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Dedicated to L. D. Faddeev
on the occasion of his 60th birthday

CONSTRUCTIONS OF UNIFORM DISTRIBUTIONS IN TERMS OF GEOMETRY OF NUMBERS

M. M. Skriganov

Abstract. In the paper the author proves that the points of admissible lattices in the Euclidean space are distributed very uniformly in parallelepipeds. In particular, the remainder terms in the correspoinding lattice point problem are found to be logarithmically small. As an application of these results point sets with the lowest possible discrepancies in the unit cube and quadrature formulas with the smallest possible errors in the classes of functions with anisotropic smoothness are given in terms of admissible lattices.

Introduction

In the present paper we study admissible lattices with respect to the norm form $\operatorname{Nm} X = x_1 \dots x_s$, $X = (x_1, \dots, x_s) \in \mathbb{R}^s$, i.e., the lattices Γ which satisfy the condition

$$\inf_{\gamma \in \Gamma \setminus \{0\}} |\operatorname{Nm} \gamma| > 0.$$

Well-known relationships between algebraic number fields and the lattices in the Euclidean space give broad classes of admissible lattices for all dimensions $s \geqslant 2$. A series of remarkable, properties of these lattices has been studied in geometry of numbers and in algebraic number theory (see, for example, Borevich and Schafarevich [1], Cassels [2], Gruber and Lekkerkerker [3]).

In the present paper we obtain new results concerning admissible lattices. It turns out that the points of an admissible lattice have an amazingly uniform distribution in parallelepipeds with edges parallel to the coordinate axes. More precisely, one of our main results can be formulated as follows (cf. Theorem 1.1 below). Let an admissible lattice $\Gamma \subset \mathbb{R}^s$ and a parallelepiped $\Pi \subset \mathbb{R}^s$ of the indicated type be given. Let $t\Pi$ be the dilatation of Π by a factor $t > 0$, let $t\Pi + X$ be the translation of $t\Pi$ by a vector $X \in \mathbb{R}^s$, and let $N(t\Pi + X, \Gamma)$ be the number of points of Γ lying inside $t\Pi + X$. Then the following asymptotic formula holds as $t \to \infty$:

$$N(t\Pi + X, \Gamma) = \frac{\operatorname{vol} \Pi}{\det \Gamma} t^s + O(\ln^{s-1} t). \tag{0.1}$$

Key words and phrases. Lattice point problem, uniform distributions, quadrature formulas.

Moreover, this formula is valid uniformly both for $X \in \mathbb{R}^s$ and for parallelepipeds Π of bounded volume.

It is reasonable to expect that the bound for the remainder term in formula (0.1) is best possible. However, we can prove this conjecture only in the dimension $s = 2$. At the same time, for X outside a set of very small measure the bound for the remainder in (0.1) can be improved up to a magnitude of order $O(\ln^{\frac{s-2}{2}} t)$.

It should be pointed out that if we take the integer point lattice $\mathbb{Z}^s$ instead of an admissible lattice Γ, then the remainder term in the corresponding asymptotic formula will be of order $O(t^{s-1})$. The phenomenon of such strong dependence of the remainder term upon the choice of the lattice as well as the possibility of logarithmically small bounds for the remainders in the lattice point problem were first discovered and studied in the author's papers [4, 5, 6]. In the present paper these results are given in a final form.

As an application of our results we give new and very simple constructions of point sets with the lowest possible discrepancies. Let $\mathbb{K}^s$ be the unit cube in $\mathbb{R}^s$. Let $\Gamma - X$ be the translation of the lattice Γ by a vector $-X$ and let $t^{-1}(\Gamma - X)$ be the contraction of the shifted lattice $\Gamma - X$ by a factor $t > 0$. Let us introduce the following point set in the unit cube:

$$\Omega_{t,x}(\Gamma) = t^{-1}(\Gamma - X) \cap \mathbb{K}^s. \tag{0.2}$$

The number of elements in the set (0.2) is equal to $N_{t,X} = N(t\mathbb{K}^s + X, \Gamma)$.

We are interested in the uniformity of the distribution of the points of the set (0.2) as $t \to \infty$. To control the irregularity of distributions, one uses the so-called extremal discrepancy $\Delta(\cdot)$ and the L_q-discrepancies $\Delta_q(\cdot)$. For example, the extremal discrepancy for the set (0.2) can be defined in the following way:

$$\Delta(\Omega_{t,X}(\Gamma)) = \sup_{\Pi \subset \mathbb{K}^s} |N(t\Pi + X, \Gamma) - \operatorname{vol} \Pi \, N(t\mathbb{K}^s + X, \Gamma)|, \tag{0.3}$$

where the supremum is taken over the parallelepipeds Π contained in $\mathbb{K}^s$ and whose edges are parallel to the coordinate axes (all details will be given in Section 2).

In general, for arbitrary lattices, say for the lattice $\mathbb{Z}^s$, the sets (0.2) are distributed as $t \to \infty$ with very poor quality of uniformity. It is remarkable that for admissible lattices, the sets (0.2) behave completely differently. Let Γ be an admissible lattice, then the comparison of the asymptotic formula (0.1) and definition (0.3) gives the following bound as $t \to \infty$ (cf. Corollary 2.1):

$$\Delta(\Omega_{t,X}(\Gamma)) \ll \ln^{s-1} t \ll \ln^{s-1} N_{t,X}. \tag{0.4}$$

Moreover, this bound is valid uniformly for $X \in \mathbb{R}^s$.

In addition we prove that for X outside a set of very small measure, the following bounds for L_q-discrepancies hold as $t \to \infty$:

$$\Delta_q(\Omega_{t,X}(\Gamma)) \ll \ln^{\frac{s-1}{2}} t \ll \ln^{\frac{s-1}{2}} N_{t,X}. \tag{0.5}$$

It is well known that for $q > 1$, bounds (0.5) are best possible, i.e. these bounds cannot be improved for any set with the same number of elements. Hypothetically, bounds (0.4) are also best possible. However, at present, this conjecture is proved only in the dimension $s = 2$ (all necessary references will be given in Section 2).

We see from bounds (0.4) and (0.5) that the point sets (0.2) constructed in terms of admissible lattices fill the unit cube very uniformly as $t \to \infty$. Moreover, it is worth noting that the least bounds for the discrepancies can be reached by very regular sets

(0.2), that is in contrast with a conventional point of view when uniformly distributed sets are regarded as disordered and chaotic.

Bounds (0.4) and (0.5) enable us to use the points of the sets (0.2) as the nodes for numerical evaluation of multiple integrals. In the present paper we give very simple constructions of quadrature formulas by means of admissible lattices, and we prove that these formulas have the best possible accuracy in the general classes of functions with anisotropic smoothness (cf. Theorem 2.1).

