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KERNEL DETERMINATION PROBLEM FOR ONE PARABOLIC EQUATION WITH MEMORY

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Abstract: This paper studies the inverse problem of determining a multidimensional kernel function of an integral term which depends on the time variable t and $(n - 1)$ -dimensional space variable $x' = (x_1, \dots, x_{n-1})$ in the n -dimensional diffusion equation with a time-variable coefficient at the Laplacian of a direct problem solution. Given a known kernel function, a Cauchy problem is investigated as a direct problem. The integral term in the equation has convolution form: the kernel function is multiplied by a solution of the direct problem's elliptic operator. As an overdetermination condition, the result of the direct question on the hyperplane $x_n = 0$ is used. An inverse question is replaced by an auxiliary one, which is more suitable for further investigation. After that, the last problem is reduced to an equivalent system of Volterra-type integral equations of the second order with respect to unknown functions. Applying the fixed point theorem to this system in Hölder spaces, we prove the main result of the paper, which is a local existence and uniqueness theorem.

Keywords: Inverse problem, Resolvent, Integral equation, Fixed point theorem, Existence, Uniqueness.

1. Introduction

The constitutive relations for a linear nonhomogeneous heat propagation and diffusion processes in a medium with memory contain a time- and space-dependent kernel in an integral term of time variable convolution type [11, 14–16, 19]. Often, in practical applications, these kernels are unknown functions, and it is required to determine them. Memory function determination problems in heat equations have been the object of study since the end of the last century. The nonlinear inverse source and linear inverse coefficient problems with different types of over-determination conditions can be mostly found in the literature (see, for example, [1–3, 8, 10, 12, 13, 17, 20, 21] and the references therein). The authors of these researches argued solutions by the special solvability and stability estimates as well as the numerical outlook for solving this type of problems.

Among works devoted to finding the kernel depending on one time variable (one-dimensional inverse problem), we note [4, 14, 16, 19]. Multidimensional inverse problems, when a kernel, in addition to the time variable, also depends on all or a part of spatial variables, are few studied. In this direction, we observe [4, 5, 7, 9, 16]. In [7], the problem of determining a kernel depending on a time variable t and an $(n - 1)$ -dimensional spatial variable $x' = (x_1, \dots, x_{n-1})$ was investigated. The principal part of the integrodifferential equation in [7] is an n -dimensional heat conduction operator and the integral part has a form of time-convolution with respect to unknown functions:

the solutions of direct and inverse problems. However, in applications, the study of kernel determination problems is of great interest when the kernel in a convolution type integral is multiplied by an elliptic operator of a solution to the direct problem (see [12]). The present paper considers this kind of parabolic integrodifferential equations, for which the inverse problem will be studied.

Consider the problem of determining functions $u(x, t)$ and $k(x', t)$, $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$, $t > 0$, from the equations

$$u_t = a(t)\Delta u - \int_0^t k(x', t - \tau)a(\tau)\Delta u(x, \tau)d\tau, \quad (x, t) \in \mathbb{R}_T^n, \quad (1.1)$$

$$u(x, 0) = \varphi(x), \quad x \in \mathbb{R}^n, \quad (1.2)$$

$$u(x', 0, t) = f(x', t), \quad (x', t) \in \mathbb{R}_T^{n-1}, \quad f(x', 0) = \varphi(x', 0), \quad (1.3)$$

where Δ is the Laplace operator with respect to spatial variables $x = (x_1, \dots, x_n)$,

$$\mathbb{R}_T^n = \{(x, t) | x = (x', x_n) \in \mathbb{R}^n, 0 < t < T\}$$

is a strip of thickness T , $T > 0$ is an arbitrary fixed number, $a(t) \in C^2[0, T]$, $0 < a_0 \leq a(t) \leq a_1 < \infty$, and a_0 and a_1 are given numbers.

Our investigations were devoted to the results of [4, 5, 7, 9] under the condition of the integrodifferential heat equation of parabolic type with a variable coefficient and a particular convolution integral.

In this paper, we use the Hölder space H^α with exponent α , where α is a positive integer, for functions depending only on spatial variables. We also use the space $H^{\alpha, \alpha/2}$ with exponents α and $\alpha/2$ for functions depending on both time and spatial variables.

Throughout this paper, we require that

$$\begin{aligned} \varphi(x) \in H^{l+8}(\mathbb{R}^n), \quad \varphi(x) \geq \varphi_0 = \text{const} > 0, \quad f(x', t) \in H^{l+6, (l+6)/2}(\bar{\mathbb{R}}_T^{n-1}), \\ \bar{\mathbb{R}}_T^{n-1} = \{(x', t) | x' \in \mathbb{R}^{n-1}, 0 \leq t \leq T\}. \end{aligned}$$

The spaces $H^l(Q)$ and $H^{l, l/2}(Q_T)$ and their norms are defined in [6, p. 16–27]. In what follows, we denote by $|\cdot|_T^{l, l/2}$ the norm of functions in the space $H^{l, l/2}(Q_T)$ (in the particular cases $Q_T = \mathbb{R}_T^n$ or $Q_T = \mathbb{R}_T^{n-1}$) depending on time and spatial variables and by $|\cdot|^l$ the norms of functions depending only on spatial variables (for $Q = \mathbb{R}^n$ or $Q = \mathbb{R}^{n-1}$).

The paper is organized as follows. In Section 2, we reduce the inverse problem (1.1)–(1.3) to an auxiliary problem with the additional unknown k outside the integral. In Section 3, using the Poisson formula, we reduce the auxiliary problem to an equivalent system of integral equations with respect to unknown functions. In Section 4, we study the inverse problem as the problem of determining functions $k(t)$ from problem (1.1)–(1.3) using the contraction mapping principle.

