q be the transitive maximal solvable subgroup of the symmetric group S_q of degree q.

Proposition 1. *A subgroup $H \subset G$ is contained in M_{q1} if and only if H is conjugate in G to the subgroup $H_{q1} = (k^* wr L_q) \cap G$ except for the cases $q = 2$, $p^l = 2, 3, 5, 7, 9$ and $q = 3$, $p^l = 2, 4$ then M_{q1} is empty.*

1.1. Let $q > 2$, β be a positive integer such that multiplicative order of β modulo q is equal to $q - 1$. Define the monomial matrix $a = [a_{ij}]$, $1 \le i, j \le q$, as follows: $a_{qq} = -1$, $a_{x(i)i} = -1$, where $x(i)$ $(i = \overline{1, q-1})$ are satisfy the following system of equations:

$$\begin{cases} x(i) + 1 = x(i+\beta), & \overline{1, q-\beta-1} \\ x(q-\beta) + 1 = q & \\ x(i) + 1 = x(i+\beta-q), & \overline{q-\beta+1, q-1} \end{cases}$$

and $a_{ij} = 0$ for all the other values of i and j.

Let $b = \begin{pmatrix} 0 & 1 \\ E_{q-1} & 0 \end{pmatrix}$, $c_i = \operatorname{diag}(\underbrace{1, \ldots 1}_{i-1}, \alpha, 1, \ldots \alpha^{-1})$, $i = \overline{1, q-1}$.

Then a polycyclic presentation for H_{q1} is

$$\begin{aligned} < a, b, c_1, \ldots c_{q-1} \mid\ & a^{q-1} = E_q, \\ & b^a = b^\beta,\ b^q = E_q, \\ & c_1^a = c_{x(1)},\ c_1^b = c_{q-1}^{-1},\ c_1^{p^l-1} = E_q, \\ & c_i^a = c_{x(i)},\ c_i^b = c_{i-1}c_{q-1}^{-1},\ c_i^{c_j} = c_i, \\ & c_i^{p^l-1} = E_q \ (1 \le j < i \le q-1) > . \end{aligned}$$

1.2. Let $q = 2$.

1.2.1. $p = 2$. Then

$$\begin{aligned} H_{21} = < a, b \mid\ & a^2 = E_2, \\ & b^a = b^{-1},\ b^{p^l-1} = E_2 >, \end{aligned}$$

where $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b = \operatorname{diag}(\alpha, \alpha^{-1})$.

1.2.2. $p \neq 2$. Let m be a positive integer such that 2^m divides $p^l - 1$ and 2^{m+1} does not divide $p^l - 1$.

Then a polycyclic presentation for H_{21} is

$$\begin{aligned} < a, b, c \mid\ & a^2 = c^{2^{m-1}}, \\ & b^a = b^{-1},\ b^{(p^l-1)2^{-m}} = E_2, \\ & c^a = c^{-1},\ c^b = c,\ c^{2^m} = E_2 >, \end{aligned}$$

where $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $b = \mathrm{diag}(\alpha^{2^m}, \alpha^{-2^m})$, $b = \mathrm{diag}(\alpha^{(p^l-1)2^{-m}}, \alpha^{-(p^l-1)2^{-m}})$. In particular, if $p^l \equiv 3 \pmod 4$ then $b = \mathrm{diag}(\alpha^2, \alpha^{-2})$, $c = -E_2$.

2. The normalizer of a Singer cycle

The set M_{q2} consists of normalizers of irreducible maximal abelian subgroups of G.

Proposition 2. *Let B be a maximal irreducible abelian subgroup of G. Subgroup $H \subset G$ is contained in M_{q2} if and only if H is conjugate in G to the group $H_{q2} = N_G(B)$ except for the cases $q = 2, p^l = 4, 7$ then M_{q2} is empty.*

