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SINGULARITIES OF MOMENTUM MAPS OF INTEGRABLE HAMILTONIAN SYSTEMS WITH TWO DEGREES OF FREEDOM

§1. Introduction

Let (M^4, ω) be a symplectic manifold. A Hamiltonian system

$$\dot{x}^i = (\operatorname{sgrad} H)^i = \omega^{ij} \frac{\partial H}{\partial x^j}$$

is called *integrable,* if there exists a function f which is constant on the integrale trajectories of the system and functionally independent with H (i.e., the differentials df and dH are independent almost everywhere). The function f is called *an additional integral.*

The mapping $F = (H, f) : M^4 \to \mathbb{R}^2$ is called *the momentum mapping* of the integrable Hamiltonian system. The foliation of the manifold M^4 into connected components of level surfaces of the momentum mapping F is called *a Liouville foliation.* We will assume that for any a and b the common level surface $H^{-1}(a) \cap f^{-1}(b)$ is compact. According to the well-known Liouville theorem [1], any non-singular fiber of this foliation is a two-dimensional torus (Liouville torus). The solutions of our equation on each of these tori are periodic or conditionally periodic winding. If the Hamiltonian system is non-resonant then the Liouville foliation does not depend on the choice of the additional integral f.

Definition 1.1. *A set $U \subset M^4$ is called saturated, if it consists of the whole fibers, that is, for any $x \in U$ the fiber of the Liouville foliation through x belongs to U.*

Definition 1.2. *A simple closed curve $C \subset M^4$ is called critical, if at any its point the vectors* $\operatorname{sgrad} H$ *and* $\operatorname{sgrad} f$ *are tangent to C.*

If $\operatorname{sgrad} H \neq 0$ *on C we supply C with the orientation induced by* $\operatorname{sgrad} H$. *Otherwise we leave C nonoriented.*

Definition 1.3. *Two saturated sets V and $\widetilde{V}$ are called topologically equivalent, if there exists a homeomorphism $\tau : V \to \widetilde{V}$ which preserves the Liouville foliation structure (i.e., taking fibers to fibers) and takes critical curves to critical curves, preserving their orientations.*

The aim of this paper is to describe the topological structure of integrable Hamiltonian systems in saturated neighborhoods of singular points of the momentum map. A similar problem was formulated by S. Smale in his well-known paper [2]. The papers [3, 4] by L. M. Lerman, Ya. L. Umanskii are also devoted to this problem. Many effects that appear at singular points of a momentum map were discovered by M. P. Kharlamov in his analysis of concrete integrable problems in rigid body dynamics [5], see also [6–8]. Besides, many papers have been devoted to the investigation of the local structure of momentum map singularities [9–11].

Moreover, the analysis of singular points is useful because of the following problem. Suppose h is a regular value of H and the restriction of f to the isoenergy surface $Q_h = \{H = h\}$ is a Bott function. A complete topological invariant of such objects was constructed in [12–16]. According to [16], the topology of an integrable Hamiltonian system on Q_h can be coded in the form of the so-called marked molecule $W^*(Q^3, H)$.

The calculation of marked molecules for concrete examples of integrable Hamiltonian systems is not an easy problem, and the analysis of saturated neighborhoods of singular points can be very helpful here. Actually, the problem of describing the topological structure is close to the following question.

Let $Q_h = \{H = h\}$ be an isoenergy surface of an integrable Hamiltonian system and $W^*(Q_h, H)$ the corresponding molecule describing the topology of the Liouville foliation on Q_h. The question is : what is happening to the molecule $W^*(Q_h, H)$ when the energy level h changes? Usually $h_0 \in \mathbb{R}$ can be a "bifurcation value" (in other words, $W^*(Q_{h_0}, H)$ changes when the energy level passes through h_0) in the two following cases: 1) the level $\{H = h_0\}$ contains a critical point x_0 at which $dF = 0$, 2) the additional integral $f : Q_{h_0} \to \mathbb{R}$ is not a Bott function, in other words, in Q_{h_0} there is a degenerate critical manifold of f. Thus, to understand how the molecule changes we should first find out what happen near critical points and degenerate critical manifolds.

§2. Basic notions and the statement of the problem

Let $\dot{x} = \operatorname{sgrad} H(x) = \omega^{-1}(dH(x))$ be an integrable Hamiltonian system on a symplectic manifold $(M^4, \omega), f : M^4 \to \mathbb{R}$ an additional integral, i.e. $\{H, f\} = \operatorname{sgrad} H\,(f) = 0$, and $F = (H, f) : M^4 \to \mathbb{R}^2$ the corresponding momentum map. By $L(x)$ we denote the fiber of the Liouville foliation passing through $x \in M^4$ or, which is the same, the connected component of the inverse image $F^{-1}(F(x))$, containing x. Since the flows sgrad H and sgrad f commute we can define the Poisson action of the Abelian group $\mathbb{R}^2$ on M^4, generated by flowing along the vector fields sgrad H and sgrad f :

$$\Phi : \mathbb{R}^2 \to \mathrm{Diff}(M^4).$$

By $O(x)$ we denote the orbit of this action through $x \in M^4$. It is well known that the orbit $O(x)$ must be homeomorphic to one of the following manifolds: $\mathbb{R}^0, \mathbb{R}^1, \mathbb{R}^2, S^1, S^1 \times \mathbb{R}^1, T^2 = S^1 \times S^1$.

Let $K_1 = \{x \in M^4 | \mathrm{rk} dF(x) < 2\}$ be the set of critical points of the momentum map, $\Sigma = F(K)$ be the bifurcation diagram. In general, singularities of the momentum map can be rather complicated. Therefore, first of all we describe non-degenerate orbits.

Let us consider a point $x_0 \in K$ such that $\mathrm{rk} dF(x_0) = 1$. Let $dH(x_0) \neq 0$ (the case $df \neq 0$ is analogous). It is clear that in this case the orbit $O(x_0)$ is one-dimensional, and x_0 is critical for the restriction of f to the level surface $\{H = H(x_0)\}$.

Definition 2.1. *We call the one-dimensional orbit $O(x_0)$ non-degenerate if the rank of the Hessian of the function $f\big|_{\{H=H(x_0)\}}$ at the point x_0 is equal to 2. This condition means exactly that locally $O(x_0)$ is a non-degenerate critical submanifold of the function f restricted to the level $\{H = H(x_0)\}$.*

Commentary. This definition corresponds to the notion of the Bott integral introduced by A. T. Fomenko [12]. But sometimes non-degeneracy of orbits is defined in another way (see, for example, [3, 4]). By our definition non-degeneracy means that the orbit is elliptic or hyperbolic. According to [3, 4] the so-called parabolic orbits are also considered as non-degenerate ones.

Proposition 2.1. ([3, 11]) Let $O(x_0)$ be a non-degenerate one-dimensional closed orbit and $dh(x_0) \neq 0$. Then there exists a tubular neighborhood U of $O(x_0)$ such that

(1) there exists a pair $(\lambda, \mu) \in \mathbb{R}^2$ such that $(\lambda dH + \mu df) = 0$;

(2) the set $U \cap K = P$ is a smooth symplectic submanifold in M^4;

(3) the function $H : P \to \mathbb{R}$ has no singularities and stratifies P into one-dimensional closed non-degenerate orbits, in particular, $P \approx S^1 \times (0,1)$;

(4) the image of P under the momentum map is a smooth curve $F(P) \subset \Sigma$.

Consider $p \in K_1$ such that $\mathrm{rk}(dF(p)) = 0$. Recall that the Poisson bracket $\{,\}$ allows to define the structure of a Lie algebra on the space of symmetric bilinear forms $\mathrm{Sym}(T_p^* M^4 \otimes T_p^* M^4)$ in the following natural way. If g_1 and g_2 are two smooth functions having singularities at $p \in M^4$, then we put by definition

$$[d^2 g_1(p), d^2 g_2(p)] = d^2\{g_1, g_2\}(p).$$

It is easily seen that the Lie algebra defined on $\mathrm{Sym}(T_p^* M^4 \otimes T_p^* M^4)$ by operation $[,]$ is isomorphic to the symplectic Lie algebra $sp(2,\mathbb{R})$. Since f and H commute, the Lie subalgebra $L_{f,H}$ generated by $d^2 f(p)$ and $d^2 H(p)$ is commutative.

Definition 2.2. [3] *The singular point $p \in K$ is called non-degenerate if $L_{f,H}$ is a Cartan subalgebra.*

Remark. Everywhere below, speaking about non-degenerate critical points we mean the points $p \in K$ such that $\mathrm{rk}(dF(p)) = 0$. If $\mathrm{rk}(dF(p)) = 1$ then we speak about degeneracy or non-degeneracy of the one-dimensional orbit $O(p)$.

