K. N. Ilinski, A. S. Stepanenko, q-Deformed superspace and q-extended supersymmetric Hamiltonian with arbitrary superpotential, *Zap. Nauchn. Sem. POMI*, 1996, Volume 235, 260–272

https://www.mathnet.ru/eng/znsl3653

K. N. Ilinski, A. S. Stepanenko

q–DEFORMED SUPERSPACE AND q–EXTENDED SUPERSYMMETRIC HAMILTONIAN WITH ARBITRARY SUPERPOTENTIAL

§1. Introduction

Recently deformations of standard mathematical objects have attracted a lot of attention ([1,2] and refs. in [2]). In particular, in [3] the quantum group representation theory and q-deformed spaces have been studied and in [4] the procedure of supergroup quantization has been presented.

The attempts to construct the description of dynamics on q-deformed spaces were undertaken in [5, 6, 7]. In [8] q-deformed quantum spaces were considered as a gradedcommutative algebra. On this basis the classical and quantum dynamics on q-deformed spaces were proposed. The particles appeared after the quantization satisfy the q-deformed commutation relations:

$$a_i a_i^+ \mp a_i^+ a_i = 1 \qquad a_i a_j^+ \mp q a_j^+ a_i = 0$$
$$a_i^+ a_j^+ \mp q^{-1} a_j^+ a_i^+ = 0 \qquad a_i a_j \mp q^{-1} a_j a_i = 0$$

$$q^m = 1 \quad , \quad m \in N \quad , \quad 1 \le i \le j \le n$$

$${a_i^+}^2 = a_i^2 = 0 \quad \text{for} q\text{-deformed fermions,}$$

where the upper sign corresponds to the quantization on q-deformed usual spaces and lower one corresponds to the quantization on q-deformed superspaces. The particles can be naturally called q-bosons and q-fermions. It is interesting to consider the generalization of supersymmetric theory for the case when q-fermions are included into the theory instead of fermions.

As we can see later such supersymmetric theory will differ from the deformed supersymmetry (q-supersymmetry or q-SUSY) by V. P .

Permanent address: Theoretical Department, St.-Petersburg Nuclear Physics Institute, Gatchina, St.-Petersburg, 188350, Russian Federation.

Spiridonov [9] where the following relations for the supercharges Q^+, Q^- were used:

$$qQ^+Q^- + q^{-1}Q^-Q^+ = H \quad {Q^+}^2 = {Q^-}^2 = 0$$

$$qHQ^- - q^{-1}Q^-H = qQ^+H - q^{-1}HQ^+ = 0 \ .$$

In our approach it is possible to discover the q-deformation only for the pair of different q-fermions and that is why this theory is q-extended supersymmetry (q-ESUSY). It is possible to coincide such theory in the framework of colour supersymmetry [10] (we are grateful N. Borisov for this remark).

In this paper we consider one of the possible generalization of SUSY. For this purpose we formulate a q-superspace formalism and construct the q-supertransformation group on this q-superspace. After this we start from a standard action for a scalar d-dimensional real superfield and build q-extended supersymmetric Lagrangian for component variables of the q-superfield. After the quantization procedure corresponding conserved quantities become q-supercharges. In difference from [11] where only general constructions of q-extended supersymmetry and simplest examples were considered in this paper we give exact expressions for q-supercharges and q-extended supersymmetric Hamiltonian with arbitrary superpotential.

§2. q-EXTENDED SUPERSYMMETRY AND q-DEFORMED SUPERSPACE FORMALISM

As in usual case [13, 14, 15] the most natural way to build supersymmetric Lagrangians is by employing the superspace formalism. For the sake of simplicity we will consider $(0+1)$-dimensional case, but all constructions can be generalized for multidimensional one.

For the $(0+1)$-dimensional case superspace reduces to the supertime. In the standard way besides time t complex anticommuting variables $\theta_i, \bar\theta_i, i = 1,2$ are introduced. Instead of that we will consider the variables $\theta_i, \bar\theta_i \in HQ_m^{0|2}, i = 1,2$, i.e.,

$$\theta_i\bar\theta_i = -\bar\theta_i\theta_i \qquad {\theta_i}^2 = \bar\theta_i^2 = 0 \qquad q^m = 1 \quad , \quad m \in N$$

$$\theta_1\theta_2 = -q^{-1}\theta_2\theta_1 \qquad \bar\theta_1\theta_2 = -q\theta_2\bar\theta_1$$

$$\bar\theta_1\bar\theta_2 = -q^{-1}\bar\theta_2\bar\theta_1 \qquad \theta_1\bar\theta_2 = -q\bar\theta_2\theta_1$$

It was shown in [8] that the algebra $HQ_m^{0|2}$ can be described as a graded-commutative algebra. For that we began from the algebra

