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Non-standard Logic and Reliability of Information

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Abstract. We study intransitive temporal multi-agent logic with agents' multi-valuations for formulas letters. This logic is defined in a semantic as a set of formulas which are true at linear models with multi-valued variables and clusters of states. We propose a background for such approach and a technique for computation truth values of formulas. Main results concerns solvability problem, we prove that the resulting logic is decidable.

Keywords: frame and Kripke model, multi-agent logics, NLM-logics, decidability problem.

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Introduction

Nowadays mathematical logics widely applied in research concerning computer science, information sciences overall. The interaction here is of both side and tasks and problems in computer science generates new areas in mathematical logic and induces creation new technique and tools in mathematical logic itself. Conception of knowledge arise in the analysis of distributed systems, and leaded to development multi-agent and multi-valued logical systems. More details about this can be found in the works of Halpern, Vardy (Reasoning About Knowledge [3]), Rybakov (Refined common knowledge logics or logics of common information, [4]).

In resent time research concerning knowledge was combined with implementation of temporal logic (cf. Rybakov [5–8]). An automata-theoretic approach to multi-agent planning was evolved at Wooldridge, [9].

In this our short paper we study intransitive temporal multi-agent logic with agents' multi-valuations for formulas letters. Earlier common knowledge in [3] was modeled at Kripke models with structure looking as simply a single time cluster with multi-relations for agents accessibility relations. This brought interesting strong results correlating well with observed examples and intuition. Here we wish to develop this approach towards modeling knowledge with Kripke frames which are linear time models with states represented by arbitrary clusters for agents accessibility relations. Here time is intransitive and it acts to only connect clusters. We propose a background for such approach and a technique for computation truth values of formulas. Main

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results concerns solvability problem, we prove that the resulting logic is decidable, prove existence of some deciding algorithm.

1. Definitions, preliminary facts

It is assumed that the reader is familiar with algebraic and Kripke semantics for modal logics. By *logic* we mean the set of all theorems that can be proven in a given axiomatic system, or all formulas which are true at a chosen set K of models. By propositional logic we mean algebraic propositional logic, although the reader may consider λ to be just a modal logic. Necessary information and all known facts used further, can be found in widely known literature [10].

Let us recall some definitions. A *frame* $\mathcal{F} := \langle W, R\rangle$ is a pair, where W is a non-empty set and R is a binary relation on W.

A non-empty set $C \subseteq W$ is called a *cluster* if: 1) for any x, y, $x \neq y$ xRy holds, 2) for any $x \in C$ and $y \in W$, $((xRy \,\&\, yRx) \Longrightarrow y \in C)$ holds. Frame is said to be intransitive if the accessibility relation R on W is intransitive.

A sequence of elements (or clusters) $\{a_0; a_1; \ldots; a_n\}$ of an intransitive frame is called a chain of length $n+1$ if, for all $i < n$, element a_{i+1} is R-accessible from element a_i (that is $a_i R a_{i+1}$) and there are no other frame elements (or clusters) between them.

The depth of an element x of a model (frame) F is the maximal number of clusters in chains of clusters starting from the cluster $C(x)$ containing x. The set of all elements (worlds) in a frame (model) F of depth not more than n will be denoted by $S_{\leq n}(F)$, and the set of elements of depth n we denote by $S_n(F)$.

A logic λ is said to be decidable if there is an algorithm which for any formula may compute if this formula belongs to λ. A logic λ satisfies the finite model property (FMP) if for any $\alpha \notin \lambda$ there exists a finite λ-model at which α is not valid.

2. Logic NML

NML-frame is a tuple

$$\mathcal{F}_{NML} = \langle W, Next, R_1, \ldots R_n\rangle,$$

where

$$W = \bigcup_{i \in J} C_i, J = [0, 1, \ldots m] \vee J = N,$$

and any C_i is the cluster, that is R_j, $1 \leqslant j \leqslant n$ is the equivalence relation on C_i such that: $xR_jy \iff \exists i \in J : (x \in C_i \;\&\; y \in C_i)$.

Temporal binary relation $Next$ is defined as follows:

$$(x \; Next \; y) \iff \exists i \in J : (x \in C_i \;\&\; y \in C_{i+1}).$$

In a case $J = [0, \ldots, m]$ we assume $\forall x, y \in C_m (x \; Next \; y)$.