The local structure of singularities of a momentum map $F = (H, f) : M^4 \to \mathbb{R}^2$ at non-degenerate singular points has been explored in the paper by L. M. Lerman and Ya. L. Umanskii [3]. Recall briefly some basic results from [3], which we will need below.

Let $x_0 \in M^4$ be a non-degenerate critical point of F and $L_{H,f}$ the Cartan subalgebra generated by $d^2 f(x_0)$ and $d^2 H(x_0)$. It is known that there are four different types of Cartan subalgebras in the real Lie algebra $sp(2,\mathbb{R})$ (see [18]). In accordance with this we can distinguish four types of non-degenerate critical points: a) center-center, b) center-saddle, c) saddle-saddle and d) focus-focus.

Theorem 2.1 (Lerman, Umanskii [3]). *Let $x_0 \in M^4$ be a non-degenerate critical point of the momentum map $F = (H, f)$. Then in some neighborhood of x_0 it there exists a canonical coordinate system (p_1, q_1, p_2, q_2) such that the functions H and f have the following form:*

1) *the center-center type*

$$H = h_0 + (p_1^2 + q_1^2)H_1 + (p_2^2 + q_2^2)H_2$$

$$f = f_0 + (p_1^2 + q_1^2)F_1 + (p_2^2 + q_2^2)F_2$$

2) *the saddle-center type*

$$H = h_0 + p_1 q_1 H_1 + (p_2^2 + q_2^2)H_2$$

$$f = f_0 + p_1q_1F_1 + (p_2 + q_2)F_2$$

3) *the saddle-saddle type*

$$H = h_0 + p_1q_1H_1 + p_2q_2H_2$$

$$f = f_0 + p_1q_1F_1 + p_2q_2F_2$$

4) *the focus-focus type*

$$H = h_0 + p_1q_1H_{11} + p_1q_2H_{12} + p_2q_1H_{21} + p_2q_2H_{22}$$

$$f = f_0 + p_1q_1F_{11} + p_1q_2F_{12} + p_2q_1F_{21} + p_2q_2F_{22}$$

where H_i, H_{ij}, F_i, F_{ij} are smooth functions and

$$H_{11}(x_0) = H_{22}(x_0), F_{11}(x_0) = F_{22}(x_0),$$
$$H_{12}(x_0) = -H_{21}(x_0), F_{12}(x_0) = -F_{21}(x_0),$$
$$H_1(x_0)F_2(x_0) - H_2(x_0)F_1(x_0) \neq 0,$$
$$H_{11}(x_0)F_{12}(x_0) - H_{12}(x_0)F_{11}(x_0) \neq 0.$$

Corollary 2.1 ([3]). *Let $x_0 \in M^4$ be a non-degenerate critical point of the momentum mapping F of saddle-saddle, saddle-center or center-center type, K be the set of critical points of F. Then there exists a neighbourhood $U(x_0)$ of the point x_0 such that*

1) *$K_1 \cap U(x_0) = P_1 \cup P_2$, where P_1 and P_2 are two-dimensional symplectic submanifolds which intersect transversally at the point x_0;*

2) *all one-dimensional orbits lying in P_1 and P_2 are non-degenerate.*

Definition 2.3. *Consider four integrable Hamiltonian systems (with noncompact fibers) on the symplectic space $\mathbb{R}^4$ with the standard form $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, generated by the Hamiltonians*

1) $H_c = p_1^2 + q_1^2$ *(center-center case),*
2) $H_s = p_1q_1$ *(saddle-saddle case),*
3) $H_c = p_1q_1 + p_2^2 + q_2^2$ *(center-saddle case),*
4) $H_f = p_1q_1 + p_2q_2$ *(focus-focus case).*

The additional independent integrals for these systems are
$f_c = p_1^2 + q_1^2 + p_2^2 + q_2^2$,
$f_s = p_1q_1 + p_2q_2$,
$f_{cs} = p_1q_1 - (p_2^2 + q_2^2)$,
$f_f = p_1q_2 + p_2q_1$, respectively. These systems together with their additional integrals are called *canonical* systems for critical points of center-center, saddle-saddle, center-saddle and focus-focus type respectively.

Corollary 2.2. *Let $x_0 \in M^4$ be a nondegenerate critical point of the momentum mapping F. Then there exists a neighborhood U_{x_0} and a homeomorphism $\tau : U_{x_0} \to \mathbb{R}^4$ such that*

1) τ takes fibers of F to those of the canonical system which has the same type as the point x_0 has,

2) τ preserves the orientation of critical curves by sgrad H.

Definition 2.4. *A smooth curve $\gamma(t)$ in $\mathbb{R}^2$ is called admissible if the following conditions hold.*

1) γ intersects Σ transversally.

2) If A is a point of intersection then critical points in $F^{-1}(A)$ form a collection of non-degenerate orbits.

3) For any point of intersection A there exists a neighborhood U_A such that $\Sigma \cap U_A$ can be represented as a graph of some smooth function of H.

Proposition 2.2. *Let $\gamma(t)$ be an admissible curve. Then its preimage $Q_\gamma = F^{-1}(\gamma)$ is a smooth three-dimensional submanifold in M^4 and the function $f_\gamma : Q_\gamma \to \mathbb{R}$ defined by $\gamma(f_\gamma(x)) = F(x)$ is a Bott function, i.e. its critical points form non-degenerate critical submanifolds.*

Following the papers [15, 16], for each admissible curve $\gamma(t)$ we can define marked molecule $W^*(Q_\gamma)$, which describe the structure of the Liouville foliation on the "isoenergy surface" $Q_\gamma = F^{-1}(\gamma)$. One can easily see that a continuous deformation of the curve $\gamma(t)$ within the class of admissible curves does not change the structure of the Liouville foliation on Q_γ and, consequently, the molecule $W^*(Q_\gamma)$.

Definition 2.5. *A point $y \in \Sigma$ is called an isolated singular point of Σ if the preimage $F^{-1}(y)$ contains at least one critical point and any circle γ_ϵ of a sufficiently small radius ϵ with the center at this point is admissible.*

Definition 2.6. *Let y be an isolated singular point of Σ. The marked molecule $W^*(Q_{\gamma_\epsilon})$ for some small circle γ_ϵ with center y is called the circle molecule and denoted by $W^*(y)$.*

Bifurcation diagrams of classical integrable systems have simple structure near their isolated singular points (Figures $1a, 1b, 1c, 1d$ correspond to center-center, center-saddle, saddle-saddle and focus-focus cases respectively). We will assume below that near singular points the bifurcation diagram looks like the figures $1a, 1b, 1c$ or $1d$.

Now we are able to formulate the main question we are interested in. Our aim is to describe the topological structure of the Liouville foliation

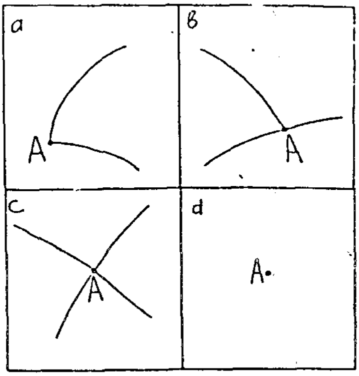

Fig. 1.

on some saturated neighborhood of isolated singular points and to show how the circle molecule can be constructed.

§3. Saddle-saddle case

The aim of the following two sections is to classify top'ologically integrable Hamiltonian systems in saturated regular neighborhoods of singular fibers containing one or several focus-focus or saddle-saddle points.

Definition 3.1. *Let a saturated open set $U^4 \subset M^4$ satisfy the following conditions.*

(1) *The bifurcation diagram of U^4 consists of a pair of tranversally intersecting curves (the point of intersection being denoted by A); each of this curves on the plane $\mathbb{R}^2(H, f)$ is the graph of some smooth function $f = f(H)$.*

(2) *The preimage $F^{-1}(A)$ of A is connected.*

(3) *Any critical point from U^4 is nondegenerate, $F^{-1}(A)$ contains one or several saddle-saddle points.*

(4) *U^4 is a regular neighborhood of $F^{-1}(A)$.*

Such sets are called *saddle neighborhoods.*

Lemma 3.1. *The singular fiber $F^{-1}(A)$ is a 2-dimensional CW-complex whose sells are all orbits of the Poisson action Φ.*

This lemma was proved, for example, in [20, 21].

Plan of the proof.

Suppose that the closure $\bar{v}$ of a two-dimensional orbit v contains a saddle-saddle point. Then the orbit v is homeomorphic to $\mathbb{R}^2$.

Recall that we denote the set of critical points of M^4 by K.