To describe matrices generating H_{q2} we need the following notation. Let $K = GF(p^{ql})$, $\sigma : x \to x^{p^l}$ be the k-automorphism of K (i.e. $<\sigma> = Gal(K/k)$). Denote by d_ρ matrix $\mathrm{diag}(\rho, \sigma(\rho), \dots \sigma^{q-1}(\rho))$, where $\rho \in K^*$ is an element of order $(p^{ql}-1)(p^l-1)^{-1}$. Let $\alpha_1, \dots \alpha_q$ be a basis of the extension K/k, $h = [\sigma^{j-1}(\alpha_i)] \in GL(q, K)$, $I_q = \begin{pmatrix} 0 & 1 \\ E_{q-1} & 0 \end{pmatrix}$. If $\beta_1, \dots \beta_q$ is the basis dual to $\alpha_1, \dots \alpha_q$, i.e. $\mathrm{tr}(\alpha_i\beta_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ then

$$hI_qh^{-1} = [\mathrm{tr}(\alpha_i\sigma(\beta_j))] \in GL(q, k) \tag{3}$$

and

$$hd_\rho h^{-1} = [\mathrm{tr}(\alpha_i\beta_j\rho)] \in GL(q, k). \tag{4}$$

2.1. Let $q > 2$. Then a polycyclic presentation for H_{q2} is
$< a, b \mid a^q = E_q,$
$b^a = b^{p^l},\ b^{(p^{ql}-1)(p^l-1)^{-1}} = E_q >$.

It is obvious that one can take matrices hI_qh^{-1}, $hd_\rho h^{-1}$ as a, b respectively and, by varying the basis $\alpha_1, \dots \alpha_q$, we will obtain different matrix representations for elements a, b. In what follows we consider in more detail some special cases.

2.1.1. Let $p^l \equiv 1 \pmod q$, $\varepsilon \in k^*$ be an element of order q and δ be a root of the polynomial $x^q - \alpha$. We take the polynomial basis $1, \delta, \dots \delta^{q-1}$ as $\alpha_1, \dots \alpha_q$. Then $q^{-1}, (q\delta)^{-1}, \dots, (q\delta)^{1-q}$ is the dual basis and it follows from (1), (2) that

$$a = \mathrm{diag}(1, \varepsilon, \dots \varepsilon^{q-1})$$
$$b = [q^{-1}\mathrm{tr}(\delta^{i-j}\rho)], 1 \le i, j \le q.$$

2.1.2. Let $q = p$ and δ be a root of an irreducible polynomial $x^q - x - \gamma$. We take the polynomial basis $1, \delta, \dots \delta^{q-1}$ as $\alpha_1, \dots \alpha_q$. Then $1 - \delta^{q-1}, -\delta^{q-2}, \dots - \delta, -1$ is the dual basis and it follows from (1), (2) that $a = [a_{ij}]$, where

$$a_{ij} = \begin{cases} -\mathrm{tr}(\delta^{i-1}(\delta+1)^{q-j}, & j > 1 \\ \mathrm{tr}(\delta^{i-1}) - \mathrm{tr}(\delta^{i-1}(\delta+1)^{q-1}, & j = 1 \end{cases}$$

and $b = [b_{ij}]$, where

$$b_{ij} = \begin{cases} -\mathrm{tr}(\delta^{q+i-j-1}\rho), & j > 1 \\ \mathrm{tr}(\delta^{i-1}\rho) - \mathrm{tr}(\delta^{q+i-2}\rho), & j = 1. \end{cases}$$

These expressions can be simplified. In particular, a_{ij} satisfy the following system of equations over $GF(q)$:

$$\begin{cases} a_{ij} = 0, & 1 \le i < j \le q \\ a_{ii} = 1, & 1 \le i \le q \\ a_{ij} = a_{i-1j} + a_{i-1j-1}, & 1 < j \le i \le q, \end{cases}$$

(i.e., nonzero elements of a form the Pascal triangle over $GF(q)$). For example, for $q = 5$

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 1 & 4 & 1 \end{pmatrix}.$$

2.1.3. Let $p \ne q$ and $p^l \not\equiv 1 \pmod q$. Take, e.g., the polynomial basis $1, \rho, \dots \rho^{q-1}$ as $\alpha_1, \dots \alpha_q$. Then $a = hI_qh^{-1}$, where $h = [\rho^{p^{(j-1)l}\cdot(i-1)}]$, $1 \le i, j \le q$ and

$$b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ \mathrm{tr}(\rho^q\beta_1) & \mathrm{tr}(\rho^q\beta_2) & \mathrm{tr}(\rho^q\beta_3) & \dots & \mathrm{tr}(\rho^q\beta_q) \end{pmatrix}.$$

2.2. Let $q = 2$.

2.2.1. Let $p^l \equiv 1 \pmod 4$, i.e. k^* contains an element i of order 4. Then a polycyclic presentation for H_{22} is

$$< a, b \mid a^4 = E_2,$$
$$b^a = b^{p^l},\ b^{(p^l+1)2^{-1}} = E_2 > .$$

If δ is a root of $x^2 - \alpha$ then as well as in 2.1.1. $a = \mathrm{diag}(i, -i)$ and $b = 2^{-1}\begin{pmatrix} \mathrm{tr}\rho^2 & \mathrm{tr}(\delta^{-1}\rho^2) \\ \mathrm{tr}(\delta\rho^2) & \mathrm{tr}(\rho^2) \end{pmatrix}$.