2. Preliminaries. Auxiliary problem

Lemma 1. *Let $\{k(t), r(t)\} \in C[0, T]$, and let $k(t)$ and $r(t)$ satisfy the integral equation*

$$r(t) = k(t) + \int_0^t k(t - \tau)r(\tau)d\tau, \quad t \in [0, T].$$

Then a solution of the integral equation

$$\varphi(t) = \int_0^t k(t - \tau)\varphi(\tau)d\tau + f(t), \quad f(t) \in C[0, T],$$

is defined by the formula

$$\varphi(t) = \int_0^t r(t-\tau)f(\tau)d\tau + f(t).$$

P r o o f. We can prove this assertion using a resolvent kernel method for linearly integral equations (see, for example [6]).

Let $u(x, t)$ be the classical solution to the Cauchy problem (1.1)–(1.2). We solve equation (1.1) with respect to $a(t)\Delta u$ and obtain

$$a(t)\Delta u = \int_0^t k(x', t-\tau)a(\tau)\Delta u(x, \tau)d\tau + u_t. \quad (2.1)$$

Then, applying Lemma 1 to (2.1), we obtain for every fixed $x \in \mathbb{R}^n$

$$u_t - a(t)\Delta u = - \int_0^t r(x', t-\tau)u_\tau(x, \tau)d\tau. \quad (2.2)$$

The function $r(x', t)$ in (2.2) is related to $k(x', t)$ as follows:

$$r(x', t) = k(x', t) + \int_0^t k(x', t-\tau)r(x', \tau)d\tau, \quad (x, t) \in \mathbb{R}_T^n. \quad (2.3)$$

We study the question of finding functions $u(x, t)$ and $r(x', t)$ that satisfy equations (2.2), (1.2), and (1.3). To solve this problem, we first will find $k(x', t)$ from (2.3).

Consider a new function $\vartheta^{(1)}(x, t) = u_{x_n x_n}(x, t)$. Differentiating equations (2.2) and (1.2) twice with respect to x_n , we obtain the following relation for $\vartheta^{(1)}(x, t)$:

$$\vartheta_t^{(1)} - a(t)\Delta\vartheta^{(1)} = - \int_0^t r(x', t-\tau)\vartheta_\tau^{(1)}(x, \tau)d\tau, \quad (2.4)$$

$$\vartheta^{(1)}(x, 0) = \varphi_{x_n x_n}(x). \quad (2.5)$$

We obtain an overdetermination condition as follows. Introduce the term $a(t)u_{x_n x_n}$ into the expression $a(t)\Delta u$ of (2.2) and set $x_n = 0$. Then, taking into account that $a(t)u_{x_n x_n} = a(t)\vartheta^{(1)}$ and using (1.2), we get

$$\vartheta^{(1)}(x', 0, t) = \frac{1}{a(t)}f_t(x', t) - \sum_{i=1}^{n-1} f_{x_i x_i}(x', t) + \frac{1}{a(t)} \int_0^t r(x', t-\tau)f_\tau(x', \tau)d\tau. \quad (2.6)$$

For the continuity of the function $\vartheta^{(1)}(x, t)$ for $x_n = t = 0$, $x \in \mathbb{R}^{n-1}$, we require the following matching condition:

$$\varphi_{x_n x_n}(x', 0) = \frac{1}{a(0)}f_t(x', 0) - \sum_{i=1}^{n-1} f_{x_i x_i}(x', 0). \quad (2.7)$$

We understand the values of the functions $a(t)$ and $f(x', t)$ and of their derivatives at $t = 0$ as the limit as $t \rightarrow +0$.

Consider another transformation of the question. Let $\vartheta^{(2)}(x, t)$ be the derivative of $\vartheta^{(1)}(x, t)$ with respect to t , i.e., let $\vartheta^{(2)}(x, t) := \vartheta_t^{(1)}(x, t)$, and let $h(x', t) := r_t(x', t)$. From (2.4)–(2.6), we get

$$\vartheta_t^{(2)} - a(t)\Delta\vartheta^{(2)} = a'(t)\Delta\vartheta^{(1)} - r(x', 0)\vartheta^{(2)} - \int_0^t h(x', t - \tau)\vartheta^{(2)}(x, \tau)d\tau, \quad (2.8)$$

$$\vartheta^{(2)}(x, 0) = a(0)\Delta\varphi_{x_n x_n}(x), \quad (2.9)$$

$$\begin{aligned} \vartheta^{(2)}(x', 0, t) &= \frac{a'(t)}{a^2(t)}f_t(x', t) + \frac{1}{a(t)}f_{tt}(x', t) - \sum_{i=1}^{n-1}f_{tx_i x_i}(x', t) \\ &- \frac{a'(t)}{a^2(t)}\int_0^t r(x', t - \tau)f_\tau(x', \tau)d\tau + \frac{1}{a(t)}\int_0^t h(x', \tau)f_\tau(x', t - \tau)d\tau + \frac{1}{a(t)}r(x', 0)f_t(x', t). \end{aligned} \quad (2.10)$$

Here, we are obtained the initial condition (2.8) using (2.4) by setting $t = 0$ and (2.5). The unknown function $r(x', 0)$ is a term of equations (2.8) and (2.10). One can define this function as follows. Similarly to obtaining equality (2.7), we need the continuity of the function $\vartheta^{(2)}(x, t)$ for $x_n = t = 0$, $x \in \mathbb{R}^{n-1}$. Then, (2.9) and (2.10) give some equation, solving which with respect to $r(x', 0)$ leads to

$$r(x', 0) = \frac{1}{f_t(x', 0)} \left[a^2(0)\Delta\varphi_{x_n x_n}(x', 0) - \frac{a'(0)}{a(0)}f_t(x', 0) - f_{tt}(x', 0) + a(0)\sum_{i=1}^{n-1}f_{tx_i x_i}(x', 0) \right]. \quad (2.11)$$

In the following calculations, we assume that $r(x', 0)$ is known.