Consider $G = K \cap F^{-1}(A)$. One can see that G is a 1-dimensional CW-complex and each cell of G is an orbit of Φ.

Let C be the union of G and such 2-dimensional orbits that attached G. The above statements imply that C is CW-complex, each cell is an orbit.

The proof is completed because it is evident that C is a compact set.

Note that the vector field sgrad H has no singularities on edges of C. Hence we can naturally orient the edges of C by sgrad H. Thus if we have a saddle neighborhood we can naturally construct CW-complex C with an oriented 1-skeleton.

Definition 3.2. *CW-complex C with the orientation on its 1-skeleton is called the saddle complex of the saddle neighborhood.*

Let us recall the definition of an atom introduced in [16].

Definition 3.3. *An atom is a pair (B^2, G) where B^2 is a two-dimensional surface, G is a graph, embedded in B^2 such that*

(1) *Each vertex of G has degree 0,2 or 4.*

(2) *The subtraction $B^2 \backslash G$ is a collection of rings $S^1 \times I$.*

(3) *There exists a decomposition of rings of $B^2 \backslash G$ into two classes (positive and negative) such that any edge of G attached to one positive and one negative ring.*

(4) *G is a connected set on each connected component of B^2.*

Usually vertices of G with degree 2 are called and denoted by *stars*.

Unlike the original definition we do not require here the connectedness of the surface B^2 in the definition of the atom and we assume that the decomposition of $B^2 \backslash G$ into positive and negative rings is fixed. In section 8 we deal with connected atoms.

Lemma 3.2. *$K \cap U^4$ is a pair of transversally intersecting 2-dimensional smooth surfaces, each of which has a natural structure of atoms (B_w, G_w) and (B_b, G_b), where $G_w = B_w \cap F^{-1}(A), G_b = B_b \cap F^{-1}(A)$.*

This lemma becomes evident if we consider the local structure of the set K.

Definition 3.4. *The pair of atoms $(B_w, G_w), (B_b, G_b)$ is called the L-type of the saddle neighborhood.*

It is clear that the L-type and saddle complex C are both topological invariants of saddle neighborhoods. But there are examples of nonequivalent saddle neighborhoods with the identical L-types and different complexes, or, conversely, with the identical complexes and different L-types.

Definition 3.5. *The triple $[C, (B_w, G_w), (B_b, G_b)]$, where C is the saddle complex and the pair $((B_w, G_w), (B_b, G_b))$ is the L-type of the saddle neighborhood is called the CL-type of the saddle neighborhood.*

Note that $G_w \cap G_b$ consists of saddle-saddle points.

Let $[C, (B_w, G_w), (B_b, G_b)]$ and $[\widetilde{C}, (\widetilde{B}_w, \widetilde{G}_w), (\widetilde{B}_b, \widetilde{G}_b)]$ be CL-types of saddle neighborhoods U^4 and $\widetilde{U}^4$ respectively.

Definition 3.6. *CL-types $[C, (B_w, G_w), (B_b, G_b)]$ and $[\widetilde{C}, (\widetilde{B}_w, \widetilde{G}_w), (\widetilde{B}_b, \widetilde{G}_b)]$ are called equivalent if there exists a homeomorphism $X : C \cup B_w \cup G_b \to \widetilde{C} \cup \widetilde{B}_w \cup \widetilde{B}_b$ such that $X(C) = \widetilde{C}$ and X preserves the orientation of edges of C.*

Theorem 3.1. *CL-type is a complete topological invariant of saddle neighborhoods (i.e., two saddle neighborhoods are topologically equivalent if and only if their CL-types are equivalent).*

Plan of the proof. Let CL-types $[C, (B_w, G_w), (B_b, G_b)]$ and $[\widetilde{C}, (\widetilde{B}_w, \widetilde{G}_w), (\widetilde{B}_b, \widetilde{G}_b)]$ be equivalent. We can extend the homeomorphism X to that between U^4 and $\widetilde{U}^4$ so that the new homeomorphism $\widetilde{X}$ we obtain preserves the fibers and the orientation of critical circles by the vector field sgrad H.

First we extend X to the homeomorphism between some spherical neighborhoods of saddle-saddle points. We can do it in such a way that this homeomorphism preserves fibers. Indeed, each saddle neighborhood has the same topological structure near saddle- saddle points. Denote the result of such extension by X_1.

Then we extend X_1 to a homeomorphism between a regular neighborhood of edges (denote it by X_2) and finally we extend X_2 to a homeomorphism between a neighborhood of the 2-cells, see [21] for details.

§4. Focus-focus case

Definition 4.1. *A saturated open set $U^4 \subset M^4$ is called a focus neighborhood if the following conditions hold.*

(1) *The bifurcational diagram of U^4 consists of a single point (denoted by A).*

(2) The preimage $F^{-1}(A)$ is connected.

(3) *Any critical point of U^4 is nondegenerate.*

(4) *U^4 is a regular neighborhood of $F^{-1}(A)$.*

It is evident that all critical points of U^4 have a focus-focus type.

Definition 4.2. *The number of critical points in the focus neighborhood is called the weight of the focus neighborhood.*

Theorem 4.1. *The weight is a complete topological invariant of a focus neighborhood (i.e., two focus neighborhoods are topologically equivalent if and only if their weights are equal).*

Plan of the proof. First, we describe the topological structure of $F^{-1}(A)$.

Let $S_1, S_2, \dots, S_n$ be 2-dimensional spheres. It is useful to suppose that index belongs to $\mathbb{Z}_n$, so $(n-1)+1=0$. Choose two points A_i and B_i on each sphere S_i and glue A_i and $B_{i+1}$for any i. As a result we obtain some topological space V.

Definition 4.3. *V is called an n-chain.*

Lemma 4.1. [21] *Let the weight of the focus neighborhood be equal to n. Then the preimage $F^{-1}(A)$ is the n-chain.*

Then, if U_1^4 and U_2^4 are focus neighborhoods with equal weights then their singular fibers are homeomorphic. If we stepwise extend this homeomorphism to a homeomorphism between some sufficiently small spherical neighborhood of vertex, next to a homeomorphism between some neighborhoods of the union of this vertex and their neighboring "rings" $R^1$and R^2, then to some neighborhood of the union $R^1 \cup R^2$ and vertices that touched R^1 and R^2 and so on we obtain homeomorphism between U_1^4 and U_2^4. It is evident that on each step we may choose an extension in such a way that it preserves the Liouville fibers.

§5. Admissible CL-type

Let $[C,(B_w,G_w),(B_b,G_b)]$ be the CL-type of some saddle neighborhood. Suppose that G_w is colored in white, G_b is colored in black.

Definition 5.1. *Let $[C,(B_w,G_w),(B_b,G_b)]$ be the CL-type of some saddle neighborhood. We say that a 2-cell $W \subset C$ is a square if the following conditions hold.*

(1) W is a quadrangle.

(2) The opposite sides of W are colored in the same colors: two in white, two in black.

(3) The opposite sides of W have parallel orientation (see Fig. 2).

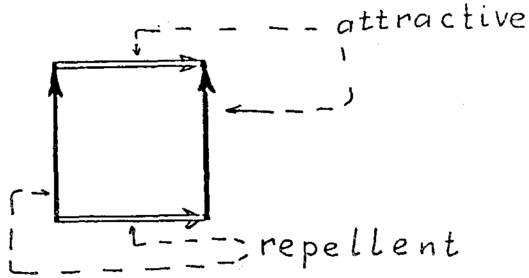

Fig. 2.

A square is shown in Fig. 2. Two of its sides are called repellent, another two sides are called attractive (see Fig. 2).

Sometimes we will assume that every square has the same canonical linear structure as the square $\{0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ has.

Property 1. Each 2-cell of the saddle complex C (with colored 1-skeleton C_1) is a square.

Let P be a vertex of C.

Property 2. Some neighborhood of P is homeomorphic to the direct product of two pairs of intersecting segments (i.e, of two crosses). If we assume that one of these pairs is black, another is white and the rays of these crosses are oriented as in Fig. 3, we can choose this homeomorphism in such a way that it will preserve color and orientation.

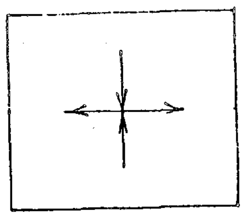

Fig. 3.

This property is evident.

Corollary. *Each edge of C is a result of attaching together of 4 sides of the squares.*

Let G be the critical graph of the atom (B^2, G) (B^2 may be nonconnected, the decompositions of $B^2 \backslash G$ into positive and negative rings is fixed).

Let e_1, e_2 be the edges of G with the common vertex Q.