2.2.2. Let $p^l \equiv 3 \pmod 4$. Then the polynomial $x^2 + 1$ is irreducible over k and $1, i$ is a basis of the extension K/k. Denote by γ, θ elements of k such that $\gamma^2 + \theta^2 = -1$. Let n be the positive integer such that $2^n | p^l + 1,\ 2^{n+1} \not| p^l + 1$. Define $\rho_j,\ j = 1, 2$ as follows: $\rho_1 = \rho^{2^n},\ \rho_2 = \rho^{(p^l+1)2^{-n}}$. Then setting in (2) $\delta = i$ we find

$$H_{22} =< a, b_1, b_2 \mid a^2 = b_2^{2^{n-1}},$$
$$b_1^a = b_1^{p^l},\ b_1^{(p^l+1)2^{-n}} = E_2,$$
$$b_2^a = b_2^{p^l},\ b_2^{b_1} = b_2,\ b_2^{2^n} = E_2 >,$$

where $a = \begin{pmatrix} \gamma & \theta \\ \theta & -\gamma \end{pmatrix}$, $b_j = 2^{-1}\begin{pmatrix} \mathrm{tr}\rho_j & -\mathrm{tr}i\rho_j \\ \mathrm{tr}i\rho_j & \mathrm{tr}\rho_j \end{pmatrix}$, j = 1,2.

2.2.3. Let $q = p = 2$, δ be a root of an irreducible polynomial $x^2 - x - \gamma$. Then similar to 2.1.2., a polycyclic presentation for H_{22} is

$$< a, b \mid a^2 = E_2,$$
$$b^a = b^{p^l},\ b^{p^l+1} = E_2 >,$$

where $a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} \mathrm{tr}(1-\delta)\rho & -\mathrm{tr}\rho \\ \mathrm{tr}(\delta(1-\delta)\rho) & -\mathrm{tr}(\delta\rho) \end{pmatrix}$.

3. Maximal primitive solvable subgroups with a scalar maximal abelian normal subgroup

The structure of subgroups which are contained in M_{q3} is more complicated than for the case of the sets M_{q1}, M_{q2}. Similar to the case of $GL(q, k)$, description of subgroups of M_{q3} can be reduced to the well-known classification of irreducible maximal solvable subgroups of $SL(2, q)$, i.e. to subgroups of M_2. Generating elements for groups of M_{21}, M_{22} were determined in 1.2 and 2.2. The set M_{23} was considered in detail in [2, 3]. We present these results here to create more detailed picture.

3.1. Let $q = 2$, $i \in k$ be an element of order 4, $s_1 = \mathrm{diag}(1, \alpha)$. If $p^l \equiv \pm 1 \pmod 8$ let τ be an element of k^* such that $\tau^2 = 2$.

3.1.1. Let $p^l \equiv 1 \pmod 4$, $a = i\tau^{-1}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $b = (i+1)2^{-1}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$, $u = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $v = \mathrm{diag}(i, -i)$, $z = -E_2$.

If $p^l \equiv 1 \pmod 8$ then according to [3] a polycyclic presentation for H_{23} is

$$\begin{aligned} < a, b, u, v, z \mid\ & a^2 = z, \\ & b^a = b^2,\ b^3 = E_2, \\ & u^a = zv,\ u^b = zv,\ u^2 = z, \\ & v^a = zu,\ v^b = uv,\ v^u = zv,\ v^2 = z, \\ & z^a = z,\ z^b = z,\ z^u = z,\ z^v = z,\ z^2 = E_2 > . \end{aligned}$$

If $p^l \equiv -3 \pmod 8$ then $H_{23} =< b, u, v >$. The group H_{24} exists only if $p^l \equiv 1 \pmod 8$ and $H_{24} = s_1 H_{23} s_1^{-1}$.