Let $\vartheta(x, t) := \vartheta_t^{(2)}(x, t)$. Then, we obtain the main problem of determining $\vartheta(x, t)$ and $h(x', t)$ satisfying the equations

$$\begin{aligned} \vartheta_t - a(t)\Delta\vartheta &= 2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)} \\ -r(x', 0)\vartheta - h(x', t)a(0)\Delta\varphi_{x_n x_n}(x) &- \int_0^t h(x', \tau)\vartheta(x, t - \tau)d\tau, \end{aligned} \quad (2.12)$$

$$\vartheta(x, 0) = \Psi(x), \quad (2.13)$$

$$\begin{aligned} \vartheta(x', 0, t) &= F(x', t) + \left(2\frac{a'(t)^2}{a^3(t)} - \frac{a''(t)}{a^2(t)} \right) \int_0^t r(x', t - \tau)f_\tau(x', \tau)d\tau \\ &- 2\frac{a'(t)}{a^2(t)}\int_0^t h(x', \tau)f_\tau(x', t - \tau)d\tau - \frac{1}{a(t)}\int_0^t h(x', \tau)f_{tt}(x', t - \tau)d\tau + \frac{1}{a(t)}h(x', t)f_t(x', 0), \end{aligned} \quad (2.14)$$

where

$$\Psi(x) = a^2(0)\Delta^2\varphi_{x_n x_n}(x) + a'(0)\Delta\varphi_{x_n x_n}(x) - r(x', 0)a(0)\Delta\varphi_{x_n x_n}(x),$$

and therefore we get

$$\begin{aligned} F(x', t) &= \left(\frac{a''(t)}{a^2(t)} - \frac{a'(t)^2}{a^3(t)} \right) f_t(x', t) + \frac{1}{a(t)}f_{ttt}(x', t) - \sum_{i=1}^{n-1}f_{ttx_i x_i}(x', t) \\ &- 2\frac{a'(t)}{a^2(t)}r(x', 0)f_t(x', t) + \frac{1}{a(t)}r(x', 0)f_{tt}(x', t). \end{aligned}$$

Equation (2.12) contains $2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)}$ on the right-hand side. Taking into consideration $\vartheta_t^{(1)} = \vartheta^{(2)}$ and using (2.4), we replace it by $\vartheta^{(2)}$:

$$a''(t)\Delta\vartheta^{(1)} = \frac{a''(t)}{a(t)}\vartheta^{(2)} + \frac{a''(t)}{a(t)} \int_0^t r(x', t - \tau)\vartheta^{(2)}(x, \tau)d\tau. \quad (2.15)$$

Similarly, from (2.4) and (2.8), we obtain

$$\begin{aligned} 2a'(t)\Delta\vartheta^{(2)} = 2(\ln a(t))' \left[\vartheta - (\ln a(t))' \left(\vartheta^{(2)} + \int_0^t r(x', t - \tau)\vartheta^{(2)}(x, \tau)d\tau \right) \right. \\ \left. - r(x', 0)\vartheta^{(2)} - \int_0^t h(x', t - \tau)\vartheta^{(2)}(x, \tau)d\tau \right]. \end{aligned} \quad (2.16)$$

Further, we will deduce that the relation $2a'(t)\Delta\vartheta^{(2)} + a''(t)\Delta\vartheta^{(1)}$ in equation (2.12) is eliminated with the help of (2.15) and (2.16).

In case (2.7) and (2.11), it does not bring difficulties following out the inverse changes to derive the equations (1.1)–(1.3) from (2.8), (2.9), and (2.12)–(2.14) [7]. So, the inverse problem (1.1)–(1.3) is similar to problem (2.8), (2.9), and (2.12)–(2.14) of determining the functions $\vartheta^{(2)}(x, t)$, $\vartheta(x, t)$, $h(x', t)$, and $r(x', t)$. \square

3. Reduction of the auxiliary problem

The following statement is the main result of this section.

Lemma 2. *The auxiliary problems (2.8)–(2.9), (2.12)–(2.13), and the equality $h(x', t) := r_t(x', t)$, are equivalent to the problem of finding the functions $\vartheta^{(2)}(x, t)$, $\vartheta(x, t)$, $h(x', t)$, and $r(x', t)$ from the following system of integral equations:*

$$\begin{aligned} \vartheta^{(2)}(x, t) = \int_{\mathbb{R}^n} a(0)\Delta\varphi_{\xi_n\xi_n}(\xi)G(x - \xi, \theta(t))d\xi + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \\ \times \int_{\mathbb{R}^n} \left[(\ln a(\theta^{-1}(\tau)))' \left(\vartheta^{(2)}(\xi, \theta^{-1}(\tau)) + \int_0^{\theta^{-1}(\tau)} r(\xi', \theta^{-1}(\tau) - \alpha)\vartheta^{(2)}(\xi, \alpha)d\alpha \right) \right. \\ \left. - r(\xi', 0)\vartheta^{(2)}(\xi, \theta^{-1}(\tau)) - \int_0^{\theta^{-1}(\tau)} h(\xi', \theta^{-1}(\tau) - \alpha)\vartheta^{(2)}(\xi, \alpha)d\alpha \right] G(x - \xi, \theta(t) - \tau)d\xi, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \vartheta(x, t) = \int_{\mathbb{R}^n} \Psi(\xi)G(x - \xi, \theta(t))d\xi + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \left[\left(\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} \right) \right. \\ \left. - 2((\ln a(\theta^{-1}(\tau)))')^2 \right) \vartheta^{(2)}(\xi, \theta^{-1}(\tau)) + \left(2(\ln a(\theta^{-1}(\tau)))' - r(\xi', 0) \right) \vartheta(\xi, \theta^{-1}(\tau)) \end{aligned}$$