Definition 5.2. *The angle $(e_1, e_2)_Q$ is called positive if there exist a positive ring R^2 such that $\overline{(R^2 \cap U_Q)} \supset [(e_1 \cup e_2) \cap U_Q]$ for any neighborhood U_Q of Q.*

Let a square $xay*$ belong to C. We will say that the edge x corresponds to the edge y along the edge a.

Property 3. Suppose that the edge x corresponds to x_1 along a, the edge y corresponds to y_1 along a, and A_1 and A_2 are the vertices of the edge a. Then the angles $(x, y)_A$ and $(x_1, y_1)_{A_1}$ are simultaneously positive or nonpositive.

Definition 5.3. *The triple $[C, (B_w, G_w), (B_b, G_b)]$, where C is a finite CW-complex with oriented 1-skeleton, (B_w, G_w) and (B_b, G_b) are atoms, $C \cap B_w = G_w, C \cap B_b = G_b$, is called an admissible CL-type if it satisfies the properties 1, 2 and 3.*

§6. Realization theorems for saddle and focus neighborhoods

Theorem 6.1. *For any natural number n there exists a focus neighborhood with the weight n.*

Theorem 6.2. *For any admissible CL-type $[C, (B_w, K_w), (B_b, K_b)]$ there exists a saddle neighborhood with CL-type $[C, (B_w, K_w), (B_b, K_b)]$.*

Proofs. Let us suppose that quadruple (A^4, ω, H, f) sets an integrable Hamiltonian system (here (A^4, ω) is a symplectic manifold, H is a Hamiltonian and f is an additional integral), we do not require the compactness of fibers. Suppose that a discrete group G acts on the manifold A and preserves the symplectic form ω, the hamiltonian H and the additional integral f. It is clear that $(A^4/G, \omega, H, f)$ sets an integrable Hamiltonian system.

To prove the theorems 1, 2 we will construct the integrable Hamiltonian systems (F, ω, H, f) and (S, ω, H, F) and two systems of discrete groups $FG = \{G_i\} (i = 1, 2, \dots)$ and $SG = \{G_\alpha\}$ (α runs over the set of all admissible CL-types) such that

$(F/G_i, \omega, H, f)$ is a focus neighborhood with the weight i,
$(S/G_\alpha, \omega, H, F)$ is a saddle neighborhood with the Cl-type α.

The construction of the integrable Hamiltonian system (F,ω,H,f) **and the system of groups FG.** It is known that there exists a focus neighborhood $(\widetilde{F}, \widetilde{\omega}, \widetilde{H}, \widetilde{f})$ with weight 1. For example, the focus-focus point appearing in the famous Lagrange case in rigid body dynamics has such a neighborhood. It is easy to see that the fundamental group $\pi_1(\widetilde{F})$ is $\mathbb{Z}$. Indeed, the singular fiber of this system is the deformation retract of $\widetilde{F}$. The singular fiber is a 1-chain, so its fundamental group is $\mathbb{Z}$.

Let (F, p) be the universal covering of $\widetilde{F}$ and $f = p^*(\widetilde{f}), H = p^*(\widetilde{H}), \omega = p^*(\widetilde{\omega})$.

Note that the group $\mathbb{Z}$ acts on F as the fundamental group of the base. This action preserves H, f, ω. Let G_i be the subgroup of $\mathbb{Z}$, generated by the number i. By definition, put $FG = \{G_i\}(i = 1, 2, \dots)$.

To complete the proof of the theorem 1, it is sufficient to note that $(F/G_i, \omega, H, f)$ is a focus neighborhood with the weight i.

The construction of the integrable Hamiltonian system (S,ω,H,f) **and the system of groups** FS**.** Let (P^2, K) be an atom (possibly nonconnected). Mark (for example, by arcs) all the positive angles of the graph K. One can see that the graph K and marked positive angles define the atom (P^2, K) uniquely. Therefore, CL-type is defined uniquely by the saddle complex C and marked positive angles.

Let $[C, (B_w, K_w), (B_b, K_b)]$ be an admissible CL-type.

Definition 6.1. *The saddle complex C together with coloring of the edges of its 1-skeleton (as above, see the previous section) and marking positive angles is called the marked complex of the CL-type* $[C, (B_w, K_w), (B_b, K_b)]$.

So, if we prove that for any marked complex C^* there exists a saddle neighborhood with the marked complex C^* we have proved Theorem 2.

Let C_1^* and C_2^* be marked complexes (possibly, infinite).

Definition 6.2. *A cell mapping $\vartheta : C_1^* \to C_2^*$ is called a right mapping if the following conditions hold.*

(1) *ϑ preserves color and orientation.*

(2) *If an angle $(e_1, e_2)_A$ is positive, then the angle $(\vartheta(e_1), \vartheta(e_2))_{\vartheta(A)}$ is positive too.*

(3) *ϑ is a linear mapping on the cells of C_1^*.*

Lemma 6.1. *Let C^* be a simply connected marked complex, C_1^* an arbitrary marked complex, $\alpha \in C^*, \alpha_1 \in C_1^*$ their 2-cells. Then there exists a unique right mapping $\kappa : C^* \to C_1^*$ such that $\kappa(\alpha) = \alpha_1$.*

Proof. Let α' be an arbitrary 2-cell of C. Choose a path γ on C^* such that

(1) γ goes from the center of α to the center of α'.

(2) γ transversally intersects 1-skeleton of C^* at the interior points of edges.

Let $e_1, e_2, .., e_n$ be the edges of C^*, sequentially intersected by γ. Let γ go sequentially through 2-cells $\beta_1 = \alpha, \beta_2, \beta_3, .., \beta_n = \alpha'$. We can associate the sequence of quadruples (t_i, β_i, s_i, n_i) with any path, where

t_i is a coordinate of the point of intersection of e_i and γ,

β_i=1 if e_i is a black edge, $\beta_i = -1$ if e_i is a white edge,

$s_i = 1$ if the edge e_i is the repellent edge with respect to $\beta_i, s_i = -1$ in another case,

$n_i = 0$ if e_{i-1} and e_i are both either repellent or attractive with respect to $\beta_i, n_i = -1$ if β_{i-1} and β_i form a positive angle and $n = -1$ in another case.

It is easy to see that there is a path γ_1 on C^* such that it begins in the center of α' and the sequence of quadruples of γ_1 is equal to the sequence of quadruples of γ.

Let the end of γ_1 belong to the cell α_1'. Construct the linear mapping $\kappa_{\alpha_1} : \alpha_1 \to \alpha_1'$ preserving color and orientation. The complex C^* is simply connected, so the definition of κ_{α_1} is correct. (It does not depend on the path.) It is evident that if two cells α_1 and α_2 have a common edge or vertex then $\kappa_{\alpha_1}|_{\overline{\alpha}_1 \cap \overline{\alpha}_2} = \kappa_{\alpha_2}|_{\overline{\alpha}_1 \cap \overline{\alpha}_2}$. The set of mappings $\{\kappa_\alpha\}$ yields the mapping κ we need.

Corollary. *Let C_1^*, C_2^* be simply connected. Then there exist a right homeomorphism between C_1^* and C_2^*.*

Consider the symplectic space $\mathbb{R}^4$ with coordinates p_1, q_1, p_2, q_2 and the form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Functions $\widetilde{H} = ((p_1-1)^2+q_1^2)((p_1+1)^2+q_1^2)$ and $\widetilde{f} = ((p_2-1)^2+q_2^2)((p_2+1)^2+q_2^2) + ((p_1-1)^2+q_1^2)((p_1+1)^2+q_1^2)$ commute, so $(\mathbb{R}^4, \widetilde{\omega}, \widetilde{H}, \widetilde{f})$ sets an integrable Hamiltonian system. Level curves of H are shown in Fig. 4. The set $\widetilde{S} = \{0 < \widetilde{H} < 2, 0 < \widetilde{f} - \widetilde{H} < 2\}$ is a saddle neighborhood. The marked complex $\widetilde{C}^*$ of this neighborhood is the direct product of white and black figures eight curves by assuming that the orientation of edges and marking positive angles (by arcs) on figures eight curves is shown in Fig. 5.

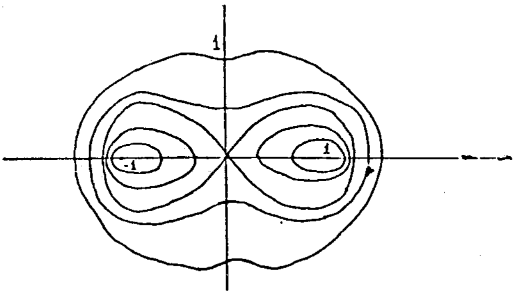

Fig. 4.