$C(\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2)$ which is freely generated by $\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2$ and used a special graduation map. In this case the graduation map $\varphi : HQ_m^{0|2} \to Z_2 \oplus Z_m \oplus Z_m$ was defined by

$$\varphi(\theta_1^{k_1} \ldots \bar{\theta}_2^{k_4}) = k_1\varphi(\theta_1) + \cdots + k_4\varphi(\bar{\theta}_2)$$
$$\varphi(\theta_1) = (1,1,0)\ ; \quad \varphi(\theta_2) = (1,0,1)\ ; \quad \varphi(\bar{\theta}_i) = -\varphi(\theta_i)$$

and the graded-commutator for the homogeneous elements A, B had the form:

$$[A, B]_q^{0|2} = AB - (-1)^{\vartheta_2(\varphi(A),\varphi(B))} q^{-\vartheta_m(\varphi(A),\varphi(B))} BA$$
$$\vartheta_2(\varphi(A), \varphi(B)) = \varphi(A)|_{Z_2}\ \varphi(B)|_{Z_2}$$

and $\vartheta_m(\alpha, \beta)$ was defined by the following relations: $\vartheta_m : Z_2 \oplus Z_m \oplus Z_m \to Z$ being biadditive antisymmetric map,

$$\vartheta_m(\alpha, \beta) = \sum_{i,j} \alpha_i \beta_j \vartheta_m(\xi_i, \xi_j) \qquad \alpha = \sum_i \alpha_i \xi_i \qquad \beta = \sum_i \beta_i \xi_i$$

$$\vartheta_m(\xi_i, \xi_j) = 1 \quad i < j \qquad \xi_1 = (l, 1, 0) \qquad \xi_2 = (k, 0, 1) \qquad \forall k, l = 0, 1$$

Our algebra $HQ_m^{0|2}$ can be described as a factor-algebra of $C(\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2)$ under the two-side ideal, generated by the elements $\{[C_1, C_2]_q^{0|2} \mid C_1, C_2 \in C(\theta_1 \ldots \bar{\theta}_2)\}$ and in this sense $HQ_m^{0|2}$ is exactly the algebra with zero graded-commutators:

$$f, g \in HQ_m^{0|2} \qquad [f, g]_q^{0|2} = 0.$$

We note that q is a particular case of ϵ-factor [12] (and refs. therein).

A dynamic q-supervariable is a quantity depending on $t, \theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2$. The coefficients of its expansion in $\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2$ have the commutation relations according to their graduations. In what follows we will use the following notations:

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \qquad \bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2) \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$$

$$\bar{\theta}\psi \equiv \bar{\theta}_1\psi_1 + \bar{\theta}_2\psi_2 \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \qquad \bar{\chi} = (\bar{\chi}_1, \bar{\chi}_2) \qquad \partial_t = \frac{\partial}{\partial t}$$

The supersymmetry group is implemented in terms of $t, \theta, \bar{\theta}$ transformations in the following way:

$$Q_{\bar{\eta}} \ : \quad t \to t + \frac{i}{2}\bar{\eta}\theta \qquad \bar{\theta} \to \bar{\theta} + \bar{\eta}$$

$$Q_\eta \ : \quad t \to t - \frac{i}{2}\bar{\theta}\eta \qquad \theta \to \theta + \eta$$

where $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, $\bar{\eta} = (\bar{\eta}_1, \bar{\eta}_2)$ are transformation parameters, such as $\varphi(\eta_i) = \varphi(\theta_i)$.

The generators of the supersymmetry transformations can be written in a differential form:

$$Q_{\bar{\eta}} = \frac{\partial}{\partial\bar{\theta}} + \frac{i}{2}\theta\partial_t \equiv \begin{pmatrix} Q^1_{\bar{\eta}} \\ Q^2_{\bar{\eta}} \end{pmatrix} \qquad Q_\eta = -\frac{\partial}{\partial\theta} - \frac{i}{2}\bar{\theta}\partial_t \equiv (Q^1_\eta, Q^2_\eta) \qquad (1)$$

Now we note that

$$[Q^\alpha_{\bar{\eta}}, Q^\alpha_\eta]^{0|2}_q = -i\partial_t \qquad \alpha = 1,2$$

and

$$[Q^1_\eta, Q^2_\eta]^{0|2}_q = Q^1_\eta Q^2_\eta + q^{-1} Q^2_\eta Q^1_\eta = 0 \qquad Q^{\alpha\,2}_{\bar{\eta}} = Q^{\alpha\,2}_\eta = 0$$

$$[Q^1_{\bar{\eta}}, Q^2_{\bar{\eta}}]^{0|2}_q = [Q^\alpha_{\bar{\eta}}, Q^{\alpha'}_\eta]^{0|2}_q = 0 \qquad \alpha' \neq \alpha. \qquad (2)$$

Above relations give the connections between the supersymmetry generators and the Hamiltonian of the system. So, graded-commutators $[Q^\alpha_{\bar{\eta}}, Q^\alpha_\eta]^{0|2}_q, \alpha = 1,2$ are the generators of time translations (up to a factor $-i$).