NML-model is a tuple $\mathcal{M}_{NML} = \langle \mathcal{F}, V_1, V_2, \ldots V_n, V_{av}\rangle$, where $\mathcal{F}$ is NML-frame, $V_j \subseteq 2^W$ is a valuation of j-th agent. If $a \in \mathcal{M}_{NML}$, and $a \in V_j(p)$ then write $(\mathcal{M}_{NML}, a) \models_{V_j} p$ and say that p is true at element a w.r.t V_j valuation of j-th agent.

The language L of logic NML consists of:

1. $P := \{p_1; p_2; \ldots\}$ — countable set of propositional variables;

2. $\{\neg, \vee, \wedge, \rightarrow\}$ — Boolean operations;
3. K_j — operator of knowledge of j-th agent, $1 \leqslant j \leqslant n$;
4. N_C — temporal operator ;
5. Stl_j — operator of informational stability for j-th agent, $1 \leqslant j \leqslant n$;
6. AV – operator of authoritative statement;
7. Rel_j — operator of informational reliability for j-th agent, $1 \leqslant j \leqslant n$;
8. $\{(,)\}$ — auxiliary symbol (brackets).

A formula in the NML-language is defined in a standard way, that is a term constructed out of letters by chosen logical operations.

Any valuation of agents V_j in the frame $\mathcal{F}_{NML}$ from models $\mathcal{M}_{NML}$ to be extended to formulas as follows:

- Truth of Boolean connectives $\neg, \wedge, \vee, \rightarrow$ is defined in the standard way;
- $(\mathcal{F}_{NML}, x) \models_{V_j} K_j(\alpha) \iff \forall y \in W((xR_jy) \Longrightarrow$
  $\Longrightarrow (\mathcal{F}_{NML}, y) \models_{V_j} \alpha)$;
- $(\mathcal{M}_{NML}, x) \models \mathcal{N}(\alpha) \iff \exists k \in N : x \in C_k \ \wedge \ \forall y \in C_{k+1}$
  $(i.e.\ x\ Next\ y)\ (\mathcal{M}_{NML}, y) \models \alpha$;
- $(\mathcal{F}_{NML}, x) \models_{V_j} Stl_j(\alpha) \iff$
  $| \{y \in W : (xR_jy) \wedge (\mathcal{F}_{NML}, y) \models_{V_j} \alpha\} | \geqslant$
  $\geqslant | \{y \in W : (xR_jy) \wedge (\mathcal{F}_{NML}, y) \models_{V_j} \neg\alpha\} |$;
- $(\mathcal{F}_{NML}, x) \models_{V_{av}} AV(\alpha) \iff V_{av}(\alpha)$
  (That is the truth of the formula $AV(\alpha)$ coin?ides with valuation $V_{av}(\alpha)$ on x);
- $(\mathcal{F}_{NML}, x) \models_{V_j} Rel_j(\alpha) \iff (\mathcal{M}_{NML}, x) \models_{V_j} Stl_j(\alpha) \wedge AV(\alpha) \wedge$
  $\wedge\, \alpha$.

Each element of the NML-frame $\mathcal{F}_{NML}$ can be understood as an information point. All information points that are potentially available at the same moment of time form clusters: C_0, C_1, C_2, . . .

The relation R_j connects with each other those points of cluster that are accessible to some agent j at the considered moment of time.

$\forall x, y \in W : xR_jy \iff$

$\iff \forall \varphi \in L \ [(\mathcal{F}_{NML}, x) \models_{V_j} K_j(\varphi) \iff (\mathcal{F}_{NML}, y) \models_{V_j} K_j(\varphi)]$,

where L is the language of logic NML.

The time relation Next links information points into a linear and discrete time stream. For two points x and y, the expression $xNexty$ means that y will be available at the next moment of time relative to x. The time is thought of as discrete. The discreteness of the time flow is understood as follows: between two moments of time C_n and C_{n+1} there are no other time points. The time chain of states has a beginning C_0.

$$C_0 \overset{Next}{\longrightarrow} C_1 \overset{Next}{\longrightarrow} C_2 \overset{Next}{\longrightarrow} C_3 \overset{Next}{\longrightarrow} \dots$$

Also important is the assumption of the linearity of the time flow, that is, the absence of branching: after each time moment C_n there can only be one moment C_{n+1} such that $C_n \overset{Next}{\longrightarrow} C_{n+1}$.