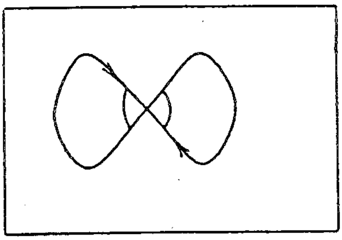

Fig. 5.

Let (S, p) be the universal covering of $\widetilde{S}$. Consider $H = p^*(\widetilde{H}), f = p^*(\widetilde{f}), \omega = p^*(\widetilde{\omega})$. (S, ω, H, f) sets an integrable Hamiltonian system (with noncompact fibers). Consider the marked complex of this system. It is the direct product of two (black and white) infinite degree 4 trees, for orientation and marking these trees see Fig. 6.

Let G be the group of right automorphisms of C^*. One can note that G is the direct product of groups of right automorphisms of the white and the black tree. Each of these two groups is generated by three elements $(a, b, s$, for example) with one relation: $s^2 = 1$. Actually, such marked and oriented tree is the universal covering of the figure eight curve from Fig. 6. So the group of automorphisms includes the fundamental group of figure eight curve (elements a and b) and a central symmetry (s). And it is clear that any automorphism can be obtained as a composition of some elements above.

Let $\widetilde{G}$ be the group of automorphisms of (S, ω, H, f). There exist natural homomorphism $\vartheta : G \to \widetilde{G}$. Actually, the subgroup of $\widetilde{G}$, generated

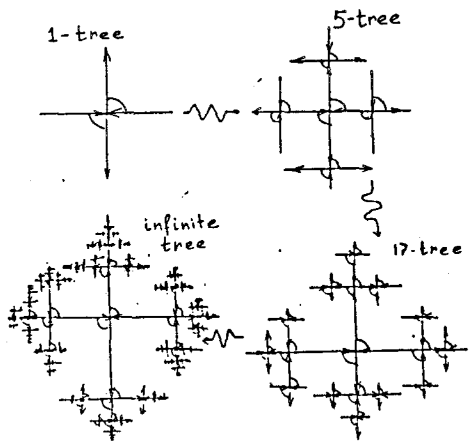

Fig. 6.

by the fundamental group of base and the symmetries relatively the planes (p_1, q_1) and (p_2, q_2) is exactly the group G.

The last step. Let C_1^* be an arbitrary marked complex. Consider its universal covering (C^*, p'). (It follows the above corollary that the simply connected covering space is S). Consider the subgroup $G_{C_1^*}$ such that $C^*/G_{C_1^*} = C_1^*$. Let $\vartheta(G_{C_1^*}) = \widetilde{G}_{C_1^*}$.

It is easy to see that $S/\widetilde{G}_{C_1^*}$ is a saddle neighborhood with the marked complex C_1^*, so $SG = \{\widetilde{G}_{C_\alpha^*}\}$ and S are the objects we need.

§7. How many

Saddle neighborhoods can be algorithmically listed. This has been done for saddle neighborhoods with 1, 2 and 3 singular points.

Theorem 7.1. *Let n be the number of the saddle-saddle points in a saddle neighborhood.*

If $n = 1$ then there exist exactly 4 saddle neighborhoods.

If $n = 2$ then there exist exactly 39 saddle neighborhoods.

If $n = 3$ then there exist exactly 256 saddle neighborhoods.

This theorem was proved by L. Lerman and Ya. Umanskii ($n = 1$), A. Bolsinov ($n = 2$) and N. Maximova ($n = 3$). The list of all possible admissible CL-types with one and two vertices is given in Table 1. In this table the positive rings on non-connected atoms are shaded, the horizontal sides of the squares are directed from left to right, the vertical ones are directed upward. The simplest atoms A, B, C, D have been described in [16] and we omit their description here. Some new atoms appearing within the framework of this theory are described in Table 2.

§8. Center-center and center-saddle cases

The aim of this section is to classify topologically integrable Hamiltonian systems in the saturated regular neighborhood of a singular fiber containing center-center or center-saddle points.

Theorem 8.1. [4,20] *Each center-center point has a saturated neighborhood such that it is topologically equivalent to* (U^4, ω, H, f), *where*

$U^4 = \{p_1^2 + q_1^2 < 1, p_2^2 + q_2^2 < 1\}$,

$H = p_1^2 + q_1^2$, $f = p_2^2 + q_2^2 + H$, $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$.

Definition 8.1. *A saturated open set $U^4 \subset M^4$ is called a center-saddle neighborhood if the following conditions hold.*

(1) *Exactly one fiber of U^4 includes zero rank points and those points have the center-saddle type (denote this fiber by L).*

(2) *U^4 is a regular neighborhood of L.*

(3) *Any curve on the bifurcation diagram of U^4 can be represented as a graph of some smooth function of H. The following lemma follows from the local structure in a small neighborhood of nondegenerate points.*

Lemma 8.1. *L is a connected graph whose vertices are all of degree 4.*

Consider the set of critical points K. Let us recall that Corollary 2.1 guarantees the existence of two critical two-dimensional manifolds P_1 and P_2 in some neighborhood of L. It is easy to see that one of them (for example, P_2) contains L.

Lemma 8.2. *(P_2, L) is a connected atom without stars.*

Since the vector field sgrad H is not equal to 0 on the edges of L then we can naturally orient the edges of L.

Definition 8.2. *The atom (P^2, L) (together with the natural orientation on its edges) is called the L-type of the center-saddle neighborhood.*

Theorem 8.2. *L-type is a complete topological invariant of center-saddle neighborhoods (i. e., two center-saddle neighborhoods are topologically equivalent if and only if their L-types are identical).*

Definition 8.3. *Let (P^2, L) be an atom. An orientation of a 4-degree graph L is called admissible if it is induced by some orientation of boundaries of rings of decomposition $P^2 \backslash L$.*

Theorem 8.3. *Let (P^2, L) be a connected atom without stars and the orientation of L be admissible. Then there exists a center-saddle neighborhood such that its L-type is equal to (P^2, L) with the orientation.*

Proof. It is clear that there exists a function $g : P^2 \to \mathbb{R}$ such that its single critical level coincides L. Consider an area form $\tilde{\omega}$ on P^2. It is evident that either sgrad(g) or sgrad$(-g)$ orients the graph L as it is necessary. Consider the disc

$$D^2 = \{p_1 + q_1 < 1\},$$

and put $V^4 = D^2 \times P^2$, $f = p_1^2 + q_1^2$, $H = \pm g + f$, $\omega = dp_1 \wedge dq_1 + \tilde{\omega}$. It is evident that $\{V^4, \omega, H, f\}$ is an integrable Hamiltonian system with the desired L-type.

§9. Circle molecules

Theorem 9.1. *The circle molecule of a center-center point is*

$$A \xrightarrow{0} A$$

Theorem 9.2. *The circle molecule of a center-saddle neighborhood with L-type $V = (P^2, G)$ is shown in Fig. 7.*

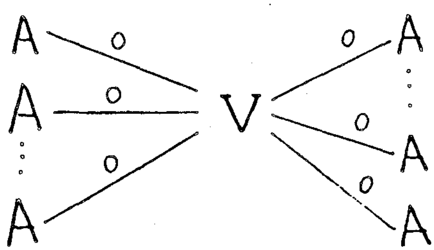

Fig. 7.

These theorems are evident.

Theorem 9.3. *There exists an algorithm of construction of circle molecule by the CL-type.*

This algorithm has been proposed by A. Bolsinov.

The idea of the algorithm. Consider the saddle neighborhood $\widetilde{S}$ from the proof of the Theorem 2. Cut $\widetilde{S}$ along the sets $\{| p_1 |< 1, q_1 = 0\}, \{p_1 = 0\}$, $\{| p_2 |< 1, q_2 = 0\}, \{p_2 = 0\}$. Each level becomes decomposed into the collection of squares. The structure of the squares is induced by the decomposition of the saddle complex. Since the subgroup $\vartheta(G)$ preserves such squares, we see that the fibers on each saddle neighborhood are decomposed in squares, and regluing of squares comply with some rules of reglueing of squares of $\widetilde{S}$ case. Bolsinov's algorithm is exactly the formalization of such rules.

The list of circle molecules of saddle neighborhoods with 1 and 2 vertices is given in Table 1.

Consider finally the focus-focus case. Let U be a focus neighborhood. The circle molecule of U is not a molecule in the usual sense because it has no atoms. Indeed, the preimage of a small circle on the bifurcation diagram is a fiber bundle over the circle whose fibers are 2-dimensional tori. Every such bundle can be given by a conjugacy class of 2×2 matrices.

Theorem 9.4. *For a focus neighborhood with the weight n this fiber bundle is determined by the conjugacy class of the matrix* $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

This theorem can be proved by standard methods of covering mapping theory.