Representation of the supersymmetric group can be implemented, for example, by means of d-dimensional scalar real q-superfield $\phi = (\phi^1 \ldots \phi^d)$. Its expansion in $\theta, \bar{\theta}$ looks like

$$\begin{aligned} \phi^k(t) = {} & x^k(t) + \frac{1}{\sqrt{2}}\bar{\theta}\psi^k(t) + \frac{1}{\sqrt{2}}\bar{\psi}^k(t)\theta + \\ & + \bar{L}^k(t)\theta_1\theta_2 + \bar{\theta}_2\bar{\theta}_1 L^k(t) + \mathrm{Tr}\,(\widehat{\theta}\mathbf{F}^k(t)) + \frac{1}{\sqrt{2}}\bar{\chi}^k(t)\widetilde{\theta} + \\ & + \frac{1}{\sqrt{2}}\bar{\widetilde{\theta}}\chi^k(t) + H^k(t)\bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 \ , \qquad k = 1 \ldots d \end{aligned} \qquad (3)$$

where

$$\widehat{\theta} = \begin{pmatrix} \bar{\theta}_1\theta_1 & \bar{\theta}_2\theta_1 \\ \bar{\theta}_1\theta_2 & \bar{\theta}_2\theta_2 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \quad \widehat{\theta} = \begin{pmatrix} \theta_1\bar{\theta}_2\theta_2 \\ \theta_2\bar{\theta}_1\theta_1 \end{pmatrix}.$$

Q-supersymmetry transformations are realized in component variables as follows:

for $\bar{\eta}Q_{\bar{\eta}}$

$$\delta_1 x^k = \frac{1}{\sqrt{2}}\bar{\eta}\psi^k \quad , \quad \delta_1 H^k = -\frac{i}{2\sqrt{2}}\bar{\eta}\dot{\chi}^k$$

$$\delta_1\bar{\psi}^k = \sqrt{2}\left\{\frac{i}{2}\bar{\eta}\dot{x}^k + (\bar{\eta}_1 F^k_{11} + \bar{\eta}_2 q^{-1}F^k_{21} \ , \ \bar{\eta}_2 F^k_{22} + \bar{\eta}_1 q F^k_{12})\right\},$$

$$\delta_1\psi^k = \sqrt{2}\begin{pmatrix} -q\bar{\eta}_2 \\ \bar{\eta}_1 \end{pmatrix} L^k$$

$$\delta_1\bar{\chi}^k = \sqrt{2}\left\{\bar{\eta}H^k + \frac{i}{2}(\bar{\eta}_1 \dot{F}^k_{22} - \bar{\eta}_2 q^{-1}\dot{F}^k_{21} \ , \ \bar{\eta}_2\dot{F}^k_{11} - \bar{\eta}_1 q \dot{F}^k_{12})\right\},$$

$$\delta_1\chi^k = -\frac{i}{\sqrt{2}}\begin{pmatrix} -q\bar{\eta}_2 \\ \bar{\eta}_1 \end{pmatrix} \dot{L}^k$$

$$\delta_1 L^k = 0, \quad \delta_1\bar{L}^k = \frac{1}{\sqrt{2}}\left(\bar{\eta}_2(\frac{i}{2}\dot{\bar{\psi}}^k_1 + \bar{\chi}^k_1) + (\frac{i}{2}\dot{\bar{\psi}}^k_2 + \bar{\chi}^k_2)\bar{\eta}_1\right)$$

$$\delta_1\mathbf{F}^k = \delta_1\begin{pmatrix} F^k_{11} & F^k_{12} \\ F^k_{21} & F^k_{22} \end{pmatrix} = \tag{4}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{\eta}_1(-\frac{i}{2}\dot{\psi}^k_1) + \bar{\eta}_2\chi^k_2 & -\bar{\eta}_2(\frac{i}{2}\dot{\psi}^k_1 + \chi^k_1) \\ -\bar{\eta}_1(\frac{i}{2}\dot{\psi}^k_2 + \chi^k_2) & \bar{\eta}_2(-\frac{i}{2}\dot{\psi}^k_2) + \bar{\eta}_1\chi^k_1 \end{pmatrix}$$

and for $Q_\eta\eta$ it is possible to obtain transformation laws using (1) and (4).

The covariant derivatives D and $\bar{D}$ have the form

$$D = \frac{\partial}{\partial\bar{\theta}} - \frac{i}{2}\theta\partial_t \equiv \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \qquad \bar{D} = -\frac{\partial}{\partial\theta} + \frac{i}{2}\bar{\theta}\partial_t \equiv (\bar{D}_1, \bar{D}_2)$$

satisfying the usual conditions

$$D\delta_i\phi = \delta_i D\phi \ , \qquad \bar{D}\delta_i\phi = \delta_i\bar{D}\phi$$

where $\delta_i\phi, i = 1,2$ are the variances of the q-supervariables under a supersymmetric transformations.