3.1.2. Let $p^l \not\equiv 1 \pmod 4$, γ, θ be the same as in 2.2.2. Define matrices a, b, u, v, z as follows: $a = \tau^{-1}\begin{pmatrix} \gamma & \theta+1 \\ \theta-1 & -\gamma \end{pmatrix}$, $b = 2^{-1}\begin{pmatrix} \gamma-\theta-1 & \gamma+\theta-1 \\ \gamma+\theta+1 & -\gamma+\theta-1 \end{pmatrix}$, $u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $v = \begin{pmatrix} \gamma & \theta \\ \theta & -\gamma \end{pmatrix}$, $z = -E_2$.

If $p^l \equiv -1 \pmod 8$ then it follows from [3] that

$$\begin{aligned} H_{23} =< a, b, u, v, z \mid\ & a^2 = z, \\ & b^a = b^2,\ b^3 = E_3, \\ & u^a = vz,\ u^b = v,\ u^2 = z, \\ & v^a = uz,\ v^b = uv,\ v^u = vz,\ v^2 = z, \\ & z^a = z,\ z^b = z,\ z^u = z,\ z^v = z,\ z^2 = E_2 > . \end{aligned}$$

If $p^l \equiv 3 \pmod 8$ then $H_{23} =< b, u, v >$. The group H_{24} exists only if $p^l \equiv -1 \pmod 8$ and $H_{24} = s_1 H_{23} s_1^{-1}$.

3.2. Let $q > 2$. Note that M_{q3} is not empty if and only if $p^l \equiv 1 \pmod q$, i.e. k^* contains an element ε of order q. Let

$$a = \mathrm{diag}(1, 1, \varepsilon, \varepsilon^3, \ldots \varepsilon^{\frac{(i-1)(i-2)}{2}} \ldots \varepsilon^{\frac{(q-1)(q-2)}{2}}),$$

$$b = \delta^{-1}\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \varepsilon & \ldots & \varepsilon^{q-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \varepsilon^{q-1} & \ldots & \varepsilon^{(q-1)^2} \end{pmatrix},$$

where

$$\delta^2 = \begin{cases} q, & \text{if } q \equiv 1 \pmod 4 \\ -q, & \text{if } q \equiv 3 \pmod 4), \end{cases},$$

$$u = \begin{pmatrix} 0 & 1 \\ E_{q-1} & 0 \end{pmatrix}, \quad v = \mathrm{diag}(1, \varepsilon, \dots \varepsilon^{q-1}).$$

Denote by N the normalizer in $GL(q,k)$ of the subgroup $A =< u, v >$. It is well known that for each $h \in N$ the following holds:

$h^{-1}uh = \mu u^{\alpha} v^{\beta}$, $h^{-1}vh = \lambda u^{\gamma} v^{\delta}$, where $\mu, \lambda \in k^*$ and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,q).$$

Define the homomorphism φ as follows: $\varphi : N \to SL(2,q), \quad h \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Let $i_q = \begin{cases} 1, q = 3 \\ 3, q \equiv \pm 3 \pmod 8 \\ 4, q \equiv \pm 1 \pmod 8 \end{cases}$, $s_m = \mathrm{diag}(\underbrace{1, \dots 1}_{q-1}, \alpha^m)$, $m = \overline{0, q-1}$, H_{qi} be the subgroup of G such that $H_{qi} \triangleright A$ and $H_{qi}/A \cong H_{2i-2}$, $i = \overline{3, 2+i_q}$. One can prove the existence of such subgroups for each $i = \overline{3, 2+i_q}$ except for the case $q = 3, p^l \not\equiv 1 \pmod 9$. In this case we define H_{33} as a subgroup of G such that $H_{33} \triangleright A$ and H_{33}/A is isomorphic to the Sylow 2-subgroup of $SL(2,3)$.

Proposition 3. *Every subgroup $H \in M_{q3}$ is conjugate in G to one and only one of subgroups $s_m H_{qi} s_m^{-1}$, $m = \overline{0, q-1}, i = \overline{3, i_q + 2}$, except for the case $q = 3, p^l \not\equiv 1 \pmod 9$ then H is conjugate in G to H_{33}.*

Note that for some q, i groups $H_{qi}, i \geq 3$, are not maximal solvable subgroups of G. Table 1 contains the complete list of exceptions.

We now proceed to finding generating elements for H_{qi}. Since $a^{-1}ua = uv^{-1}, a^{-1}va = v$ and $b^{-1}ub = v^{-1}, b^{-1}vb = u$ then $\varphi(a) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\varphi(b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is clear that $\varphi(a), \varphi(b)$ are generators for $SL(2,q)$ and, therefore, the following approach to determining generators for $H_{qi} \in M_{q3}$ is suggested here.