As is evident from the foregoing, the subject matter of the present paper is at the crossroads of several areas. Two of them are geometry of numbers and uniform distribution theory, respectively. Advanced function theory can be mentioned as the third area because rather refined estimates in the spirit of the Littlewood–Paley theory will be used in proving our theorems.

Earlier, the most part of this paper was presented in the author's note [7] and the preprints [8, 9].

Applications of our results and methods to algebraic number theory are given recently by N. A. Nikichine and the author [10].

Added in proofs. Applications to stectral theory are given very recently in the paper: Skriganov M. M., *Anomalies in spectral asymptotics*, Dokl. Akad. Nauk of Russia (1994, to appear). In this paper, on the base of geometry of numbers, the author gives examples of elliptic pseudo-differential operators on compact manifolds with logarithmically small error terms in spectral asymptotics.

Our paper consists of eight sections. In Sections 1 and 2 the main results are stated. Sections 3 and 4 contain certain auxiliary results from the geometry of numbers and from calculus. A version of the Littlewood–Paley theory is described in Section 5. Basic bounds for sums over admissible lattices are given in Section 6. Sections 7 and 8 devoted to a proof of our main results.

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§1. Statement of the results: distributions of lattice points inside parallelepipeds

First we introduce some notation and definitions. Let $\mathbb{R}^s = \{X : X = (x_1, \dots, x_s), x_j \in \mathbb{R}\}$ be the s-dimensional Euclidean space. We define the product of vectors $X = (x_1, \dots, x_s)$ and $Y = (y_1, \dots, y_s)$ as the vector $X \cdot Y = (x_1 y_1, \dots, x_s y_s)$. We set

$$\operatorname{Nm} X = \prod_{j=1}^{s} x_j, \qquad \operatorname{Tr} X = \sum_{j=1}^{s} x_j. \tag{1.1}$$

It is obvious that $\operatorname{Nm} X \cdot Y = \operatorname{Nm} X \operatorname{Nm} Y$. In this notation the standard metric and inner products in $\mathbb{R}^s$ can be written as follows:

$$|X|^2 = \operatorname{Tr} X^2, \qquad X^2 = X \cdot X, \qquad \langle X, Y \rangle = \operatorname{Tr} X \cdot Y.$$

The space $\mathbb{R}^s$ with the above vector multiplication $X \cdot Y$ becomes a commutative ring with the unity $\mathbb{I} = (1, \dots, 1)$. The zero divisors in this ring satisfy the relation $\operatorname{Nm} X = 0$. Let

$$\mathbf{E}^s = \{ X \in \mathbb{R}^s : \operatorname{Nm} X \neq 0 \}$$

be the set of invertible elements of the ring. Note that $\mathbb{E}^s$ coincides with $\mathbb{R}^s$, in which the coordinate hyperplanes are removed. For $X \in \mathbb{E}^s$ we define $X^{-1} = (x_1^{-1}, \dots, x_s^{-1})$, so that $X \cdot X^{-1} = X^{-1} \cdot X = \mathbb{1}$. Let

$$\mathrm{Um} = \{ U \in \mathbb{E}^s : |\mathrm{Nm}\, U| = 1 \} \tag{1.2}$$

be the group of unimodular points of $\mathbb{R}^s$. For any $X \in \mathbb{E}^s$, one has a unique representation

$$X = tU, \quad t = |\mathrm{Nm}\, X|^{\frac{1}{s}} > 0, \quad U = t^{-1}X \in \mathrm{Um}. \tag{1.3}$$

Let $\mathcal{O} \subset \mathbb{R}^s$ be a compact body, $\mathrm{vol}\, \mathcal{O}$ the volume of $\mathcal{O}$, $\mathcal{O} + X$ the translation of $\mathcal{O}$ by a vector $X \in \mathbb{R}^s$, and $T \cdot \mathcal{O}$ the body obtained by multiplying every point of $\mathcal{O}$ by a vector $T \in \mathbb{E}^s$, where we have $\mathrm{vol}\, T \cdot \mathcal{O} = |\mathrm{Nm}\, T| \mathrm{vol}\, \mathcal{O}$. Note that multiplying $\mathcal{O}$ by $T = (t_1, \dots t_s)$ may be regarded as a nonhomogeneous dilatation of $\mathcal{O}$ by a factor t_j in the direction of the j-th coordinate axis. The homogeneous dilatation of $\mathcal{O}$ by a factor $t > \mathcal{O}$, which corresponds to $T = t\mathbb{1} = (t, \dots, t)$, will be denoted by $t\mathcal{O}$. Let $\mathbb{K}^s = \left[\frac{1}{2}; \frac{1}{2}\right]^s$ be the unit cube in $\mathbb{R}^s$ centered at the origin and with edges parallel to the coordinate axes. Then

$$T \cdot \mathbb{K}^s = \prod_{j=1}^{s} \left[-\frac{|t_j|}{2}; \frac{|t_j|}{2} \right]$$

is a parallelepiped centered at the origin with edges parallel to the codordinate axes and with the volume equal to $|\mathrm{Nm}\, T|$.

Let $\Gamma \subset \mathbb{R}^s$ be a lattice, i.e., a discrete subgroup of the group of translations of $\mathbb{R}^s$ with compact fundamental set $\mathcal{F}(\Gamma) = \mathbb{R}^s/\Gamma$ (see [2] and [3] for details); let $\det \Gamma$ be the determinant of Γ, where we have $\det \Gamma = \mathrm{vol}\, \mathcal{F}(\Gamma)$; let $\Gamma + X$ be the translation of Γ by a vector X, and let $T \cdot \Gamma$ be the lattice obtained by multiplying every point of Γ by a vector $T \in \mathbb{E}^s$, where we have $\det T \cdot \Gamma = |\mathrm{Nm}\, T| \det \Gamma$. Suppose that $t\Gamma$ is an homogeneous dilatation of Γ by a factor $t > 0$, $\mathbb{Z}^s$ is the integer coordinate lattice, and $\mathbb{Z}$ is the ring of integers.

Let

$$\mathrm{Nm}\, \Gamma = \inf_{\gamma \in \Gamma \setminus \{0\}} |\mathrm{Nm}\, \gamma| \tag{1.4}$$

be the homogeneous minimum of the lattice Γ with respect to the form $\mathrm{Nm}\, X$. We recall (cf. [2, 3]) that a lattice Γ is called admissible if $\mathrm{Nm}\, \Gamma > 0$. It is obvious that for an admissible lattice Γ, all lattices $T \cdot \Gamma$, $T \in \mathbb{E}^s$, are also admissible, since $\mathrm{Nm}\, T \cdot \Gamma = |\mathrm{Nm}\, T| \mathrm{Nm}\, \Gamma$. Well-known examples of admissible lattices can be described as follows.