$$\begin{aligned}
& - \int_0^{\theta^{-1}(\tau)} h(\xi', \alpha) \vartheta(\xi, \theta^{-1}(\tau) - \alpha) d\alpha + \left(\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} + 2(\ln a(\theta^{-1}(\tau)))' \right) \\
& - 2((\ln a(\theta^{-1}(\tau)))')^2) \int_0^{\theta^{-1}(\tau)} r(\xi', \theta^{-1}(\tau) - \alpha) \vartheta^{(2)}(\xi, \alpha) d\alpha + 2(\ln a(\theta^{-1}(\tau)))' \quad (3.2) \\
& \times \int_0^{\theta^{-1}(\tau)} h(\xi', \theta^{-1}(\tau) - \alpha) \vartheta^{(2)}(\xi, \alpha) d\alpha - h(\xi' \theta^{-1}(\tau)) a(0) \Delta \varphi_{\xi_n \xi_n}(\xi) \Big] G(x - \xi, \theta(t) - \tau) d\xi,
\end{aligned}$$

$$\begin{aligned}
h(x', t) &= \frac{a(t)}{f_t(x', 0)} \left[\int_{\mathbb{R}^n} \Psi(\xi) G(x' - \xi', \xi_n, \theta(t)) d\xi - F(x', t) \right] \\
& + \frac{a(t)}{f_t(x', 0)} \left[\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \left(\left[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} - 2((\ln a(\theta^{-1}(\tau)))')^2 \right] \vartheta^{(2)}(\xi, \theta^{-1}(\tau)) \right. \right. \\
& \left. \left. + \left[2(\ln a(\theta^{-1}(\tau)))' - r(\xi', 0) \right] \vartheta(\xi, \theta^{-1}(\tau)) - \int_0^{\theta^{-1}(\tau)} h(\xi', \alpha) \vartheta(\xi, \theta^{-1}(\tau) - \alpha) d\alpha \right. \right. \\
& \left. \left. + \left[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} + 2(\ln a(\theta^{-1}(\tau)))' - 2((\ln a(\theta^{-1}(\tau)))')^2 \right] \int_0^{\theta^{-1}(\tau)} r(\xi', \tau - \alpha) \vartheta^{(2)}(\xi, \alpha) d\alpha \right. \right. \quad (3.3) \\
& \left. \left. + 2(\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} h(\xi', \theta^{-1}(\tau) - \alpha) \vartheta^{(2)}(\xi, \alpha) d\alpha - h(\xi', \theta^{-1}(\tau)) a(0) \Delta \varphi_{\xi_n \xi_n}(\xi) \right) \right. \\
& \left. \times G(x' - \xi', \xi_n, \theta(t) - \tau) d\xi \right] - f_t(x', 0) (2((\ln(a(t)))')^2 - \frac{a''(t)}{a(t)}) \int_0^t r(x', t - \tau) f_\tau(x', \tau) d\tau + \\
& + 2f_t(x', 0) (\ln(a(t)))' \int_0^t h(x', \tau) f_\tau(x', t - \tau) d\tau + f_t(x', 0) \int_0^t h(x', \tau) f_{tt}(x', t - \tau) d\tau,
\end{aligned}$$

$$r(x', t) = r(x', 0) + \int_0^t h(x', \tau) d\tau. \quad (3.4)$$

P r o o f. To prove Lemma 2, we use the formula [3]

$$p(x, t) = \int_{\mathbb{R}^n} \varphi(\xi) G(x - \xi; \theta(t)) d\xi + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} F(\xi, \theta^{-1}(\tau)) G(x - \xi; \theta(t) - \tau) d\xi, \quad (3.5)$$

which provides a solution to the following Cauchy problem for the heat equation with a time-variable coefficient of thermal conductivity:

$$\begin{aligned}
p_t - a(t) \Delta p &= F(x, t), \quad x \in \mathbb{R}^n, \quad t > 0, \\
p(x, 0) &= \varphi(x), \quad x \in \mathbb{R}.
\end{aligned}$$

In (3.5),

$$\theta(t) = \int_0^t a(\tau) d\tau$$

and $\theta^{-1}(t)$ is the inverse function of $\theta(t)$;

$$G(x - \xi; \theta(t) - \tau) = \frac{1}{(2\sqrt{\pi(\theta(t) - \tau)})^n} e^{-|x-\xi|^2/4(\theta(t)-\tau)}$$

is the fundamental solution related to the operator of heat with the coefficient of thermal conductivity that depends on time:

$$\frac{\partial}{\partial t} - a(t)\Delta, \quad \xi = (\xi_1, \dots, \xi_n), \quad \xi' = (\xi_1, \dots, \xi_{n-1}), \quad d\xi = d\xi_1 \cdots d\xi_n, \quad |x|^2 = x_1^2 + \cdots + x_n^2.$$

Equations (3.1) and (3.2) follow from the Cauchy problems (2.8), (2.9) and (2.12), (2.13) with (3.5), independently. In (3.2), we set $x_n = 0$ and use another case of (2.14). After that, we get equation (3.3). Equality (3.4) is clear.