The action for the q-extended supersymmetric quantum mechanics can be written in the following way

$$S = \int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \ \left(\bar{D}\phi^k D\phi^k - 2V(\phi)\right) = \int dt \ \mathcal{L}$$

where $V(\cdot)$ is an arbitrary function (superpotential). We imply the integration over $\theta_i, \bar{\theta}_i$ in the following sense

$$\int 1 \mathrm{d}\theta_i = \int 1 \mathrm{d}\bar{\theta}_i = \int \theta_j \mathrm{d}\bar{\theta}_i = \int \bar{\theta}_j \mathrm{d}\theta_i = 0 \qquad \forall i, j = 1, 2$$

$$\int \theta_i \mathrm{d}\theta_k = \int \bar{\theta}_i \mathrm{d}\bar{\theta}_k = \delta_{ik} \;, \qquad \phi(\theta_i) = \phi(\mathrm{d}\theta_i) \;, \qquad \phi(\bar{\theta}_i) = \phi(\mathrm{d}\bar{\theta}_i)$$

As a result the Lagrangian takes the form

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2}\dot{x}^k \left(\dot{F}^k_{11} + \dot{F}^k_{22}\right) - 2H^k \left(F^k_{11} + F^k_{22}\right) + \frac{1}{4}\dot{\bar{\psi}}^k \dot{\psi}^k + \bar{\chi}^k \chi^k + \\
& + \frac{i}{4}\left(\dot{\bar{\psi}}^k \chi^k - \bar{\psi}^k \dot{\chi}^k + \dot{\bar{\chi}}^k \psi^k - \bar{\chi}^k \dot{\psi}^k\right) + 2i\bar{L}^k \dot{L}^k - 2\frac{\partial V}{\partial x^k} H^k + \\
& + 2\frac{\partial^2 V}{\partial x^k \partial x^l}\left(\frac{1}{2}\bar{\psi}^l \chi^k + \frac{1}{2}\bar{\chi}^l \psi^k + F^k_{12}F^l_{21} - F^k_{11}F^l_{22} - \bar{L}^l L^k\right) - \\
& - \frac{\partial^3 V}{\partial x^l \partial x^m \partial x^n}\left(F^l_{12}\bar{\psi}^m_1 \psi^n_2 + F^l_{21}\bar{\psi}^m_2 \psi^n_1 - F^l_{11}\bar{\psi}^m_2 \psi^n_2 - F^l_{22}\bar{\psi}^m_1 \psi^n_1 + \right. \\
& \left. + \bar{L}^l \psi^m_1 \psi^n_2 + L^l \bar{\psi}^m_2 \bar{\psi}^n_1\right) - \frac{1}{2}\frac{\partial^4 V}{\partial x^l \partial x^m \partial x^n \partial x^k}\bar{\psi}^l_1 \bar{\psi}^m_2 \psi^n_2 \psi^k_1 \qquad (6)
\end{aligned}$$

where full derivatives have been neglected. The auxiliary variables F^k_{ij}, χ^k, $\bar{\chi}^k$, H^k may be excluded in the usual way if matrix $\partial_{lk}V = K$ is invertible. Omitting spatial symbols to simplify the expressions we can put down the expressions for auxiliary fields

$$\begin{aligned}
\chi_i(t) &= (-2\psi_i(t)V''[x(t)] + \dot{\psi}_i(t)i)/2 \\
\bar{\chi}_i(t) &= -(2\bar{\psi}_1(t)V''[x(t)] + \dot{\bar{\psi}}_i(t)i)/2 \\
F_{i,j\neq i}(t) &= K^{-1}V'''[x(t)]\bar{\psi}_j(t)\psi_i(t)/2 \\
F_{ii}(t) &= (-1)^i\frac{1}{4}((-1)^k\bar{\psi}_k(t)\psi_k(t)K^{-1}V'''[x(t)]) - \frac{1}{2}V'[x(t)] \\
H(t) &= \frac{1}{4}(\bar{\psi}_i(t)\psi_i(t)V'''[x(t)]) + \frac{1}{2}V'[x(t)]V''[x(t)] + \frac{1}{4}\ddot{x}(t).
\end{aligned}$$

After substitutions of auxiliary fields Lagrangian has the form:

$$\mathcal{L} = -\Big(8\bar{L}(t)L(t)V''[x(t)] - 8\bar{L}(t)\dot{L}(t)i + 4\bar{L}(t)\psi_1(t)\psi_2(t)V'''[x(t)] -$$
$$-4\bar{\psi}_1(t)\bar{\psi}_2(t)L(t)V'''[x(t)]q + 3\bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)\psi_2(t)K^{-1}\left(V'''[x(t)]\right)^2 q -$$
$$-2\bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)\psi_2(t)V^{(IV)}[x(t)]q + 2\bar{\psi}_i(t)\psi_i(t)V'[x(t)]V'''[x(t)] +$$
$$+4\bar{\psi}_i(t)\psi_i(t)\left(V''[x(t)]\right)^2 - 2\bar{\psi}_i(t)\dot{\psi}_i(t)V''[x(t)]i + 2\dot{\bar{\psi}}_i(t)\psi_i(t)V''[x(t)]i +$$
$$+2\left(V'[x(t)]\right)^2 V''[x(t)] - 2V''[x(t)]\left(\dot{x}(t)\right)^2\Big)/4.$$

Let us denote the action as follows

$$S = \int \mathrm{d}t\mathrm{d}\bar{\theta}_1\mathrm{d}\theta_1\mathrm{d}\bar{\theta}_2\mathrm{d}\theta_2\ \mathcal{L} = \int \mathrm{d}t\ \widehat{\mathcal{L}}\ .$$