1. Let $\mathbb{F}$ be a totally real algebraic number field of degree s and let σ be the canonical embedding of $\mathbb{F}$ in the Euclidean space $\mathbb{R}^s$

$$\sigma \colon \mathbb{F} \ni \xi \to \sigma(\xi) = (\sigma_1(\xi), \dots, \sigma_s(\xi)) \in \mathbb{R}^s,$$

where $\{\sigma_j\}_1^s$ are s distinct embeddings of $\mathbb{F}$ in the field $\mathbb{R}$ of real numbers. It is easy to check that $\mathrm{Nm}\, \sigma(\xi)$ is the norm and $\mathrm{Tr}\, \sigma(\xi)$ is the trace in the field $\mathbb{F}$. Therefore, under the embedding σ a full $\mathbb{Z}$-module $M \subset \mathbb{F}$ corresponds to an admissible lattice $\Gamma_M = \sigma(M)$ (see [1, Chap. II] and [3, Sect. 4]).

2. In the case of dimension 2, a more general construction is known. Namely, let α and $\alpha' \neq \alpha$ be real irrational numbers with bounded partial quotients in their continued fractions, then the lattice defined by the formula

$$\Gamma_{\alpha,\alpha'} = \{ \gamma \in \mathbb{R}^2 : \gamma = (n + \alpha m, n + \alpha' mm); \ n, m \in \mathbb{Z} \}$$

is admissible (cf. [3, Sect. 43.4 and XIV.5]). The lattices $\Gamma_{\alpha,\alpha'}$ are a generalization of the lattices Γ_M given above. Indeed, $\Gamma_{\alpha,\alpha'} = \Gamma_M$ for quadratic irrationalities α, α', conjugate in some real quadratic field $\mathbb{Q}(\sqrt{D})$ and for the module $M = \{1, \alpha\} \subset \mathbb{Q}(\sqrt{D})$.

The basic property characterizing admissible lattices is the compactness property (in the sense of Mahler, see [2, Chap. V]; [3, Sect. 17]) for the following set

$$\mathcal{U}[\Gamma] = \{U \cdot \Gamma \, , U \in \mathrm{Um}\}$$

in the set of all lattices of given dimension. It would be interesting to study in details intimate relationships between the structure of the compact set $\mathcal{U}[\Gamma]$ and the distribution of points of the lattice Γ. Here these relationships are studied on the level of the following two characteristics of $\mathcal{U}[\Gamma]$ which are unimodular invariants: $\det \Gamma$ and $\mathrm{Nm}\, \Gamma$.

For any discrete set $\mathcal{A} \subset \mathbb{R}^s$, we define

$$N(\mathcal{O}, \mathcal{A}) = \operatorname{card} \mathcal{O} \cap \mathcal{A} = \sum_{\gamma \in \mathcal{A}} \chi(\mathcal{O}, \gamma), \tag{1.5}$$

where $\chi(\mathcal{O}, X)$, $X \in \mathbb{R}^s$, is the characteristic function of $\mathcal{O}$. In particular, $N(\mathcal{O}, \Gamma)$ is equal to the number of points of the lattice Γ inside the body $\mathcal{O}$, and we have

$$N(\mathcal{O} + X, \Gamma) = N(\mathcal{O}, \Gamma - X), \qquad N(T \cdot \mathcal{O}, \Gamma) = N(\mathcal{O}, T^{-1} \cdot \Gamma). \tag{1.6}$$

We define $R(\mathcal{O}, \Gamma)$ by setting (cf. [3, 11])

$$N(\mathcal{O}, \Gamma) = \frac{\operatorname{vol} \mathcal{O}}{\det \Gamma} + R(\mathcal{O}, \Gamma). \tag{1.7}$$

From (1.6) it follows that

$$R(\mathcal{O} + X, \Gamma) = R(\mathcal{O}, \Gamma - X), \qquad R(T \cdot \mathcal{O}, \Gamma) = R(\mathcal{O}, T^{-1} \cdot \Gamma). \tag{1.8}$$

In particular, the remainder $R(\mathcal{O} + X, \Gamma)$ is a periodic function of $X \in \mathbb{R}^s$ with the period lattice Γ. We introduce the following quantities:

$$r(\mathcal{O}, \Gamma) = \sup_{X \in \mathcal{F}(\Gamma)} |R(\mathcal{O} + X, \Gamma)|, \tag{1.9}$$

$$r_q(\mathcal{O}, \Gamma) = \left[(\det \Gamma)^{-1} \int_{\mathcal{F}(\Gamma)} |R(\mathcal{O} + X, \Gamma)|^q \, dx \right]^{\frac{1}{q}}, \tag{1.10}$$

where $q > 0$ is a real number.

We are interested in finding bounds for (1.9) and (1.10) when $\mathcal{O} = T \cdot \mathbb{K}^s$ and $|\mathrm{Nm}\, T| \to \infty$. In general, for arbitrary lattices (say, for the lattice $\mathbb{Z}^s$) relation (1.7) is not even an asymptotic formula as $|\mathrm{Nm}\, T| \to \infty$ since the remainder term $R(T \cdot \mathbb{K}^s + X, \Gamma)$ in (1.7) may have greater order than the principal term $|\mathrm{Nm}\, T| / \det \Gamma$. Nevertheless, the following statement holds.

Theorem 1.1. *If $\Gamma \subset \mathbb{R}^s$ is an admissible lattice, then for all $T \in \mathbb{R}^s$ one has the bounds*

$$r(T \cdot \mathbb{K}^s, \Gamma) < c(\Gamma)[\ln(2 + |\mathrm{Nm}\, T|)]^{s-1} \tag{1.11}$$

$$r_q(T \cdot \mathbb{K}^s, \Gamma) < c_q(\Gamma)[\ln(2 + |\mathrm{Nm}\, T|)]^{\frac{s-1}{2}}. \tag{1.12}$$

The constants in (1.11) *and* (1.12) *depend upon the lattice* Γ *only by means of the invariants* $\det \Gamma$ *and* $\mathrm{Nm}\, \Gamma$.

Throughout this paper the letters c and C will denote different positive constants whose exact values will not be the particular concern of us; when necessary, we indicate the dependence of these constants upon additional parameters.

Remark 1.1. Note that the above bound (0.1) is a direct corollary to bound (1.11). As was mentioned, logarithmically small bounds for remainders in the lattice point problem were given first in the author's papers [4, 5, 6]. Bounds (0.1) and (1.11) were proved in [5] in the dimension $s = 2$; at the same time in dimensions $s > 2$ similar bounds with the exponent of the logarithm equal to s were proven in [5] for the lattices Γ_M. The reduction of the exponent from s to $s-1$ given in bounds (0.1), (1.11) is of particular importance because there are reasons to expect that bounds (0.1) and (1.11) are best possible. However, we can prove this conjecture only in the dimension $s = 2$ (see Remark 2.2 below). Concerning bounds (1.12), we can prove that they are best possible for all $s \geqslant 2$ and $q > 1$ (see Remark 2.2). Earlier weaker bounds for the metrics (1.10) were given in [5]. In the present paper we shall not dwell on questions of unimprovability of our results.