We add to the equations (3.1)–(3.4) the integral equation. It can be gained from relations (2.2) and (1.2). First, we use formula (3.5) after integrating by parts in the integral on the right-hand side of (2.2). In conclusion, we get the following equivalent integral equation for $u(x, t)$:

$$\begin{aligned} u(x, t) = & \int_{\mathbb{R}^n} \varphi(\xi) G(x - \xi; \theta(t)) d\xi + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \left[r(\xi', \theta^{-1}(\tau)) \varphi(\xi) \right. \\ & \left. - r(\xi', 0) u(\xi, \theta^{-1}(\tau)) - \int_0^{\theta^{-1}(\tau)} h(\xi', \theta^{-1}(\tau) - \alpha) u(\xi, \alpha) d\alpha \right] G(x - \xi; \theta(t) - \tau) d\xi. \end{aligned} \quad (3.6)$$

□

4. Existence and uniqueness

In this section, we show that a solution to the system of integral equations (3.1)–(3.4), (3.6) exists and is unique. To this end, we use the well-known Banach's principle [18, pp. 87–97]. Our goal is to set the integral equations like a system with a nonlinear operator for unknown functions $\vartheta^{(2)}(x, t)$, $\vartheta(x, t)$, $h(x', t)$, and $r(x', t)$, and show that an operator of this type is a contraction mapping operator. The uniqueness and existence then follow straight away.

Recall that F is a contraction mapping operator in a closed set Ω , which is a subset of a Banach space, if it satisfies the following two properties:

- (1) if $y \in \Omega$, then $Fy \in \Omega$ (i.e., F maps Ω into itself);
- (2) if $y, z \in \Omega$, then $\|Fy - Fz\| \leq \rho \|y - z\|$ with $\rho < 1$ (ρ is a constant independent of y and z).

Right now, we introduce the primary result of this research.

Theorem 1. *Suppose that all cases of Section 1 on regard to the drawn functions $a(t)$, $\varphi(x)$, and $f(x', t)$ and the matching cases (1.3) and (2.7) are fulfilled except $|f_t(x', 0)| > f_0 = \text{const} > 0$, f_0 is a fixed number. Then there is a sufficiently small number $T > 0$ such that the unique answer to the inverse question (1.1)–(1.3) exists in the class of functions $u(x, t) \in H^{l+2, (l+2)/2}(\bar{\mathbb{R}}_T^n)$ and $k(x', t) \in H^{l, l/2}(\bar{\mathbb{R}}_T^{n-1})$.*

P r o o f. The system of equations (3.1)–(3.4), (3.6) is a closed system of unknown functions $\vartheta^{(2)}(x, t)$, $\vartheta(x, t)$, $h(x', t)$, $r(x', t)$, and $u(x, t)$ in \mathbb{R}_T^n . It can be written as a nonlinear operator equation

$$\psi = A\psi; \quad (4.1)$$

here $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^* = (\vartheta^{(2)}(x, t), \vartheta(x, t), h(x', t), r(x', t), u(x, t))^*$, where $*$ is the transposition symbol. According to equations (3.1)–(3.4) and (3.6), the operator $A\psi = [(A\psi)_1, (A\psi)_2, (A\psi)_3, (A\psi)_4, (A\psi)_5]$ has the form

$$\begin{aligned} (A\psi)_1 = & \psi_{01}(x, t) + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \left[(\ln a(\theta^{-1}(\tau)))' (\psi_1(\xi, \theta^{-1}(\tau))) \right. \\ & + \int_0^{\theta^{-1}(\tau)} \psi_4(\xi', \theta^{-1}(\tau) - \alpha) \psi_1(\xi, \alpha) d\alpha \left. - r(\xi', 0) \psi_1(\xi, \alpha) \right. \\ & \left. - \int_0^{\theta^{-1}(\tau)} \psi_3(\xi', \theta^{-1}(\tau) - \alpha) \psi_1(\xi, \alpha) d\alpha \right] G(x - \xi, \theta(t) - \tau) d\xi, \end{aligned} \quad (4.2)$$

$$\begin{aligned} (A\psi)_2 = & \psi_{02}(x, t) + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \left(\left[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} - 2((\ln a(\theta^{-1}(\tau)))')^2 \right] \psi_1(\xi, \theta^{-1}(\tau)) \right. \\ & + \left[2(\ln a(\theta^{-1}(\tau)))' - r(\xi', 0) \right] \psi_2(\xi, \theta^{-1}(\tau)) - \int_0^{\theta^{-1}(\tau)} \psi_3(\xi', \alpha) \times \psi_2(\xi, \theta^{-1}(\tau) - \alpha) d\alpha \\ & + \left[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} + 2(\ln a(\theta^{-1}(\tau)))' - 2((\ln a(\theta^{-1}(\tau)))')^2 \right] \int_0^{\theta^{-1}(\tau)} \psi_4(\xi', \theta^{-1}(\tau) - \alpha) \psi_1(\xi, \alpha) d\alpha \\ & + 2(\ln a(\theta^{-1}(\tau)))' \int_0^{\theta^{-1}(\tau)} \psi_3(\xi', \theta^{-1}(\tau) - \alpha) \\ & \left. \times \psi_1(\xi, \alpha) d\alpha - \psi_3(\xi', \theta^{-1}(\tau)) a(0) \Delta \varphi_{\xi_n \xi_n}(\xi) \right) G(x - \xi, \theta(t) - \tau) d\xi, \end{aligned} \quad (4.3)$$