Table 1 (page 1)

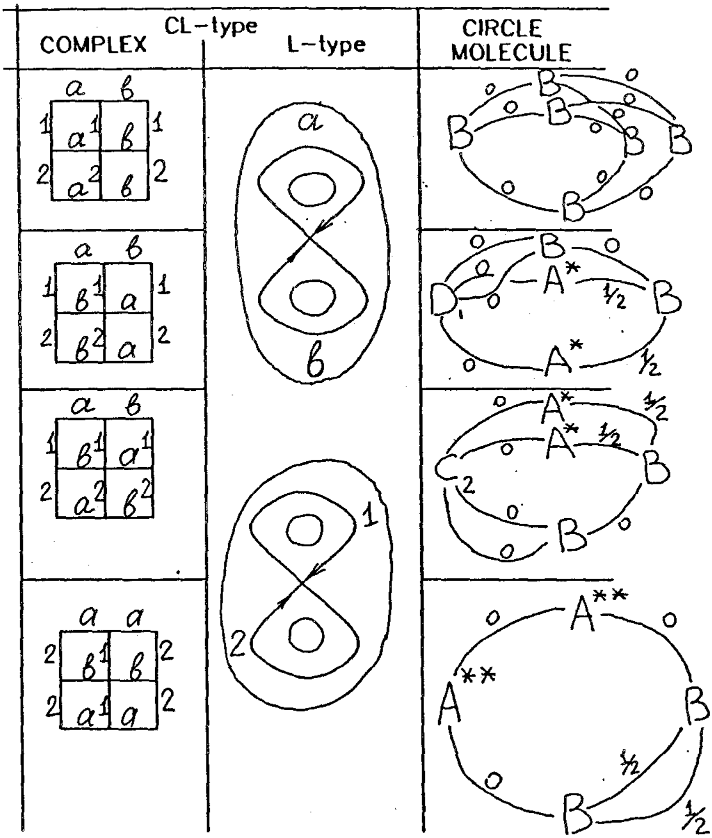

Table 1 (page 2)

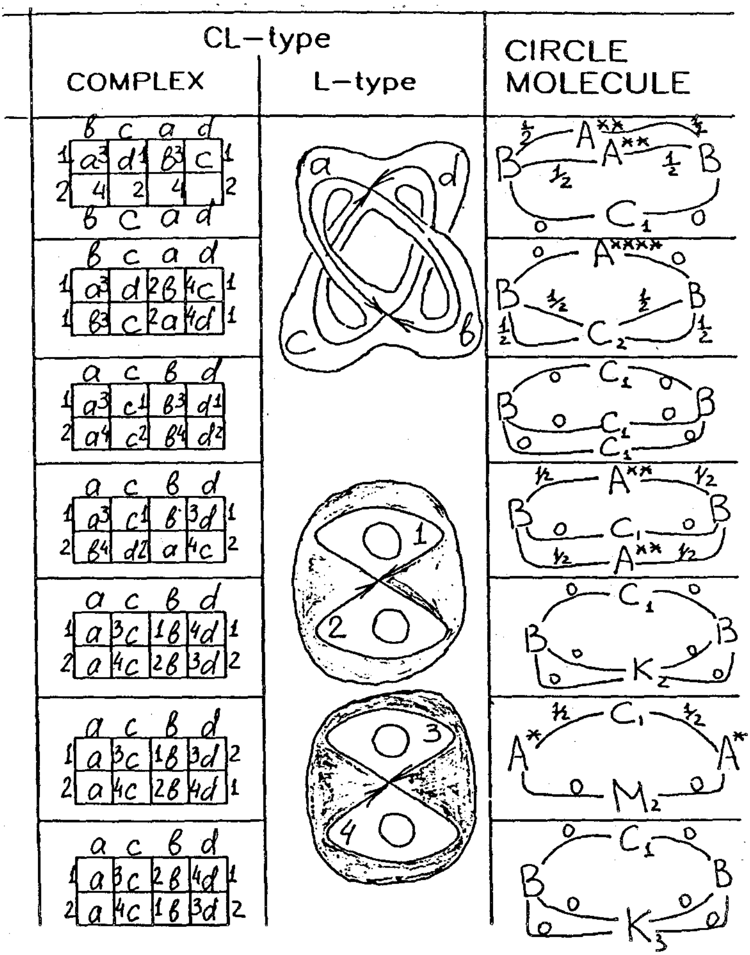

Table 1 (page 3)

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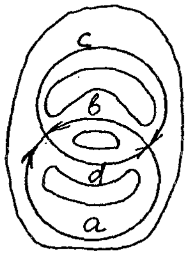
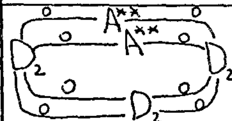
2 b 4d 2a 4c 2</td><td rowspan="3">
</td><td></td></tr>
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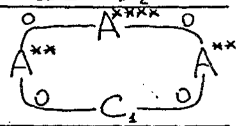
1 b 3c 2b 4c 1</td><td></td></tr>
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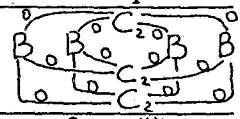
2 a 4b 2d 4c 2</td><td></td></tr>
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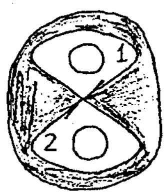
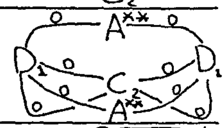
2 b 4d 2a 4c 2</td><td rowspan="2">
</td><td></td></tr>
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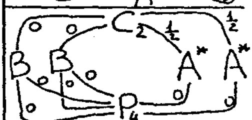
2 a 4b 1d 4c 2</td><td></td></tr>
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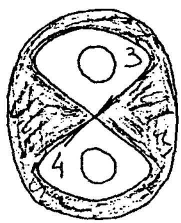
2 a 4b 1d 3c 1</td><td rowspan="2">
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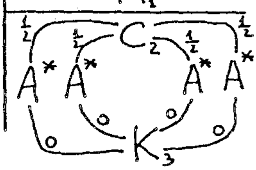
2 a 4b 1d 3c 2</td><td></td></tr>
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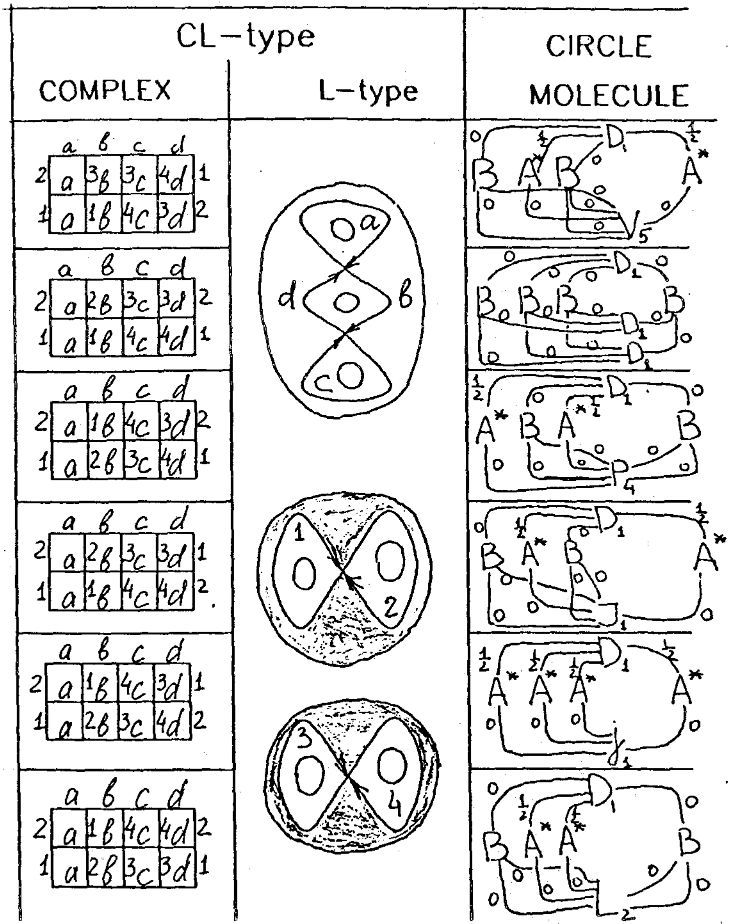