Remark 1.2. A careful analysis of Theorem 1.1 shows that the magnitude of the remainder $R(T \cdot \mathbb{K}^s + X, \Gamma)$ jumps from bound (1.12) to bound (1.11) on a set of very small measure in $X \in \mathcal{F}(\Gamma)$. More precisely, this observation can be formulated as follows. Let the assumptions of Theorem 1.1 be valid. Then for every $T \in \mathbb{E}^s$ and for arbitrary $\varepsilon > 0$, there exists a subset $\mathcal{A}_\varepsilon(T) \subset \mathcal{F}(\Gamma)$ whose measure is less than ε and such that one has the bound

$$|R(T \cdot \mathbb{K}^s + X, \Gamma)| < c_\delta(\Gamma)\varepsilon^{-\delta}[\ln(2 + |\operatorname{Nm} T|)]^{\frac{s-1}{2}}, \tag{1.13}$$

provided that $X \in \mathcal{F}(\Gamma) \setminus \mathcal{A}_\varepsilon(T)$. In estimate (1.13) $\delta > 0$ is arbitrarily small.

It would be very interesting to study in detail the structure of these exeptional sets $\mathcal{A}_\varepsilon(T)$. In particular, we would like to know:

(i) whether the sets $\mathcal{A}_\varepsilon(T)$ are nonempty;

(ii) whether the sets $\mathcal{A}_\varepsilon(T)$ have a positive measure;

(iii) whether it is possible to replace the power estimate $\varepsilon^{-\delta}$ in (1.13) with arbitrary $\delta > 0$ by an exponentially small bound.

§2. Statement of the results: uniform distributions and quadrature formulas

Now we turn to the description of our constructions of the many dimensional uniform distributions. First we recall some definitions and results from this field. For details we refer to the books [12] by Kuipers and Niederreiter, [13] by Beck and Chen.

Let $\mathcal{A}_m \subset \mathbb{K}^s$ be a set consisting of $m \geqslant 1$ points in the unit cube $\mathbb{K}^s$. We set

$$\Delta(\mathcal{A}_m) = \sup_{T \in \mathbb{I}^s} |D(T)|, \qquad \Delta_q(\mathcal{A}_m) = \left[\int_{\mathbb{I}^s} |D(T)|^q dT\right]^{\frac{1}{q}}, \quad q > 0, \tag{2.1}$$

where $\mathbb{I}^s = [0;1]^s$ is the shifted unit cube: $\mathbb{I}^s = \mathbb{K}^s + \frac{1}{2}\mathbb{I}$, and

$$D(T) = N(T \cdot \mathbb{K}^s, \mathcal{A}_m) - m|\operatorname{Nm} T|. \tag{2.2}$$

Quantities $\Delta(\cdot)$ and $\Delta_q(\cdot)$ are called the extremal discrepancy and the L_q — discrepancy, respectively. They give measures of irregularities of distributions. Namely, the points of sets $\mathcal{A}_m$ uniformly fill out the unit cube $\mathbb{K}^s$ as $m \to \infty$ if $m^{-1}\Delta(\mathcal{A}_m) \to 0$ or (that is the same) $m^{-1}\Delta_q(\mathcal{A}_m) \to 0$.

The construction of sets with the smallest discrepancies is the main problem of the theory of uniform distributions. It is known that for an arbitrary set $\mathcal{A}_m$ and for $q > 1$, the following lower bound is valid:

$$\Delta_q(\mathcal{A}_m) > c_{q,s} \ln^{\frac{s-1}{2}} m. \tag{2.3}$$

This lower bound was proved by Roth [14] for $q = 2$ and by W. M. Schmidt [15] for all $q > 1$ (cf. [12, 13]). Hypotheticaly (cf. [13, p. 6 and p. 283]), for an arbitrary set $\mathcal{A}_m$ one also has the following lower bound:

$$\Delta(\mathcal{A}_m) > c_s \ln^{s-1} m. \tag{2.4}$$

This conjecture was proved by W. M. Schmidt [16] in the dimension $s = 2$ (cf. [12, 13]). For $s > 2$, this problem is still opened.

The lower bounds (2.3) and (2.4) are reached for certain sets. Well-known examples of such sets were given by Roth [17] for the L_2-discrepancy and by Chen [18] for all L_q-discrepancies, $q > 0$, and by Halton [19] for the extremal discrepancy. In dimensions $s > 2$ these sets are suitable modifications of the socalled Halton–Hammersly–Roth sequence (cf. [12, Chap. 2, Sect. 3]; [13, Sect. 3.2]).

In the present paper we give new constructions of the sets with the smallest lower bounds (2.3) and (2.4). Let us introduce the following finite subset in the unit cube:

$$\Omega_{T,Z}(\Gamma) = T^{-1} \cdot (\Gamma - Z) \cap \mathbb{K}^s, \tag{2.5}$$

i.e., we translate the lattice Γ by a vector $-Z$, then multiply the shifted lattice $\Gamma - Z$ by a vector T^{-1}, $T \in \mathbb{E}^s$, and consider the points of $T^{-1} \cdot (\Gamma - Z)$ which lie in the cube $\mathbb{K}^s$. It is obvious that the number of elements in the set (2.5) is equal to

$$N_{T,Z} = N(\mathbb{K}^s, T^{-1} \cdot (\Gamma - Z)) = N(T \cdot \mathbb{K}^s + Z, \Gamma) \tag{2.6}$$

(cf. (1.6)).

We are interested in the uniformity of distribution of the points $\omega \in \Omega_{T,Z}(\Gamma)$ as $|\operatorname{Nm} T| \to \infty$. In general, for arbitrary lattices (say, for the lattice $\mathbb{Z}^s$) the sets (2.5) are not uniformly distributed as $|\operatorname{Nm} T| \to \infty$. Nevertheless, Theorem 1.1 gives the following.