$$\begin{aligned} (A\psi)_3 = & \psi_{03}(x', t) + \frac{a(t)}{f_t(x', 0)} \left[\int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \left(\left[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} \right. \right. \right. \\ & \left. \left. - 2((\ln a(\theta^{-1}(\tau)))')^2 \right] \psi_1(\xi, \theta^{-1}(\tau)) + \left[2(\ln a(\theta^{-1}(\tau)))' - r(\xi', 0) \right] \psi_2(\xi, \theta^{-1}(\tau)) \right. \\ & \left. - \int_0^{\theta^{-1}(\tau)} \psi_3(\xi', \alpha) \psi_2(\xi, \theta^{-1}(\tau) - \alpha) d\alpha + \left[\frac{a''(\theta^{-1}(\tau))}{a(\theta^{-1}(\tau))} + 2(\ln a(\theta^{-1}(\tau)))' \right. \right. \\ & \left. \left. - 2((\ln a(\theta^{-1}(\tau)))')^2 \right] \int_0^{\theta^{-1}(\tau)} \psi_4(\xi', \theta^{-1}(\tau) - \alpha) \psi_1(\xi, \alpha) d\alpha + 2(\ln a(\theta^{-1}(\tau)))' \times \right. \end{aligned}$$

$$\begin{aligned}
& \times \int_0^{\theta^{-1}(\tau)} \psi_3(\xi', \theta^{-1}(\tau) - \alpha) \psi_1(\xi, \alpha) d\alpha - \psi_3(\xi', \theta^{-1}(\tau)) a(0) \Delta \varphi_{\xi_n \xi_n}(\xi) \\
& \times G(x' - \xi', \xi_n, \theta(t) - \tau) d\xi \Big] - f_t(x', 0) \left(2((\ln(a(t)))')^2 - \frac{a''(t)}{a(t)} \right) \int_0^t \psi_4(x', t - \tau) f_\tau(x', \tau) d\tau \\
& + 2f_t(x', 0) (\ln(a(t)))' \int_0^t \psi_3(x', \tau) f_\tau(x', t - \tau) d\tau + f_t(x', 0) \int_0^t \psi_3(x', \tau) f_{tt}(x', t - \tau) d\tau,
\end{aligned} \tag{4.4}$$

$$(A\psi)_4 = \psi_{04}(x', t) + \int_0^t \psi_3(x', \tau) d\tau, \tag{4.5}$$

$$\begin{aligned}
(A\psi)_5 &= \psi_{05}(x, t) + \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} \left[\psi_4(\xi', \theta^{-1}(\tau)) \varphi(\xi) \right. \\
& \left. - r(\xi', 0) \psi_5(\xi, \theta^{-1}(\tau)) - \int_0^{\theta^{-1}(\tau)} \psi_3(\xi', \theta^{-1}(\tau) - \alpha) \psi_5(\xi, \alpha) d\alpha \right] G(x - \xi; \theta(t) - \tau) d\xi.
\end{aligned} \tag{4.6}$$

In (4.2)–(4.6), we introduced the notation:

$$\begin{aligned}
\psi_{01}(x, t) &= \int_{\mathbb{R}^n} a(0) \Delta \varphi_{\xi_n \xi_n}(\xi) G(x - \xi, \theta(t)) d\xi, \\
\psi_{02}(x, t) &= \int_{\mathbb{R}^n} \Psi(\xi) G(x - \xi, \theta(t)) d\xi, \\
\psi_{03}(x', t) &= \frac{a(t)}{f_t(x', 0)} \left[\int_{\mathbb{R}^n} \Psi(\xi) G(x' - \xi', \xi_n, \theta(t)) d\xi - F(x', t) \right], \\
\psi_{04}(x', t) &= r(x', 0), \quad \psi_{05}(x, t) = \int_{\mathbb{R}^n} \varphi(\xi) G(x - \xi; \theta(t)) d\xi.
\end{aligned}$$

Define

$$|\psi|_T^{l, l/2} = \max (|\psi_1|_T^{l, l/2}, |\psi_2|_T^{l, l/2}, |\psi_3|_T^{l, l/2}, |\psi_4|_T^{l, l/2}, |\psi_5|_T^{l, l/2}),$$

fix T_0 such that $T_0 > T$, and consider in the space $H^{l, l/2}(\mathbb{R}_T^n)$ the set $S(T)$ of functions $\psi(x, t)$ satisfying the inequality

$$|\psi - \psi_0|_T^{l, l/2} \leq |\psi_0|_{T_0}^{l, l/2}, \tag{4.7}$$

where $\psi_0 = (\psi_{01}, \psi_{02}, \psi_{03}, \psi_{04}, \psi_{05})$ and

$$|\psi_0|_{T_0}^{l, l/2} = \max (|\psi_{01}|_{T_0}^{l, l/2}, |\psi_{02}|_{T_0}^{l, l/2}, |\psi_{03}|_{T_0}^{l, l/2}, |\psi_{04}|_{T_0}^{l, l/2}, |\psi_{05}|_{T_0}^{l, l/2}).$$

For a sufficiently small T , the operator A is a contraction mapping operator in $S(T)$. Then the uniqueness and existence theorem follows right away from the contraction mapping principle.