Let $< a_1, \dots a_n >$ be a polycyclic generating set for $H_{2i-2} \cong H_{qi}/A$, where $a_j = \begin{pmatrix} a_{11}^{(j)} & a_{12}^{(j)} \\ a_{21}^{(j)} & a_{22}^{(j)} \end{pmatrix} \in SL(2,q)$. Without loss of generality we can take $a_{11}^{(j)} \neq 0$. Then $a_j = \begin{pmatrix} 1 & 0 \\ y_j & 1 \end{pmatrix} \begin{pmatrix} 1 & z_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_j & 1 \end{pmatrix} = \varphi(b_j)$, where $x_j = \frac{1 - a_{11}^{(j)}}{a_{12}^{(j)}}$, $y_j = a_{21}^{(j)} + \frac{a_{22}^{(j)}(1 - a_{11}^{(j)})}{a_{12}^{(j)}}$, $z_j = a_{12}^{(j)}$ and

$b_j = b^{-1}a^{y_j}ba^{-z_j}b^{-1}a^{-x_j}b$. Note that in the last formula one has to treat x_j, y_j, z_j as integers and not as elements of $GF(q)$. Then $< b_1, \ldots b_n, u, v >$ is a polycyclic generating set for H_{qi}. For each q, i, formulas for evaluating $b_j, j = \overline{1, n}$ can be simplified.

For example, if $q = 3, \xi \in k^*$ is an element of order 9, $a = \mathrm{diag}(\xi, \xi, \xi^{-2})$, $b = (1-\varepsilon)^{-1}\begin{pmatrix} 1 & \varepsilon & 1 \\ \varepsilon & 1 & 1 \\ \varepsilon & \varepsilon & \varepsilon^2 \end{pmatrix}$, $c = (1-\varepsilon)^{-1}\begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 1 & \varepsilon \\ 1 & 1 & \varepsilon^2 \end{pmatrix}$

then $H_{33} = \begin{cases} < a, b, c, u, v >, & \text{if } p^l \equiv 1 \pmod 9 \\ < b, c, u, v >, & \text{if } p^l \not\equiv 1 \pmod 9. \end{cases}$ (see [3, p. 83]).

4. GAP GROUP LIBRARY FOR IRREDUCIBLE MAXIMAL SOLVABLE SUBGROUP OF $SL(q, k)$

In previuos sections the algorithm that produces a list of the irreducible maximal solvable subgroups of $SL(q, k)$ was developed. The next objective is to implement this algorithm and to provide electronic access to the list of groups so obtained. This section contains a brief description of a GAP group library that solves the problem. The library consists of five functions returning one representative of each of conjugacy classes of a subset M_{qi}, $i = 1, 2, 3$. The list of generators of these representatives which is returned by functions is a generating polycyclic (consistent) sequence. If a subset M_{qi} is empty then the corresponding function returns an error message.

The function 'ImprimitiveMaxSolvableSubgroup' (resp. 'PrimitiveMaxSolvableSubgroup1') returns the unique up to conjugacy in $SL(q, k)$ subgroup $MSS1(q, p, l)$ of M_{q1} (resp. $MSS2(q, p, l)$ of M_{q2}). 'PrimitiveMaxSolvableSubgroup2' (resp. 'AllPrimitiveMaxSolvableSubgroups2') returns a complete and irredundant list of $GL(q, k)$ (resp. $SL(q, k)$) conjugacy class representatives of the subgroups of the set M.

Example.

```
gap> g := PrimitiveMaxSolvableSubgroup1(2, 5, 2);
MSS2(2,5,2)
gap> gen := g.generators;
[ [ [ Z(5), 0*Z(5) ], [ 0*Z(5), Z(5)^3 ] ],
 [ [ Z(5^2)^5, Z(5^2)^13], [Z(5^2)^14, Z(5^2)^5]]]
gap> h := AllPrimitiveMaxSolvableSubgroups2(2, 5, 2);
[ MSS3(2,5,2), MSS3.1(2,5,2) ]
```

Here $g = MSS2(2, 5, 2)$ is the Singer cycle normalizer of $SL(2, 25)$ and *gen* is a polycyclic generating sequence of g, h is a list of $SL(2, 25)$-

conjugacy class representatives of the maximal solvable subgroups of $SL(2,25)$ with scalar maximal abelian normal subgroup ($MSS3(2,5,2)$ is conjugate to $MSS3.1(2,5,2)$ in $GL(2,25)$ but not in $SL(2,25)$).

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