We can find conserved Noether's currents directly for $\widehat{\mathcal{L}}$, then integrate over $\theta, \bar{\theta}$ and obtain conserved supercharges. These expressions contain not only "physical" fields but also auxiliary fields. After substitutions of constrains supercharges have the following form in terms of "physical" fields:

$$Q_{\eta_i}\eta_i = -\left(4\bar{L}(t)\psi_2(t)V''[x(t)]iq^{-1} - \bar{\psi}_1(t)\bar{\psi}_2(t)\psi_2(t)V'''[x(t)]i -\right.$$
$$\left. - 2\bar{\psi}_1(t)V'[x(t)]V''[x(t)]i + 2\bar{\psi}_1(t)V''[x(t)]\dot{x}(t)\right)\eta_1/2\sqrt{2}$$
$$Q_{\eta_2}\eta_2 = \left(4\bar{L}(t)\psi_1(t)V''[x(t)]i - \bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)V'''[x(t)]iq +\right.$$
$$\left. + 2\bar{\psi}_2(t)V'[x(t)]V''[x(t)]i - 2\bar{\psi}_2(t)V''[x(t)]\dot{x}(t)\right)\eta_2/2\sqrt{2}$$
$$Q_{\bar{\eta}_1}\bar{\eta}_1 = -\left(4\bar{\psi}_2(t)L(t)V''[x(t)]iq + \bar{\psi}_2(t)\psi_1(t)\psi_2(t)V'''[x(t)]iq -\right.$$
$$\left. - 2\psi_1(t)V'[x(t)]V''[x(t)]i - 2\psi_1(t)V''[x(t)]\dot{x}(t)\right)\bar{\eta}_1/2\sqrt{2}$$
$$Q_{\bar{\eta}_2}\bar{\eta}_2 = \left(4\bar{\psi}_1(t)L(t)V''[x(t)]i + \bar{\psi}_1(t)\psi_1(t)\psi_2(t)V'''[x(t)]i +\right.$$
$$\left. + 2\psi_2(t)V'[x(t)]V''[x(t)]i + 2\psi_2(t)V''[x(t)]\dot{x}(t)\right)\bar{\eta}_2/2\sqrt{2}.$$

§3. Classical dynamics and quantization procedure for the component variables

Here we construct classical and quantum dynamics in close analogy with [8]. It is important however that the new fields $L, \bar{L}$ arise which are not q-fields in the sense [8].

To sake of simplicity let us make the change of variables:

$$\bar{L},\ L \to \frac{\bar{L}}{\sqrt{2}}, \frac{L}{\sqrt{2}}\ ,\quad \bar{\psi} \to \bar{\psi}K^{-1/2},\ \psi K^{-1/2}.$$

Then we have for Lagrangian of the system can be rewritten in the form:

$$\mathcal{L} = -\Big(4\bar{L}(t)L(t)V''[x(t)] - 2\bar{L}(t)\dot{L}(t)i + 2\dot{\bar{L}}(t)L(t)i +$$
$$+2\sqrt{2}\bar{L}(t)\psi_1(t)\psi_2(t)K^{-1}V'''[x(t)] -$$
$$-2\sqrt{2}\bar{\psi}_1(t)\bar{\psi}_2(t)L(t)K^{-1}V'''[x(t)]q +$$
$$+3\bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)\psi_2(t)K^{-3}\left(V'''[x(t)]\right)^2 q +$$
$$+2\bar{\psi}_i(t)\psi_i(t)K^{-1}V'[x(t)]V'''[x(t)] -$$
$$-2\bar{\psi}_1(t)\bar{\psi}_2(t)\psi_1(t)\psi_2(t)K^{-2}V^{(IV)}[x(t)]q +$$
$$+4\bar{\psi}_i(t)\psi_i(t)K - 2\bar{\psi}_i(t)\dot{\psi}_i(t)i +$$
$$+2\dot{\bar{\psi}}_i(t)\psi_i(t)i + 2\left(V'[x(t)]\right)^2 K - 2K\dot{x}^2(t)\Big)/4.$$

The principle of extreme action for trajectories is

$$\delta S = 0$$

$$S[x,p,\psi,\bar{\psi},L,\bar{L}] = \int \mathrm{dt}\, \Big\{\frac{i}{2}\sum_{j=1}^{2}[\bar{\psi}_j(t)\dot{\psi}_j(t) - \dot{\bar{\psi}}_j(t)\psi_j(t)] +$$
$$+\frac{i}{2}[\bar{L}(t)\dot{L}(t) - \dot{\bar{L}}(t)L(t)] + p(t)\dot{x}(t) - \mathcal{H}[x,p,\psi,\bar{\psi},L,\bar{L}]\Big\} \equiv \int \mathrm{dt}\; \mathcal{L}$$

where

$$\dot{\psi}_j{}^k = \frac{d\psi_j^k}{dt}, \quad \dot{\bar{\psi}}_j{}^k = \frac{d\bar{\psi}_j{}^k}{dt}, \quad \dot{L}^k = \frac{dL^k}{dt}, \quad \dot{\bar{L}}^k = \frac{d\bar{L}^k}{dt}, \quad \varphi(\mathcal{H}) = 0.$$

Derivatives are calculated as in the Grassmann algebra case.