First, it is seen that A has the first property of a contraction mapping operator. Let $\psi \in S(T)$, $T < T_0$. Then, from relation (4.7), we have

$$|\psi_i|_T^{l, l/2} \leq 2 |\psi_0|_{T_0}^{l, l/2}, \quad i = 1, 2, 3, 4, 5.$$

Define

$$a_1 := \|a\|_{C^2[0,T]}, \quad a_2 := \max_{t \in [0,T]} |(\ln a(t))'|$$

$$r_1 = |r(x', 0)|^l, \quad f_1 := |f(x', t - \tau)|^{l+6, (l+6)/2}, \quad \varphi_1 := |\varphi(x)|^{l+6}.$$

It is not hard to see that

$$\begin{aligned} |(A\psi)_1 - \psi_{01}|_T^{l,l/2} &= \left| \int_0^{\theta(t)} \frac{d\tau}{a(\theta^{-1}(\tau))} \int_{\mathbb{R}^n} [(\ln a(\theta^{-1}(\tau)))' (\psi_1(\xi, \theta^{-1}(\tau))) \right. \\ &\quad \left. + \int_0^{\theta^{-1}(\tau)} \psi_4(\xi', \theta^{-1}(\tau) - \alpha) \psi_1(\xi, \alpha) d\alpha \right] - r(\xi', 0) \psi_1(\xi, \alpha) \\ &\quad \left. - \int_0^{\theta^{-1}(\tau)} \psi_3(\xi', \theta^{-1}(\tau) - \alpha) \psi_1(\xi, \alpha) d\alpha \right] G(x - \xi, \theta(t) - \tau) d\xi \Big|_T^{l,l/2} \\ &\leq \int_0^{\theta(t)} \frac{d\tau}{|a(\theta^{-1}(\tau))|_T} \int_{\mathbb{R}^n} [|(\ln a(\theta^{-1}(\tau)))'|_T \\ &\quad \times (\psi_1(\xi, \theta^{-1}(\tau)))_T^{l,l/2} + \int_0^{\theta^{-1}(\tau)} |\psi_4(\xi', \theta^{-1}(\tau) - \alpha)|_T^l |\psi_1(\xi, \alpha)|_T^l d\alpha \\ &\quad + |r(\xi', 0)|^l |\psi_1(\xi, \alpha)|_T^{l,l/2} \\ &\quad \left. + \int_0^{\theta^{-1}(\tau)} |\psi_3(\xi', \theta^{-1}(\tau) - \alpha)|_T^{l,l/2} |\psi_1(\xi, \alpha)|_T^{l,l/2} d\alpha \right] G(x - \xi, \theta(t) - \tau) d\xi \\ &\leq |\psi_0|_{T_0}^{l,l/2} \frac{2T^2}{a_0} \left(a_2 + 2Ta_2 |\psi_0|_{T_0}^{l,l/2} + r_1 + 2T |\psi_0|_{T_0}^{l,l/2} \right) := |\psi_0|_{T_0}^{l,l/2} \beta_1. \end{aligned}$$

In the same way, we obtain

$$\begin{aligned} |(A\psi)_2 - \psi_{02}|_T^{l,l/2} &\leq |\psi_0|_{T_0}^{l,l/2} \left[\frac{2T^2}{a_0} \left(\frac{a_1}{a_0} + 2a_2^2 + 2a_2 + r_1 + 2T |\psi_0|_{T_0}^{l,l/2} \right) \right. \\ &\quad \left. + 2T |\psi_0|_{T_0}^{l,l/2} \left(\frac{a_2}{a_0} + 2a_2 + 2a_2^2 \right) + 4Ta_2 |\psi_0|_{T_0}^{l,l/2} + a_1 \varphi_1 \right] := |\psi_0|_{T_0}^{l,l/2} \beta_2, \\ |(A\psi)_3 - \psi_{03}|_T^{l,l/2} &\leq |\psi_0|_{T_0}^{l,l/2} \left(2 \frac{T^2}{f_1} \left[\frac{a_1}{a_0} + 2a_2^2 + 2a_2 + r_1 + 2T |\psi_0|_{T_0}^{l,l/2} \right. \right. \\ &\quad \left. \left. + 2T |\psi_0|_{T_0}^{l,l/2} \left(\frac{a_1}{a_0} + 2a_2 + 2a_2^2 \right) + 4T |\psi_0|_{T_0}^{l,l/2} a_2 + a_1 \varphi_1 \right] + T f_1^2 \left(\frac{a_1}{a_0} + 2a_2^2 + 2a_2 + 1 \right) \right) := |\psi_0|_{T_0}^{l,l/2} \beta_3 \\ |(A\psi)_4 - \psi_{04}|_T^{l,l/2} &\leq 2T |\psi_0|_{T_0}^{l,l/2} := |\psi_0|_{T_0}^{l,l/2} \beta_4, \\ |(A\psi)_5 - \psi_{05}|_T^{l,l/2} &\leq |\psi_0|_{T_0}^{l,l/2} \cdot \frac{2T^2}{a_0} (\varphi_1 + r_1 + 2T |\psi_0|_{T_0}^{l,l/2}) := |\psi_0|_{T_0}^{l,l/2} \beta_5, \end{aligned}$$

where $\beta_i(T) \rightarrow 0$ as $T \rightarrow 0$, $i = 1, 2, 3, 4, 5$. Accordingly, if we take T ($T < T_0$) such that the following relation holds:

$$\beta := \max \{ \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \} < 1,$$

then the operator A has the first property of a contraction operator of mapping, i.e., $A\psi \in S(T)$. Next, let us think about the second property of a contraction mapping operator for A . Let

$$\psi^{(1)} = \left(\psi_1^{(1)}, \psi_2^{(1)}, \psi_3^{(1)}, \psi_4^{(1)}, \psi_5^{(1)} \right) \in S(T), \quad \psi^{(2)} = \left(\psi_1^{(2)}, \psi_2^{(2)}, \psi_3^{(2)}, \psi_4^{(2)}, \psi_5^{(2)} \right) \in S(T).$$