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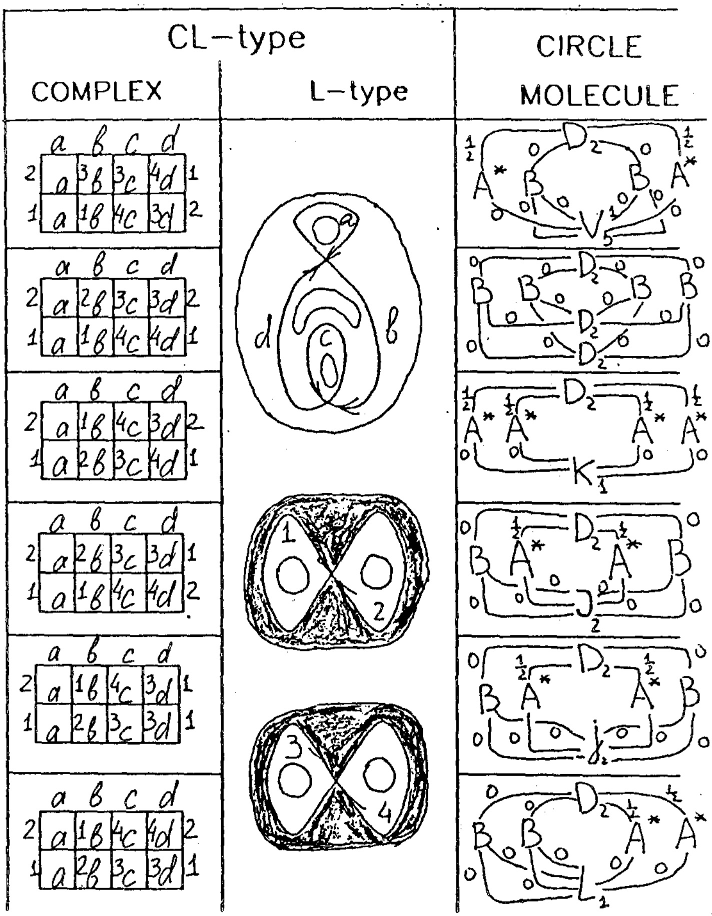

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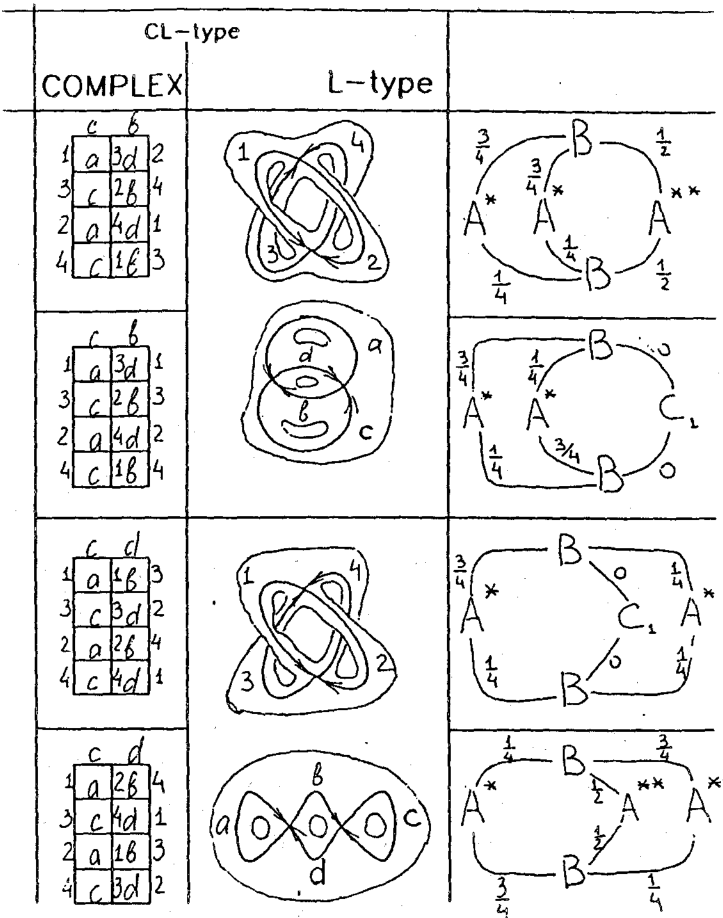

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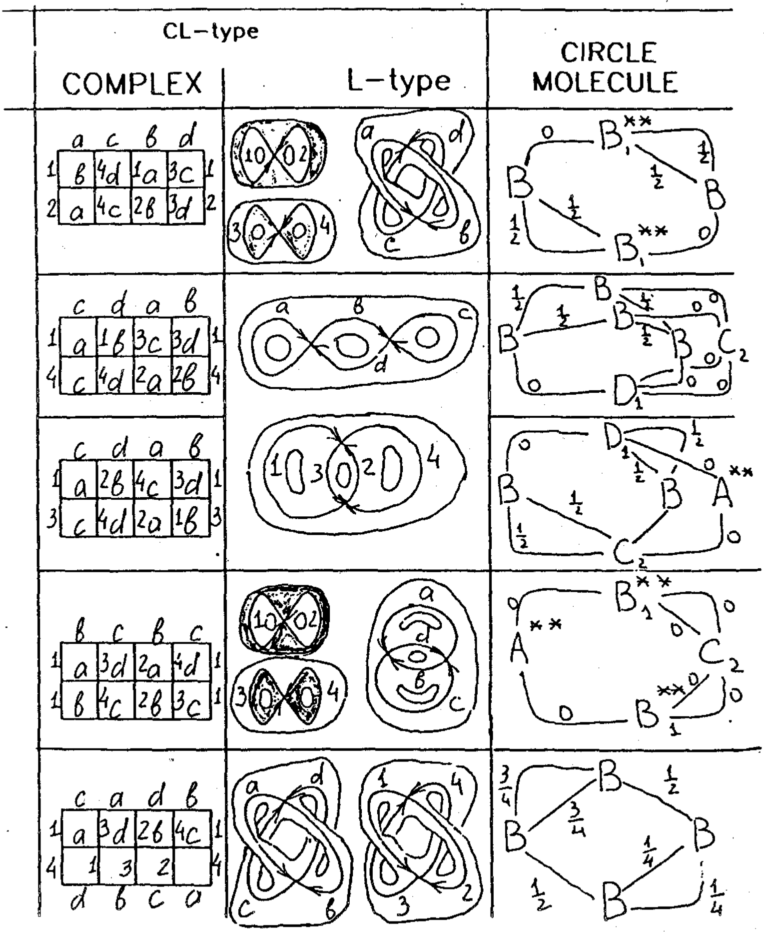

Table 1 (page 8)

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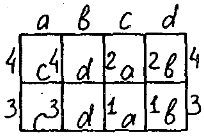
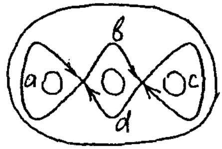
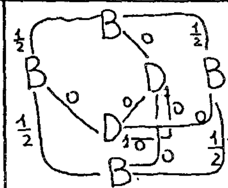
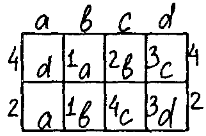
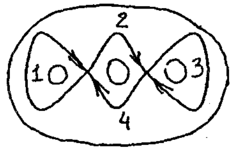
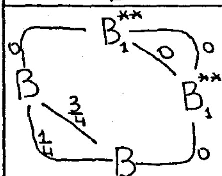
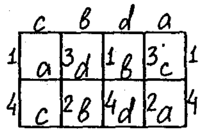
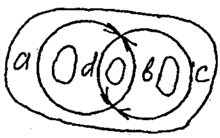
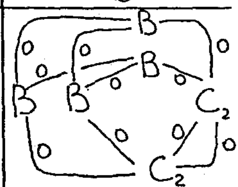
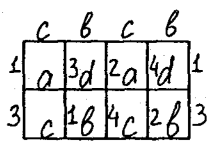
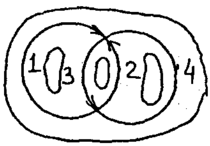
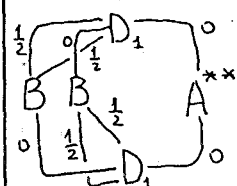
COMPLEX	L–type	
		
		
		
		

Table 2 (page 1)