The equations for extreme trajectories have the form:

$$\frac{\partial\mathcal{L}}{\partial\psi_j} = \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_j}, \quad \frac{\partial\mathcal{L}}{\partial\bar{\psi}_j} = \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}_j}, \quad \frac{\partial\mathcal{L}}{\partial L} = \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{L}},$$

$$\frac{\partial\mathcal{L}}{\partial\bar{L}} = \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{\bar{L}}}, \quad \frac{\partial\mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{x}}, \quad \frac{\partial\mathcal{L}}{\partial p} = \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{p}}.$$

Here $\frac{\partial}{\partial L^k}, \frac{\partial}{\partial\bar{L}^k}, \frac{\partial}{\partial\psi_j{}^k}, \frac{\partial}{\partial\bar{\psi}_j{}^k}$ are defined as left derivatives in close analogy with the Grassmann case. Note that they are differentiations of graduated algebra of functions, i.e.,

$$\frac{\partial}{\partial\psi_j{}^k}fg = \left(\frac{\partial}{\partial\psi_j{}^k}f\right)g + (-1)^{-\theta_2(\varphi(\psi_j^k),\varphi(f))}q^{-\theta_m(\varphi(\bar{\psi}_j{}^k),\varphi(f))}f\left(\frac{\partial}{\partial\psi_j{}^k}g\right)$$

for homogeneous elements f, g and similar for $\frac{\partial}{\partial L}, \frac{\partial}{\partial \bar{L}}, \frac{\partial}{\partial \psi}$.

Putting $p = \dot{x}K$ we can rewrite the motion equations in Hamiltonian form:

$$\dot{\chi}_k = \left(\Omega^{-1}\right)_{kj} \frac{\partial \mathcal{H}}{\partial \chi_j}$$

where

$$\begin{pmatrix} \psi_1 \\ \bar{\psi}_1 \\ \psi_2 \\ \bar{\psi}_2 \\ L \\ \bar{L} \\ x \\ p \end{pmatrix}, \quad \Omega \equiv \begin{pmatrix} \Omega_1 & 0 & 0 & 0 \\ 0 & \Omega_2 & 0 & 0 \\ 0 & 0 & \Omega_3 & 0 \\ 0 & 0 & 0 & \Omega_4 \end{pmatrix},$$

$$\Omega_{1,2} \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \Omega_3 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Omega_4 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence, the inverse matrix Ω^{-1} is

$$\Omega^{-1} = \begin{pmatrix} \Omega_1^{-1} & 0 & 0 & 0 \\ 0 & \Omega_2^{-1} & 0 & 0 \\ 0 & 0 & \Omega_3^{-1} & 0 \\ 0 & 0 & 0 & \Omega_4^{-1} \end{pmatrix},$$

$$\Omega_{1,2}^{-1} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \Omega_3^{-1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Omega_4^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is possible to introduce the Poisson structure in close analogy with the Grassmann case:

$$\{f, g\}_{\mathrm{cl}} = -\sum_{s,s'} \left(\Omega^{-1}\right)_{s,s'} \frac{\partial}{\partial \chi'_{s'}} \frac{\partial}{\partial \chi_s} f(\chi) g(\chi') \Big|_{\chi=\chi'}.$$

Such defined Poisson brackets satisfy the following relations for homogeneous elements f, g, h:

$$\{f, g\}_{\mathrm{cl}} = (-1)^{\theta_2(\varphi(f),\varphi(g))} q^{\theta_m(\varphi(f),\varphi(g))} \{g, f\}_{\mathrm{cl}}$$

$$\{fg, h\}_{\mathrm{cl}} = f\{g, h\}_{\mathrm{cl}} + (-1)^{\theta_2(\varphi(g),\varphi(h))} q^{\theta_m(\varphi(g),\varphi(h))} \{f, h\} g$$

$$\begin{gathered} \{f, \{g, h\}_{\mathrm{cl}}\}_{\mathrm{cl}} + \\ +(-1)^{\theta_2(\varphi(g),\varphi(h))+\theta_2(\varphi(f),\varphi(h))} q^{\theta_m(\varphi(g),\varphi(h))+\theta_m(\varphi(f),\varphi(h))} \{h, \{f, g\}_{\mathrm{cl}}\}_{\mathrm{cl}} + \\ +(-1)^{\theta_2(\varphi(f),\varphi(g))+\theta_2(\varphi(f),\varphi(h))} q^{\theta_m(\varphi(f),\varphi(g))+\theta_m(\varphi(f),\varphi(h))} \{g, \{h, f\}_{\mathrm{cl}}\}_{\mathrm{cl}} = 0 \end{gathered} \qquad (7)$$

which correspond to generalization of antisymmetry, differentiation property and Jacobi identity for ordinary Poisson brackets.

The equations of motion can be written by means of these Poisson brackets as

$$\dot{\chi_j} = \{H, \chi_j\}_{\rm cl}.$$

It should be noted that classical dynamics complies with the graduation structure (i.e., evolution does not change graduation of χ) due to $\varphi(H) = 0$.