Let α be some formula. The time degree $td(\alpha)$ of the formula α is determined as follows: $td(p) = td(\top) = td(\bot) = 0,\ td(\alpha \wedge \beta) =$
$= td(\alpha \to \beta) = td(\alpha \vee \beta) = \max\{td(\alpha), td(\beta)\},\ td(\neg\alpha) = td(\alpha),$
$td(\mathcal{N}(\alpha)) = td(\alpha) + 1.$

The "modal" degree is defined analogously with a difference for not-boolean formulas: $md(K_j(\alpha)) = md(\alpha) + 1.$

Now we are ready to proof of the following theorem.

Theorem 1. *The logic NML is decidable and has FMP.*

Proof. Assume that formula α is given and it is not the theorem of our logic. So α is not valid at defining model: $(\mathcal{M}_{NML}, x) \not\models \alpha$. Let α has temporal degree $td(\alpha) = k$.

Without loss of generality we can assume that the element x belongs to the cluster C_0 of the model $\mathcal{M}_{NML}$. First we define standard filtration on each cluster of our model as follows.

Let $Sub(\alpha)$ be the set of all subformulas of formula α. We define:

$$x \sim y \iff \exists\, i : x, y \in C_i \ \wedge\ \forall A \in Sub(\alpha)$$

$$[(\mathcal{M}_{NML}, x) \models A \iff (\mathcal{M}_{NML}, y) \models A].$$

Then we define

$$\forall x \in \mathcal{M}_{NML},\ \ [x] := \{t : x \sim t\};,\ W^f = \{[x] : x \in W\} \text{ and}$$

$$\forall x, y \in \mathcal{M}_{NML}\ (x\ Next\ y \iff [x] Next^f [y]);$$

$$\forall x, y \in \mathcal{M}_{NML}\ \forall j \leqslant n\ (xR_jy \implies [x] R_j^f [y]);$$

$$[x] R_j^f [y] \iff K_j(A) \in Sub(\alpha)$$

$$[(\mathcal{F}_{NML}, x) \models_{V_j} K_j(A) \implies (\mathcal{F}_{NML}, y) \models_{V_j} A];$$

$$\forall\ p \in Sub(\alpha)\ (V_j^f(p) := \{[x] : x \in V_j(p)\});$$

$$\forall\ p \in Sub(\alpha)\ (V_{av}^f(p) := \{[x] : x \in V_{av}(p)\}).$$

Let's denote obtained frame and model as

$$\mathcal{F}_{NML}^F = \langle W^f, Next^f, R_1^f, \ldots R_n^f \rangle,$$

$$\mathcal{M}^F = \langle W^f, Next^f, R_1^f, \ldots R_n^f; V_1^f \ldots V_n^f, V_{av}^f \rangle.$$

All clusters of that model turned out to be filtered separately.

$$C_0^f; \overset{Next}{\longrightarrow} C_1^f; \overset{Next}{\longrightarrow} C_2^f; \overset{Next}{\longrightarrow} C_3^f; \overset{Next}{\longrightarrow} \ldots$$

The power of each cluster in the resulting model do not exceed $2^{|Sub(\alpha)|}$.

By standard well known filtration lemma the next statement holds:

$$\forall A \in Sub(\alpha) \quad [(\mathcal{M}_{NML}, x) \models A \iff (\mathcal{M}^F, [x]) \models A].$$

Thus we get:

Statement 1. $(\mathcal{M}^F, [x]) \not\models_{V_j} \alpha$.

Let us take NML-model

$$\mathcal{M}^F = \langle \bigcup_{i\in J} C_i^f, Next^f, R_1^f, \dots R_n^f; V_1^f \dots V_n^f, V_{av}^f \rangle,$$

and a submodel

$$\mathcal{M}_k^F = \langle (C_0^f \cup C_1^f \cup, \dots, \cup C_k^f), Next^f, R_1^f, \dots R_n^f; V_1^f, \dots V_n^f, V_{av}^f \rangle$$

and let the valuation on clusters $C_0^f, C_1^f, \dots, C_k^f$ coincides on this models.

Statement 2. Let $td(\alpha) = k$ then:

$$\forall x \in C_0^f \quad [(\mathcal{M}^F, x) \models \alpha \iff (\mathcal{M}_k^F, x) \models \alpha]. \tag{1}$$

That is the truth of the formula α on any element $x \in C_0^f$ is uniquely determined by the values of all propositional variables included in the formula on elements of clusters $C_0^f,\ C_1^f,\ \dots, C_k^f$ of the model $\mathcal{M}^F$.

Proof. By induction on temporal degree of formula α.