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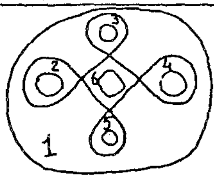
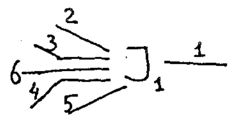
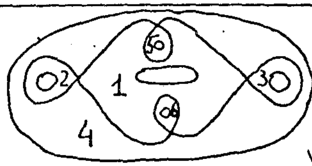
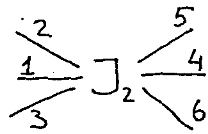
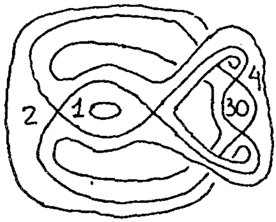
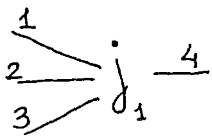
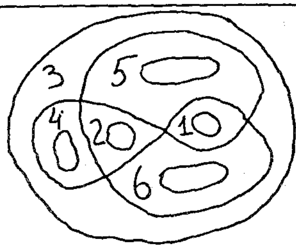
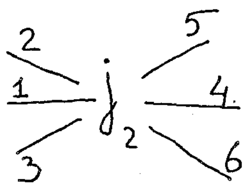
J_1		
J_2		
j_1		
j_2		

Table 2 (page 2)

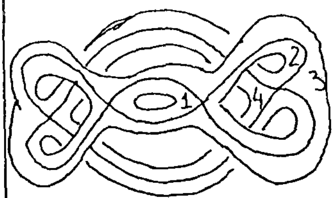
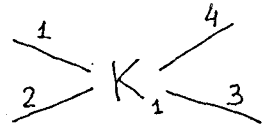
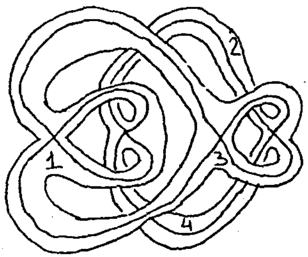
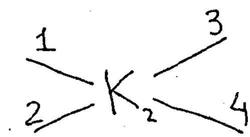
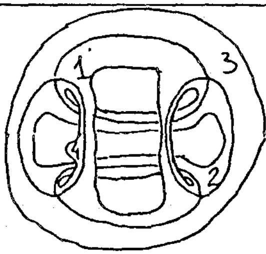
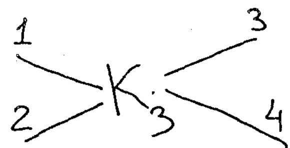
lame	B_2, G	Correspondence of picture and rings.
K_1		
K_2		
K_3		

Table 2 (page 3)

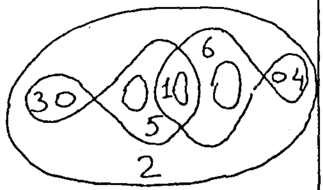
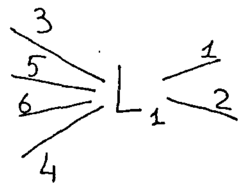
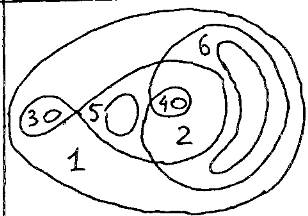
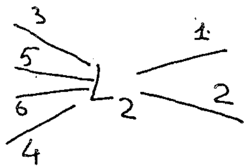
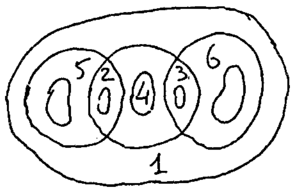
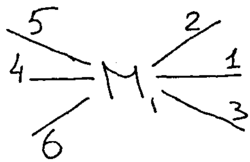
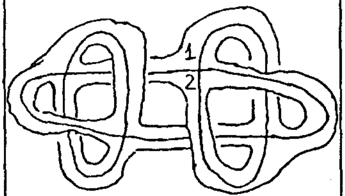
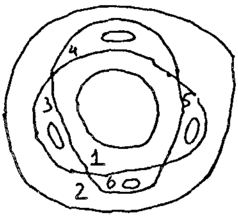
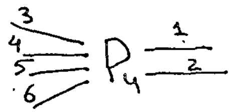
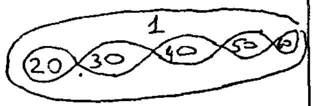
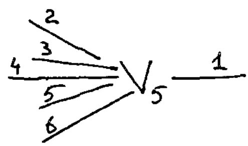
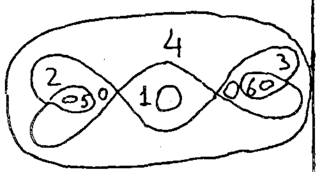
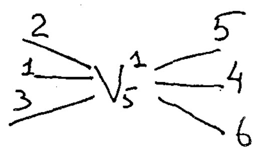
me	B^2, G	Correspondence of picture and rings
L_1		
L_2		
M_1		

Table 2 (page 4)

Name	B^2, G	Correspondence of picture and rings
M_2		
P_4		
V_5		
V_5^1		

REFERENCES

1. V. I. Arnold, *Mathematical methods in classical mechanics.* Springer-Verlag, 1978.
2. S. Smale, *Topology and mechanics.* — Invent. Math. **10** no. 6 (1970), 305-331; *II.* — ibid. **11** no. 1, 45-64.
3. L. M. Lerman, Ya. L. Umanskii, *Structure of the Poisson action of* $\mathbb{R}^2$ *on a four-dimensional symplectic manifold.* — Selecta Mathematica Sovietica **6** no. 4 (1987), 365-396; *II.* — ibid. **7** no. 1 (1988), 39-48.
4. L. M. Lerman, Ya. L. Umanskii, *The classification of four-dimensional Hamiltonian systems and Poisson actions of* $\mathbb{R}^2$ *in extended neighborhoods of simple singular points.* — Matem. Sbornik **184** no. 4 (1993), 103-138. (Russian)
5. M. P. Kharlamov, *Topological analysis of integrable problems in dynamics of rigid body.* Leningrad, Leningrad Univ., 1988. (Russian)
6. T. I. Pogosyan, *Critical integral surfaces in Clebsch problem.* — Mekhanika Tverdogo Tela no. 16 (1984), 19-24. (Russian)
7. O. I. Bogoyavlenskii, G. F. Ivakh, *Topological analysis of integrable cases of V.A.Steklov.* — Uspekhi Mat. Nauk **40** no. 4 (1985), 145-146. (Russian)
8. A. A. Oshemkov, *Fomenko invariants for the main integrable cases of the rigid body motion equations.* Topological classification of integrable Hamiltonian systems Advances in Soviet Math, Vol. 6 (A.T.Fomenko, eds.), Providence, 1991, pp. 67–145.
9. J. M. Arms, J. E. Marsden, V. Moncrief, *Symmetry and Bifurcations of Momentum Mappings.* — Commun. Math. Phys. **78** (1981), 455-478.
10. L. Bos, M. J. Gotay, *Singular angular Momentum Mappings.* — J. Diff. Geom. **24** (1986), 181–203.
11. H. Ito, *Action-angle coordinates at singularities of analytic integrable systems.* Preprint.
12. A. T. Fomenko, *The topology of surfaces of constant energy in integrable Hamiltonian systems and obstructions to integrability.* — Math. USSR Izvestiya **29** no. 3 (1987), 629-658.
13. A. T. Fomenko, *Integrability and Non-integrability in Geometry and Mechanics.* Kluwer Academic Publishers, 1988.
14. A. T. Fomenko, *Qualitative geometrical theory of integrable systems. Classification of isoenergy surfaces and bifurcation of Liouville tori at the critical energy values.* — Lect. Notes in Math. **1334** (1988), 221–245.
15. A. T. Fomenko, H. Zieschang, *Topological classification of integrable Hamiltonian systems.* Preprint IHES/M/88/62/, Institute des Hautes Etudes scientifiques, Bures-sur-Ivette.
16. A. V. Bolsinov, S. V. Matveev, A. T. Fomenko, *Topological classification of integrable Hamiltonian systems of two degrees of freedom. A list of systems of low complexity.* — Uspekhi Mat. Nauk **45** no. 2 (1990), 49–77. (Russian)
17. A. T. Fomenko, S. V. Matveev, *A Morse-type theory for integrable Hamiltonian systems with tame integrals.* — Math. Notes **43** no. 5-6 (1988), 382–386.
18. J. Williamson, *On the algebraic problem concerning the normal forms of linear dynamical systems.* — Amer. J. Math. **58** no. 1 (1936), 141–163.
19. A. T. Fomenko, H. Zieschang, *Topological invariant and criterion of equivalence of integrable Hamiltonian systems with two degrees of freedom.* — Izvestija AN

SSSR **54** no. 3 (1990), 546-575. (Russian)

20. A. V. Bolsinov, *Methods of calculation of the Fomenko-Zieschang invariant.* Topological Classification of Integrable Systems Advances in Soviet Mathematics, Vol. 6 (A.T.Fomenko, eds.), 1991, pp. 147–183.
21. V. S. Matveev, *Integrable Hamiltonian systems with two degrees of freedom. The topological structure of saturated neighborhoods of points of saddle-saddle and focus-focus type.* in preparation.

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