The commutation relations for canonical variables related to $\{\ \}_{\rm cl}$ have the form:

$$\{\psi_i, \bar{\psi}_i\}_{\rm cl} = i\ , \quad \{L, \bar{L}\}_{\rm cl} = i\ , \quad \{x, p\}_{\rm cl} = -1 \tag{8}$$

and others are zeroth.

To canonically quantize our theory one has

1 to construct the operators of canonical variables in Hilbert space, which satisfy the commutation relations (8)

2 to define the quantum Poisson brackets $\{\ \}_\hbar$ which have the same algebraic properties as the classical ones.

In this case item 2 leads to following definition of quantum Poisson brackets:

$$\{A, B\}_\hbar = \frac{i}{h}[A, B]_q^{0|2}.$$

The commutation relations for the field operators follow from the above:

$$(L^+)^+ = L\ , \quad (\psi_i^+)^+ = \psi_i\ , \quad q^* = q^{-1}\ ,$$

$$(\psi_i^+)^2 = (\psi_i)^2 = 0\ , \quad \psi_1\psi_2 + q^{-1}\psi_2\psi_1 = 0\ , \quad \psi_1\psi_2^+ + q\psi_2^+\psi_1 = 0\ ,$$

$$L\psi_1 - q\psi_1 L = 0\ , \quad L\psi_2 - q^{-1}\psi_2 L = 0\ ,$$

$$L\psi_2^+ - q\psi_2^+ L = 0\ , \quad L\psi_1^+ - q^{-1}\psi_1^+ L = 0\ ,$$

$$LL^+ - L^+L = h\ , \quad \psi_i\psi_i^+ + \psi_i^+\psi_i = h\ , \quad [x, p] = ih \tag{9}$$

and relations which can be obtained by Hermitian conjugation.

After quantization q-supercharges have the form:

$$
\begin{aligned}
Q_1 =& -(2\sqrt{2}\bar{L}\psi_2 K^{1/2} i q^{-1} - \bar{\psi}_1\bar{\psi}_2\psi_2 K^{-3/2} V''' i + \bar{\psi}_1 K^{-3/2} V''' h i + \\
&+ 2\bar{\psi}_1 K^{-1/2} p - 2\bar{\psi}_1 K^{1/2} V' i)/2\sqrt{2} \\
Q_2 =& (2\sqrt{2}\bar{L}\psi_1 K^{1/2} i - \bar{\psi}_1\bar{\psi}_2\psi_1 K^{-3/2} V''' i q - \bar{\psi}_2 K^{-3/2} V''' h i - \\
&- 2\bar{\psi}_2 K^{-1/2} p + 2\bar{\psi}_2 K^{1/2} V' i)/2\sqrt{2} \\
Q_1^+ =& (2\sqrt{2}\bar{\psi}_2 L K^{1/2} i q + \bar{\psi}_2\psi_1\psi_2 K^{-3/2} V''' i q - \\
&- 2\psi_1 K^{-1/2} p - 2\psi_1 K^{1/2} V' i)/2\sqrt{2} \\
Q_2^+ =& -(2\sqrt{2}\bar{\psi}_1 L K^{1/2} i + \bar{\psi}_1\psi_1\psi_2 K^{-3/2} V''' i + \\
&+ 2\psi_2 K^{-1/2} p + 2\psi_2 K^{1/2} V' i)/2\sqrt{2}.
\end{aligned}
$$

It is possible to check that such q-supercharges obey the q-extended supersymmetry algebra

$$
\{Q_i, Q_i^+\} = H \ , \quad (Q_i)^2 = (Q_i^+)^2 = 0 \ , \quad (Q_i^+)^+ = Q_i \ ,
$$
$$
Q_1 Q_2 = -q^{-1} Q_2 Q_1 \ , \quad Q_1 Q_2^+ = -q Q_2^+ Q_1 \ , \quad i = 1, 2
$$

with following quantum q- extended supersymmetric Hamiltonian

$$
\begin{aligned}
\mathcal{H} = h\Big(& 8\bar{L}LK + 4\sqrt{2}\bar{L}\psi_1\psi_2 K^{-1} V''' - 4\sqrt{2}\bar{\psi}_1\bar{\psi}_2 L K^{-1} V''' q + \\
& +6\bar{\psi}_1\bar{\psi}_2\psi_1\psi_2 K^{-3}(V''')^2 q - 4\bar{\psi}_1\bar{\psi}_2\psi_1\psi_2 K^{-2} V^{(IV)} q + \\
& +3\bar{\psi}_1\psi_1 K^{-3}(V''')^2 h - 2\bar{\psi}_1\psi_1 K^{-2} V^{(IV)} h + 4\bar{\psi}_1\psi_1 K^{-1} V' V''' + \\
& +8\bar{\psi}_1\psi_1 K + 3\bar{\psi}_2\psi_2 K^{-3}(V''')^2 h - 2\bar{\psi}_2\psi_2 K^{-2} V^{(IV)} h + \\
& +4\bar{\psi}_2\psi_2 K^{-1} V' V''' + 8\bar{\psi}_2\psi_2 K - 3K^{-3}(V''')^2 h^2 + \\
& +4K^{-2} V''' p h i + 2K^{-2} V^{(IV)} h^2 - 4K^{-1} V' V''' h + \\
& +4K^{-1} p^2 + 4K(V')^2 - 4Kh\Big)/8.
\end{aligned}
$$

Let us introduce the creation (annihilation) operators a_i^+, a_i, l^+, l which correspond to the fields $\psi, \bar{\psi}, L, \bar{L}$:

$$
\psi_i^k = \sqrt{h} a_i^k \qquad L^k = \sqrt{h} l^k
$$
$$
\psi_i^{k\,+} = \sqrt{h} a_i^{k\,+} \qquad L^{k\,+} = \sqrt{h} l^{k\,+}
$$

It is easy to obtain their commutation relations from (9).