Corollary 2.1. *If $\Gamma \subset \mathbb{R}^s$ is an admissible lattice, then the following assertions are valid:*

1. *For all $T \in \mathbb{E}^s$ and $Z \in \mathbb{R}^s$, one has the bound*

$$\Delta(\Omega_{T,Z}(\Gamma)) < c(\Gamma)[\ln(2 + |\operatorname{Nm} T|)]^{s-1} \leqslant C(\Gamma)[\ln(2 + N_{T,Z})]^{s-1}. \tag{2.7}$$

2. *For every $T \in \mathbb{E}^s$, there exists a vector $Z(T) \in \mathbb{R}^s$ which may depend on $q > 0$ and such that the following holds:*

$$\Delta_q(\Omega_{T,Z(T)}(\Gamma)) < c_q(\Gamma)[\ln(2 + |\operatorname{Nm} T|)]^{\frac{s-1}{2}} \leqslant C_q(\Gamma)[\ln(2 + N_{T,Z(T)})]^{\frac{s-1}{2}}. \tag{2.8}$$

Proof. Using formulas (1.6), (1.7), (2.5), (2.6), we obtain the following expression for the function (2.2) with $\mathcal{A}_m = \Omega_{T,Z}(\Gamma)$:

$$D(X) = R(T \cdot X \cdot \mathbb{K}^s + Z, \Gamma) - |\operatorname{Nm} X| R(T \cdot \mathbb{K}^s + Z, \Gamma).$$

Hence we have the following inequalities for discrepancies (2.1):

$$\Delta(\Omega_{T,Z}(\Gamma)) \leqslant \sup_{X \in \mathbb{I}^s} |R(T \cdot X \cdot \mathbb{K}^s + Z, \Gamma)| + |R(T \cdot \mathbb{K}^s + Z, \Gamma)|, \tag{2.9}$$

$$\Delta_q^q(\Omega_{T,Z}(\Gamma)) \leqslant 2^{q-1} \left\{ \int_{\mathbb{I}^s} |R(T \cdot X \cdot \mathbb{K}^s + Z, \Gamma)|^q dX + |R(T \cdot \mathbb{K}^s + Z, \Gamma)|^q \right\}. \tag{2.10}$$

Let Γ be an admissible lattice. Now bound (2.7) follows from (2.9) and (1.11). To prove (2.8), we integrate (2.10) over the fundamental set $\mathcal{F}(\Gamma)$. Using (1.12), we obtain

$$(\det \Gamma)^{-1} \int_{\mathcal{F}(\Gamma)} \Delta_q^q(\Omega_{T,Z}(\Gamma))\, dZ < c_q(\Gamma)[\ln(2 + |\operatorname{Nm} T|)]^{\frac{s-1}{2} q}.$$

This inequality immediatly implies the existence of a vector $Z(T) \in \mathcal{F}(\Gamma)$ satisfying (2.8). The proof is completed.

Remark 2.1. It is useful to keep in mind that for any admissible lattice Γ and for each fixed $Z \in \mathbf{R}^s$, the function of $T \in \mathbf{E}^s$ given by $N_{T,Z} = N(T \cdot \mathbf{K}^s + Z, \Gamma)$ takes all positive integer values. This follows from the fact that any hyperplane in $\mathbf{R}^s$ given by the equation $x_j = a$, $a \in \mathbf{R}$, contains at most one point of Γ (see Lemma 3.1 below). Thus, our construction (2.5) gives the examples of uniform distributions with an arbitrary number of elements.

Remark 2.2. The arguments used in proving Corollary 2.1 show that bounds (1.12) are best possible for all $s \geqslant 2$ and $q > 1$ and that bound (1.11) is the best possible for $s = 2$, since otherwise one could construct the sets (2.5) which would break the lower bounds (2.3) and (2.4), respectively. Similarly, bound (1.11) is best possible for all s if the conjecture (2.4) is valid in an arbitrary dimension.

Earlier weaker bounds for discrepancies of the sets similar to (2.5) were given by Frolov [20] and by the author [4, 5].

Corollary 2.1 shows that the sets (2.5) constructed in terms of admissible lattices fill out the unit cube very uniformly. This circumstance enables us to use the points $\omega \in \Omega_{T,Z}(\Gamma)$ as the modes for the numerical evaluation of s-multiple integrals. Thus, bound (2.7) combined with the Koksma–Hlawka inequality (see [12, Chap. 2, Sect. 5]) leads immediatly to the following.

Corollary 2.2. *Let $\Gamma \subset \mathbf{R}^s$ be an admissible lattice and let $f(X)$ be a function of bounded variation $\mathrm{Var}_{HK} f$ (in the sense of Hardy and Krause) defined on the cube $\mathbf{K}^s$. Then for $|\operatorname{Nm} T| \to \infty$, the following bound holds:*

$$\left| \int_{\mathbf{K}^s} f(X)dX - \frac{1}{N_{T,Z}} \sum_{\omega \in \Omega_{T,Z}(\Gamma)} f(\omega) \right| < c(\Gamma) \operatorname{Var}_{HK} f \frac{\ln^{s-1} N_{T,Z}}{N_{T,Z}} \tag{2.11}$$

(*cf.* (2.6)).

Now we describe our results concerning quadrature formulas for functions with anisotropic smoothness. Let

$$D = \frac{\partial^s}{\partial x_1 \dots \partial x_s} \tag{2.12}$$

be a differential operator. For any integer $l \geqslant 1$ and a real number $q \geqslant 1$, the anisotropic Sobolev space $\overset{\circ}{V}{}^l_q(\mathbf{K}^s)$ is defined as the space of functions f on $\mathbf{R}^s$ with compact supports lying inside the unit cube $\mathbf{K}^s$ and where the mixed partial derivative $D^l f$ exists (in the weak sense) and $D^l f \in L_q(\mathbf{K}^s)$. This space of functions can be normed by the expression

$$\|f\|_{l,q} = \left(\int_{\mathbf{K}^s} |D^l f(X)|^q dX \right)^{\frac{1}{q}}. \tag{2.13}$$

Note that expression (2.13) is a norm since the functions f are supported in $\mathbf{K}^s$. The resulting normed space is complete. The main facts about the spaces $\overset{\circ}{V}{}^l_k(\mathbf{K}^s)$ are collected in Section 4. In particular, without loss of generality we can assume that the functions $f(X) \in \overset{\circ}{V}{}^l_k(\mathbf{K}^s)$ are continuous (cf. Lemma 4.1 below).

For $f(X) \in \overset{\circ}{V}{}^l_k(\mathbf{K}^s)$ we define $\delta(f, \Gamma)$ by setting

$$\int_{\mathbf{K}^s} f(X)\, dX = \det \Gamma \sum_{\gamma \in \Gamma} f(\gamma) + \delta(f, \Gamma). \tag{2.14}$$

We regard relation (2.14) as a quadrature formula where the integral is replaces by a finite sum over the lattice Γ. We are interested in estimating the error $\delta(f,\Gamma)$ in terms of the number $N(\mathbb{K}^s,\Gamma)$ (ch. (1.5)) of nodes in the quadrature formula (2.14). Of course, such an estimate is interesting if we suitably shrink the lattice Γ, for example, replacing Γ by $t^{-1}\Gamma$ as $t\to\infty$.