Based on the inequalities

$$\begin{aligned} & \left| \psi_2^{(1)} \psi_1^{(1)} - \psi_2^{(2)} \psi_1^{(2)} \right|_T^{l,l/2} = \left| (\psi_2^{(1)} - \psi_2^{(2)}) \psi_1^{(1)} + \psi_2^{(2)} (\psi_1^{(1)} - \psi_1^{(2)}) \right|_T^{l,l/2} \\ & \leq 2 \left| \psi^{(1)} - \psi^{(2)} \right|_T^{l,l/2} \max \left(\left| \psi_1^{(1)} \right|_T^{l,l/2}, \left| \psi_2^{(2)} \right|_T^{l,l/2} \right) \leq 4 \left| \psi_0 \right|_T^{l,l/2} \left| \psi^{(1)} - \psi^{(2)} \right|_T^{l,l/2}, \end{aligned}$$

we evaluate the difference

$$\begin{aligned} |((A\psi)^{(1)} - A\psi)^{(2)}|_1|_T^{l,l/2} & \leq \int_0^{\theta(t)} \frac{d\tau}{|\alpha(\theta^{-1}(\tau))|_T} \int_{\mathbb{R}^n} \left[|(\ln \alpha(\theta^{-1}(\tau)))'|_T \left((\psi_1^{(1)}(\xi, \theta^{-1}(\tau)) \right. \right. \\ & \quad \left. \left. - \psi_1^{(2)}(\xi, \theta^{-1}(\tau)) \right) \right]_T^{l,l/2} + \int_0^{\theta^{-1}(\tau)} \left[\left| \psi_4^{(1)}(\xi', \theta^{-1}(\tau) - \alpha) \psi_1^{(1)}(\xi, \alpha) \right. \right. \\ & \quad \left. \left. - \psi_4^{(2)}(\xi', \theta^{-1}(\tau) - \alpha) \psi_1^{(2)}(\xi, \alpha) \right]_T^{l,l/2} d\alpha + |r(\xi', 0)|^l \left| (\psi_1^{(1)}(\xi, \alpha) - \psi_1^{(2)}(\alpha)) \right|_T^{l,l/2} \\ & \quad + \int_0^{\theta^{-1}(\tau)} \left[\left| \psi_3^{(1)}(\xi', \theta^{-1}(\tau) - \alpha) \psi_1^{(1)}(\xi, \alpha) - \psi_3^{(2)}(\xi', \theta^{-1}(\tau) \right. \right. \\ & \quad \left. \left. - \alpha) \psi_1^{(2)}(\xi, \alpha) \right]_T^{l,l/2} d\alpha \right] G(x - \xi, \theta(t) - \tau) d\xi \\ & \leq \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \frac{T^2}{a_0} (a_2 + 4T a_2 \left| \psi_0 \right|_{T_0}^{l,l/2} + r_1 + 4T \left| \psi_0 \right|_{T_0}^{l,l/2}) := \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \mu_1. \end{aligned}$$

For other components of A , we can write

$$\begin{aligned} |((A\psi)^{(1)} - A\psi)^{(2)}|_2|_T^{l,l/2} & \leq \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \left(\frac{T^2}{a_0} \left(\frac{a_1}{a_0} + 2a_2^2 + 2a_2 + r_1 + 4T \left| \psi_0 \right|_{T_0}^{l,l/2} \right) \right. \\ & \quad \left. + 4T \left| \psi_0 \right|_{T_0}^{l,l/2} \left(\frac{a_1}{a_0} + 2a_2 + 2a_2^2 \right) + 8T a_2 \left| \psi_0 \right|_{T_0}^{l,l/2} + a_1 \varphi_1 \right) := \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \mu_2, \\ |((A\psi)^{(1)} - A\psi)^{(2)}|_3|_T^{l,l/2} & \leq \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \left(\frac{T^2}{f_1} \left[\frac{a_1}{a_0} + 2a_2^2 + 2a_2 + r_1 + 4T \left| \psi_0 \right|_{T_0}^{l,l/2} \right. \right. \\ & \quad \left. \left. + 4T \left| \psi_0 \right|_{T_0}^{l,l/2} \left(\frac{a_1}{a_0} + 2a_2 + 2a_2^2 \right) + 8T \left| \psi_0 \right|_{T_0}^{l,l/2} a_2 + a_1 \varphi_1 \right] \right. \\ & \quad \left. + T f_1^2 \left(\frac{a_1}{a_0} + 2a_2^2 + 2a_2 + 1 \right) \right) := \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \mu_3, \\ |((A\psi)^{(1)} - A\psi)^{(2)}|_4|_T^{l,l/2} & \leq \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} T := \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \mu_4, \\ |((A\psi)^{(1)} - A\psi)^{(2)}|_5|_T^{l,l/2} & \leq \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \frac{T^2}{a_0} (\varphi_1 + r_1 + 4T \left| \psi_0 \right|_{T_0}^{l,l/2}) := \left| \psi^{(1)} - \psi^{(2)} \right|_{T_0}^{l,l/2} \mu_5. \end{aligned}$$

Hence,

$$\left| (A\psi^{(1)} - A\psi^{(2)}) \right|_T^{l,l/2} < \mu \left| \psi^{(1)} - \psi^{(2)} \right|_T^{l,l/2}$$

if T satisfies the condition

$$\mu := \max \{ \mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \} < 1.$$

It is not difficult to see that if we set $T_0 = \min(\beta, \mu)$, then, for any $T \in (0, T_0)$, the operator A has the two properties of a contraction mapping operator, i.e., A takes the set $S(T)$ onto itself. Therefore, by the Banach theorem (see, for example, [22, pp. 87–97]), there is a unique fixed point of A in $S(T)$; i.e., there exists only one solution to (4.1). \square

5. Conclusion

In this paper, we have considered the problem of finding the functions $u(x, t)$ and $k(x', t)$ from the (1.1)–(1.3). First, the above problem has been reduced to an auxiliary problem. The equivalence of the auxiliary problem to Volterra-type integral equations has been shown. The existence and uniqueness of a solution to the problem have been obtained using the fixed point principle.

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