Statement 3. If $td(\alpha) = 0$ then :

$$\forall x \in C_i^f \ \ [(\mathcal{M}^F, x) \models \alpha \iff (\mathcal{M}_k^F, x) \models \alpha)],\ \ 0 \leqslant i \leqslant k$$

holds.

Proof. (I) By induction on "modal" degree $md(\alpha)$. If $md(\alpha) = 0$, then the formulas α consist only of propositional variables and standard Boolean operations. In this case, the truth of the formulas α on the element $x \in C_i^f$ is determined only by the valuation of the propositional variables included in the formula only on the element $x \in C_i^f$. Thus, according to the construction of model $\mathcal{M}_k^F$, the statement (1) is true.

Assume that for formulas α such as $md(\alpha) \leqslant \ell$ the statement is true. Let us show that for formulas such that $md(\alpha) = \ell + 1$ the statement also holds.

Let's $\alpha = K_j(\beta)$ and if $(\mathcal{M}^F, x) \models K_j(\beta)$
then $\forall y \in C_i^f,\ \ x R_j^f y : (\mathcal{M}^F, y) \models \beta$ and $md(\beta) \leqslant \ell$.
By IH $(\mathcal{M}_k^F, x) \models \beta$ holds. Hence we conclude

$$\forall x \in C_i^f \ \ [(\mathcal{M}^F, x) \models \alpha \Longrightarrow (\mathcal{M}_k^F, x) \models \alpha)],\ \ 0 \leqslant i \leqslant k$$

The proof in the opposite direction is similar. The Statement 3 is proved.

(II) IH : Assume that for all formulas α which temporal degree
$td(\alpha) \leqslant k-1$ the statement (1) is true. Let us show that for formulas α such that $td(\alpha) = k$ the statement (1) also holds.

Without loss of generality we can consider formula $\alpha = \underbrace{\mathcal{N}(\mathcal{N}(\dots(\mathcal{N}}_{k}(\beta)))\dots)$ with $td(\beta) = 0$.

Assume that $x \in C_0^f \subseteq \mathcal{M}^F$ and $(\mathcal{M}^F, x) \models \alpha$ then

$$\forall y \in C_1^f \subseteq \mathcal{M}^F,\ (x\, Next\, y) : (\mathcal{M}^F, y) \models \underbrace{\mathcal{N}(\mathcal{N}(\dots(\mathcal{N}}_{k-1}(\beta)))...)$$

and time degree of this formula is $k-1$.

Now if we take C_1^f as initial cluster then by IH
$\forall z \in C_k^f \subseteq \mathcal{M}^F\,(\mathcal{M}^F, z) \models \beta$. Because of all clusters $C_0^f, C_1^f \ldots C_k^f$ form initial interval of the frame $\mathcal{F}_{NML}^F$ and belong to the model $\mathcal{M}_k^F$ we obtain

$$\forall x \in C_0^f \quad [(\mathcal{M}^F, x) \models \alpha \Longrightarrow (\mathcal{M}_k^F, x) \models \alpha].$$

The proof in the opposite direction is similar.

Finally with the help of contraposition we get

$$(\mathcal{M}_{NML}, x) \not\models \alpha \Longleftrightarrow (\mathcal{M}_k^F, x) \not\models_{V_j^s} \alpha$$

The power of the model $\mathcal{M}_k^F$ do not exceed $k2^{|Sub(\alpha)|}$.

The Theorem is proved. □

Conclusion

We study intransitive temporal multi-agent logic with multi-valued variables. This logic is defined semantically as a set of formulas that are true on linear models with time clusters and multi-valued valuations. It is proven that the resulting logic is solvable and has FMP.

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Нестандартная логика и достоверность информации

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Аннотация. Изучается нетранзитивная временная мульти-агентная логика с мульти-означиваниями агентов для переменных и формул. Ранее время и знание моделировались с помощью моделей Крипке, структура которых выглядела как простой единый кластер с множеством отношений достижимости для агентов. Здесь мы развиваем этот подход и используем модели Крипке, которые представляют собой линейное нетранзитивное время и состояния, представленные произвольными временными кластерами для мульти-отношений достижимости агентов. Эта логика определяется семантически, как множество формул, истинных на линейных моделях с мульти-означиваниями переменных и кластерами состояний. Мы предлагаем обоснование такого подхода и методику вычисления истинностных значений формул. Основные результаты касаются проблемы разрешимости. Мы доказываем, что полученная логика разрешима и финитно аппроксимируема.

Ключевые слова: фрейм и модель Крипке, мульти-агентные логики, NLM-логики, проблема разрешимости.