If $\Gamma=\mathbb{Z}^s$, then one has the following best possible estimate as $t\to\infty$:

$$\delta(f,t^{-1}\mathbb{Z}^s)<c_f t^{-l}\leqslant C_f[N(\mathbb{K}^s,t^{-1}\mathbb{Z}^s)]^{-\frac{l}{s}}.$$

Thus, the error becomes much worse if the dimension s increases. This dramatic circumstance is a principal obstruction for applications of simple quadrature formulas (2.14) with $\Gamma=\mathbb{Z}^s$ in high dimensions. It is remarkable that the error behaves completely differently when the lattice Γ in (2.14) is admissible. Our main result concerning quadrature formulas can be stated as follows.

Theorem 2.1. *Let* $\Gamma\subset\mathbb{R}^s$ *be an admissible lattice and let* $f(X)\in\overset{\circ}{V}{}^l_k(\mathbb{K}^s)$. *Then one has the following bounds as* $|\operatorname{Nm}T|\to\infty$:

$$\delta(f,T^{-1}\cdot\Gamma)<c_{l,1}(\Gamma)\|f\|_{l,1}\frac{\ln^{s-1}|\operatorname{Nm}T|}{|\operatorname{Nm}T|^l}\leqslant C_{l,1}(\Gamma)\|f\|_{l,1}\frac{\ln^{s-1}N_T}{N_T^l} \tag{2.15}$$

for $q=1$ *and*

$$\delta(f,T^{-1}\cdot\Gamma)<c_{l,q}(\Gamma)\|f\|_{l,q}\frac{\ln^{\frac{s-1}{2}}|\operatorname{Nm}T|}{|\operatorname{Nm}T|^l}\leqslant C_{l,q}(\Gamma)\|f\|_{l,q}\frac{\ln^{\frac{s-1}{2}}N_T}{N_T^l} \tag{2.16}$$

for $q>1$.

In (2.15) *and* (2.16) *we have used the notation* $N_T=N(T\cdot\mathbb{K}^s,\Gamma)$ *for the number of nodes. The constants in* (2.15) *and* (2.16) *depend upon the lattice* Γ *only by means of the invariants* $\det\Gamma$ *and* $\operatorname{Nm}\Gamma$.

Remark 2.3. It was proved by Bykovskii [21] that an arbitrary quadrature formula for the space $\overset{\circ}{V}{}^l_k(\mathbb{K}^s)$ with $q\geqslant 2$, gives the accuracy no better then $N^{-l}(\ln N)^{\frac{s-1}{2}}$ for any choice of $N>1$ nodes. Thus our bounds for the error are best possible for all $q\geqslant 2$. There are reasons to expect that bounds (2.15) and (2.16) are best possible for all $q\geqslant 1$ (cf. Sarygin [22]).

Earlier weaker bounds for the error with $\Gamma=\Gamma_M$ were given by Frolov [23] and by the author [4, 5]. It should be pointed out that first applications of the lattices Γ_M to construction of quadrature formulas of the type (2.14) were given in [23].

Remark 2.4. The high accuracy of the bounds in Theorem 2.2 makes it possible to observe an interesting phenomenon: the exponent of the logarithm has a sharp jump from the value $\frac{s-1}{2}$ to $s-1$ when the parameter q changes continuously from $q>1$ to $q=1$.

Remark 2.5. We see from (2.15) and (2.16) that the error in the quandrature formula (2.14) is almost independent of the dimension s. Quadrature formulas with similar properties are known in the numerical analysis. Their construction by the so-called method of optimal coefficients (or, in other terms, by the method of good lattice points) was initiated by Korobov [24, 25] and by Hlawka [26] (we refer to [12, 25] for details). Here a new approach was developed in [27] by I. H. Sloan and in [28] by I. H. Sloan and P. J. Kachoyan.

It also should be noted that for $f \in \overset{\circ}{V}{}^l_q(\mathbb{K}^s)$ with $s > 2$, the optimal coefficient method gives an accuracy of order $N^{-l}(\ln N)^{ls}$, where $N > 1$ is the number of nodes. One should also keep in mind that finding the nodes in high dimensions by this method is a nontrivial matter requiring a large amount of computation. On the other hand, admissible lattices in the quadrature formulas (2.14), say, the lattices Γ_M can be constructed explicitly in any dimension $s \geqslant 2$. To do this, it is sufficient to take an arbitrary full module M in a subfield $\mathbb{F}$ of degree s in the maximal real subfield of a cyclotomic field of a suitable degree. In this case the nodes in the quadrature formula can easily by described in terms of the Gauss sums (cf. [1, Chap. V]) However, in the present paper we shall not consider the computational aspects of our results. Nevertheless we note that the comparison of all alternative approaches to construction of multi-dimensional quadrature formulas are of especial importance for the progress in this field.

In conclusion, we would like to draw attention to a close relationship between the sequences of bounds (1.11), (2.7), (2.15), on one hand, and bounds (1.12), (2.8), (2.16) on the other. This relationship has deep reasons which induces the author to collect the above results in the same paper.

§3. Auxiliary results from geometry of numbers. Bounds for sums over lattices

We recall some well-known facts from geometry of numbers. These facts supplement the properties given in Section 1. For all details, we refer to [2, 3]. Let $\lambda_1(\Gamma) \leqslant \lambda_2(\Gamma) \leqslant \cdots \leqslant \lambda_s(\Gamma)$ be successive minima of the lattice $\Gamma \subset \mathbb{R}^s$ with respect to the Euclidean metric $|X|$, $X \in \mathbb{R}^s$. In particular,

$$\lambda_1(\Gamma) = \min_{\gamma \in \Gamma \setminus \{0\}} |\gamma| \tag{3.1}$$

is the length of the shortest vector $\gamma_1 \in \Gamma \setminus \{0\}$. The vector γ_1 can be complemented to some basis $\xi_1 = \gamma_1, \xi_2, \dots, \xi_s$ of the lattice Γ, i.e., $\Gamma \ni \gamma = q_1\xi_1 + \cdots + q_s\xi_s$, where $Q = (q_1, \dots, q_s) \in \mathbb{Z}^s$. In addition,

$$|\gamma| = \Big|\sum_{j=1}^{s} q_j\xi_j\Big| \geqslant \lambda_1(\Gamma)|Q|. \tag{3.2}$$

We have the following relations (cf. [2, Chap. VIII, Theorems V and VI]):

$$\frac{2^s}{s!}\det\Gamma \leqslant v_s \prod_{j=1}^{s} \lambda_j(\Gamma) \leqslant 2^s \det\Gamma \tag{3.3}$$

where v_s is the volume of the unit ball in $\mathbb{R}^s$, and

$$1 \leqslant \lambda_j(\Gamma)\lambda_{s+1-j}(\Gamma^*) \leqslant s!, \quad j = 1, \dots, s, \tag{3.4}$$

where Γ^* is the lattice dual to Γ. Recall that Γ^* consists of all vectors γ^* such that the inner product $\langle \gamma^*, \gamma \rangle \in \mathbb{Z}$ for each $\gamma \in \Gamma$.