Now we would like to discuss creation and annihilation operator representations. Let us consider the relations:

$$
\begin{gathered}
a_1^k = b_1^k \,, \qquad\qquad (a_1^{k+})^+ = a_1^k \\
a_2^k = e^{i\varphi \sum_{i=1}^{d} (b_1^i)^+ b_1^i} b_2^k \,, \qquad (a_2^{k+})^+ = a_2^k \,, \qquad q = e^{-i\varphi} \qquad (10) \\
l^k = e^{i\varphi \sum_{i=1}^{d} (b_1^{i+} b_1^i - b_2^{i+} b_2^i)} d^k \,, \qquad (l^{k+})^+ = l^k
\end{gathered}
$$

where b_i^{k+}, b_i^k, d^{k+}, d^k are creation (annihilation) operators of bosons and fermions respectively with usual commutation relations:

$$
\begin{gathered}
[b_i^k, d^l] = [b_i^{k+}, d^l] = [b_i^k, d^{l+}] = [b_i^{k+}, d^{l+}] = 0 \\
\{b_i^k, b_j^{l+}\} = \delta_{ij} K_{kl}^{-1} \,, \qquad [d^k, d^{l+}] = M_{kl}^{-1} \\
\{b_i^k, b_j^l\} = \{b_i^{k+}, b_j^{l+}\} = 0 \,, \qquad [d^k, d^l] = [d^{k+}, d^{l+}] = 0.
\end{gathered}
$$

Using matrix representation for fermion operators we can put down matrix representation for q-fermion operators and operators l^k, l^{k+} can be implemented by means of matrix differential operators. Hence q-supercharges and q-extended supersymmetric Hamiltonian can be represented by means of matrix differential operators as it was done for usual supercharges and supersymmetric Hamiltonian in [14].

§4. Conclusion

In this paper we have studied the generalization of the supersymmetry, which was generated by changing fermions by q-fermions. The latters can be obtained after quantization of the fields, which take values in q-deformed spaces.

For this purpose we formulated the q-superspace formalism and constructed the q-supertransformation group. On this basis q-extended supersymmetric Lagrangian was built and corresponding q-supercharges were put down. We considered the rule for the quantization such systems and fulfilled this procedure for our case. We note that all q-deformed objects (q-supercharges and q-extended supersymmetric Hamiltonian) contain arbitrary functional parameter – superpotential V. It is especially interesting for investigations of topological properties of q-extended supersymmetric Hamiltonians.

References

1. L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtajan, *Quantization of Lie groups and Lie algebras.* Preprint LOMI, 1987.
2. S. Majid, — Int. J. of Mod. Phys. A, **v.5, No. 1,** (1990), 1.
3. Yu. I. Manin, *Quantum groups and non-commutative geometry.* Preptint Montreal University, CRM-1561 1988.
4. Yu. I. Manin, — Comm. in Math. Phys. **123, No. 1** (1990), 163.
5. J. Wess, B. Zumino, Preprint CERN-TH-5697/90.
6. U. Carow-Watamura, M. Schlieker, and S. Watamura, Preprint KA-THEP-1990-15.
7. I. Ya. Aref'eva, I. V. Volovich, *Quantum group particles and Non-Archimedean geometry.* Preprint CERN-TH-6137/91.
8. N. V. Borisov, K. N. Ilinski, and V. M. Uzdin, — Phys. Lett. **A 169** (1992), 427.

no 9 V. P. Spiridonov, — Mod. Phys. Lett. **A 7** (1992), 1241.

10. V. Rittenberg, D. Wyler, Nucl. Phys..
11. K. N. Ilinski, V. M. Uzdin, *Quantum superspace, q-extended supersymmetry and parasupersymmetric quantum mecanics..* — submitted in Mod. Phys. Lett..
12. W. Marcinek, — J. Math. Phys. **33(5)** (1992), 1631.
13. V. P. Akulov, A. I. Pashkov, — Sov. J. Theor. Math. Phys. **56** (1983), 344.
14. A. A. Andrianov, N. V. Borisov, M. V. Ioffe, and M. I. Eides, — Teor. Mat. Fiz. **61, No. 1** (1984), 17.
15. A. V. Smilga, — Nucl. Phys. **B 249** (1985), 413.

St.-Petersburg
Steklov Mathematical Institute

Service de Physique Théorique
CE-Saclay, F-91191
Gif-sur-Yvette Cedex, France

Поступило 12 мая 1993г.