We have the inequality

$$\lambda_1^s(\Gamma) \geqslant s^{\frac{s}{2}} \operatorname{Nm}\Gamma > \operatorname{Nm}\Gamma, \tag{3.5}$$

which follows from the definitions (1.4), (3.1) and from the inequality between the arithmetic and geometric means. More generally, we have the formula

$$\operatorname{Nm}\Gamma = s^{-\frac{s}{2}} \inf_{U \in \mathrm{Um}} \lambda_1^s(U \cdot \Gamma), \tag{3.6}$$

where Um is the group of unimodular points (see (1.2)). Formula (3.6) can be proved as follows. Let a point $\gamma_1 \in \Gamma \setminus \{0\}$ belong to a coordinate plane, say, to the plane $x_1 = 0$. Then $\operatorname{Nm}\Gamma = 0$. Let $\gamma_1 = (0, x_1, \ldots, x_s)$ and $U_t = (t^{s-1}, t^{-1}, \ldots, t^{-1}) \in \mathrm{Um}$. Then

$$\inf_{U \in \mathrm{Um}} \lambda_1^s(U \cdot \Gamma) \leqslant \inf_{t>0} \lambda_1^s(U_t \cdot \Gamma) \leqslant \inf_{t>0} |U_t \gamma_1| = \inf_{t>0} t|\gamma_1| = 0.$$

Thus (3.6) is valid in this case, since the two sides in (3.6) are equal to zero.

Now let $\operatorname{Nm}\gamma \neq 0$ for all $\gamma \in \Gamma$, and let us consider the following surfaces

$$\Sigma_\gamma = \{ X \in \mathbb{R}^s : |\operatorname{Nm} X| = |\operatorname{Nm}\gamma| \}, \quad \gamma \in \Gamma \setminus \{0\}.$$

The group Um is transitive on each Σ_γ. Thus, for some $U_\gamma \in \mathrm{Um}$ we have $U_\gamma \gamma = s^{1/2} |\operatorname{Nm}\gamma|^{1/s} \mathbb{I}$ and $|U_\gamma \gamma|^s = s^{s/2} |\operatorname{Nm}\gamma|$. Therefore (cf. (1.4) and (3.1)),

$$\inf_{U \in \mathrm{Um}} \lambda_1^s(U \cdot \Gamma) \leqslant \inf_{\gamma \in \Gamma \setminus \{0\}} \lambda_1^s(U_\gamma \cdot \Gamma) \leqslant \inf_{\gamma \in \Gamma \setminus \{0\}} |U_\gamma \gamma|^s = s^{s/2} \inf_{\gamma \in \Gamma \setminus \{0\}} |\operatorname{Nm}\gamma| = s^{s/2} \operatorname{Nm}\Gamma.$$

At the same time we derive from (3.5)

$$\inf_{U \in \mathrm{Um}} \lambda_1^s(U \cdot \Gamma) \geqslant s^{s/2} \operatorname{Nm}\Gamma.$$

These inequalities prove formula (3.6).

Lemma 3.1. *If $\Gamma \subset \mathbb{R}^s$ is an admissible lattice, then the following assertions are valid:*

1. *Any hyperplane in $\mathbb{R}^s$ given by the equation $x_j = a$, $a \in \mathbb{R}$, contains at most one point of the lattice Γ.*

2. *The dual lattice Γ^* is also admissible.*

3. *There exists a radius $r_0 = r_0(\Gamma)$ depending upon the lattice Γ only by means of the invariants* $\det\Gamma$ *and* $\operatorname{Nm}\Gamma$ *such that the ball $B(r_0) = \{ X \in \mathbb{R}^s : |X| \leqslant r_0 \}$ contains a fundamental set for the lattice Γ.*

Proof. 1. Suppose that the hyperplane $x_j = a$ contains two different points $\gamma_1, \gamma_2 \in \Gamma$. Then the hyperplane $x_j = 0$ contains the nonzero point $\gamma_3 = \gamma_2 - \gamma_1 \in \Gamma$. Thus $\operatorname{Nm}\gamma_3 = 0$ and so $\operatorname{Nm}\Gamma = 0$, i.e., the lattice is not admissible. This proves assertion 1.

2. Suppose that the lattice Γ^* is not admissible, i.e., $\operatorname{Nm}\Gamma^* = 0$. From (3.6) we conclude that $\lambda_1(U^{-1} \cdot \Gamma^*)$ can be arbitrarily small for of a suitable choice of $U^{-1} \in \mathrm{Um}$. Thus, $\lambda_s(U^{-1} \cdot \Gamma^*)$ can be arbitrarily large by (3.3). Hence $\lambda_1(U \cdot \Gamma)$ can be arbitrarily small by (3.4), since $(U \cdot \Gamma)^* = U^{-1} \cdot \Gamma^*$. Therefore, $\operatorname{Nm}\Gamma = 0$ by (3.6), i.e., the lattice Γ is not admissible. This proves assertion 2.

3. We denote by $\gamma_1, \ldots, \gamma_3$ the linearly independent points of Γ on which the successive minima $\lambda_j(\Gamma)$ are attained, i.e., $\lambda_j(\Gamma) = |\gamma_j|, j = 1, \ldots, s$. It is obvious that the parallelepiped spanned by the vectors $\gamma_1, \ldots \gamma_s$ contains a fundamental set for the lattice Γ. At the same time this parallelepiped is contained inside the ball centered at the origin and with radius

$$r = |\gamma_1| + \cdots + |\gamma_s| = \lambda_1(\Gamma) + \cdots + \lambda_s(\Gamma) \leqslant s\lambda_s(\Gamma) \leqslant s2^s \det\Gamma \lambda_1^{1-s}(\Gamma),$$

where we have used (3.3). Using (3.5) we obtain

$$r \leqslant r_0 = r_0(\Gamma) = s2^s \det\Gamma (\operatorname{Nm}\Gamma)^{\frac{1-s}{s}}.$$

This proves assertion 3. The proof of Lemma 3.1 is completed.

Now we wish to consider certain sums over lattices. We define the nonnegative function $H(X)$, $X \in \mathbb{R}^s$, by setting

$$H(X) = H(x_1, \ldots, x_s) = \prod_{j=1}^{s} (1 + |x_j|)^{-2s}; \tag{3.7}$$

moreover,

$$H(X) \leqslant (1+|X|)^{-2s}, \qquad H^2(X) \leqslant H(X) \tag{3.8}$$

and, in addition, one has the bounds

$$\sup_{X\in B(r)} H(\gamma - X) \leqslant (1+r)^{2s^2} H(\gamma), \qquad \sum_{\gamma\in\Gamma} H(\gamma) < \infty, \tag{3.9}$$

where $B(r) = \{ X \in \mathbb{R}^s : |X| \leqslant r \}$ is a ball and $\Gamma \subset \mathbb{R}^s$ is an arbitrary lattice.

We consider the following lattice sum:

$$H(\Gamma, X) = \sum_{\gamma\in\Gamma} H(\gamma - X). \tag{3.10}$$

It is obvious that series (3.10) converges and represents a periodic function of X with period lattice Γ. Choosing the radius r so that the ball $B(r)$ contains a fundamental set $\mathcal{F}(\Gamma)$ for the lattice Γ, we derive from (3.9) and (3.10) the following sequence of estimates:

$$\begin{aligned} \sup_{X\in\mathbb{R}^s} H(\Gamma, X) &= \sup_{X\in\mathcal{F}(\Gamma)} H(\Gamma, X) = \sup_{X\in B(r)} H(\Gamma, X) \leqslant \sum_{\gamma\in\Gamma} \sup_{X\in B(r)} H(\gamma - X) \\ &\leqslant (1+r)^{2s^2} \sum_{\gamma\in\Gamma} H(\gamma) \end{aligned} \tag{3.11}$$

Let $\chi(t\mathbb{K}^s, X)$ be the characteristic function of the cube $t\mathbb{R}^s$. We have

$$\chi(t\mathbb{K}^s, X) \leqslant (1+t)^{2s^2} H(X).$$

If we take into account the definitions (1.5) and (3.10), we obtain

$$N(t\mathbb{K}^s + X, \Gamma) \leqslant (1+t)^{2s^2} H(\Gamma, X). \tag{3.12}$$

Lemma 3.2. *If $\Gamma \in \mathbb{R}^s$ is an admissible lattice, then the following assertions are valid:*

1. *The sum* (3.9) *satisfies the bound*

$$\sup_{X\in\mathbb{R}^s} H(\Gamma, X) \leqslant H_\Gamma, \tag{3.13}$$

where the constant H_Γ depends upon the lattice Γ only by means of the invariants $\det\Gamma$ *and* $\operatorname{Nm}\Gamma$.

2. *The number of points of the lattice Γ in the cube $t\mathbb{K}^s + X$ satisfies the bound*

$$N(t\mathbb{K}^s + X, \Gamma) \leqslant H_\Gamma (1+t)^{2s}, \tag{3.14}$$

with the same constant H_Γ as in (3.13).

Proof. Choose the radius r in estimate (3.11) to be equal to the radius $r_0 = r_0(\Gamma)$ from assertion 3 of Lemma 3.1. then, taking (3.2), (3.5), and (3.8) into account we continue estimates (3.11) as follows:

$$\begin{aligned} \sup_{X\in\mathbb{R}^s} H(\Gamma, X) &\leqslant C(r_0)\Big[1 + \sum_{\gamma\in\Gamma\setminus\{0\}} |\gamma|^{-2s}\Big] \leqslant C(r_0)\Big[1 + \lambda_1^{-2s}(\Gamma) \sum_{Q\in\mathbb{Z}^s\setminus\{0\}} |Q|^{-2s}\Big] \\ &\leqslant C(r_0)\Big[1 + (\operatorname{Nm}\Gamma)^{-2} \sum_{Q\in\mathbb{Z}^s\setminus\{0\}} |Q|^{-2s}\Big] = H_\Gamma. \end{aligned}$$

This proves bound (3.13). Bound (3.14) follows from (3.12) and (3.13). The proof of Lemma 3.2 is completed.

We consider the set of integer vectors with zero trace (cf. (1.1))

$$L = \{ Q \in \mathbb{Z}^s : \operatorname{Tr} Q = 0 \}. \tag{3.15}$$

The set (3.15) forms the $(s-1)$-dimensional lattice lying in the hyperplane $\mathcal{L}$ given by

$$\mathcal{L} = \{ Y \in \mathbb{R}^s : \operatorname{Tr} Y = 0 \}. \tag{3.16}$$

For any $Y = (y_1, \dots, y_s) \in \mathbb{R}^s$, we define the vector 2^Y by setting

$$2^Y = (2^{y_1}, \dots, 2^{y_s}). \tag{3.17}$$

It is obvious that $2^Y \in \mathrm{Um}$ for $Y \in \mathcal{L}$.

We shall need the following elementary inequality.

Lemma 3.3. *If $Y \in \mathcal{L}$ and $\nu > 0$, then*

$$H(\nu^{-\frac{1}{s}} 2^{-Y}) \leqslant \min\{1; \nu^2 2^{-|Y|}\}, \tag{3.18}$$

where $H(\cdot)$ is given by (3.7).

Proof. We have the inequality

$$\max_{1 \leqslant j \leqslant s} y_j \geqslant \frac{1}{2s} |Y|. \tag{3.19}$$

Indeed, let $Y \neq 0$ and, to be definite, suppose that the coordinates $y_1, \dots y_n$ are positive and $y_{n+1}, \dots, y_s$ are nonpositive. From the condition $\operatorname{Tr} Y = 0$ we find that

$$2 \sum_{j=1}^{n} y_j = \sum_{j=1}^{s} |y_j|,$$

whence

$$2s \max_{1 \leqslant j \leqslant s} y_j \geqslant \sum_{j=1}^{s} |y_j| \geqslant \Big(\sum_{j=1}^{s} |y_j|^2 \Big)^{\frac{1}{2}} = |Y|.$$

Inequality (3.19) is proved.

From the definition (3.7) we obtain

$$H(X) \leqslant \prod_{j=1}^{s} \min\{1; |x_j|^{-2s}\} \leqslant \min\{1; (\max_{1 \leqslant j \leqslant s} |x_j|)^{-2s}\} \tag{3.20}$$

for any $X \in \mathbb{R}^s$. Now with the help of estimates (3.19) and (3.20) the left side of (3.18) can be bounded as follows:

$$H(\nu^{-\frac{1}{s}} 2^{-Y}) \leqslant \min\{1; \nu^2 (\max_{1 \leqslant j \leqslant s} 2^{2s y_j})\} = \min\{1; \nu^2 2^{2s \max_{1 \leqslant j \leqslant s} y_j}\}$$
$$\leqslant \min\{1; \nu^2 2^{-|Y|}\}.$$

This proves Lemma 3.3.

We consider the following sum over the lattice $L \subset \mathcal{L}$ (cf. (3.15), (3.16)):

$$H_L(P) = \sum_{Q \in L} H(2^{-Q} \cdot P^{-1}), \tag{3.21}$$

where $P \in \mathbb{E}^s$.

Lemma 3.4. *The series* (3.21) *converges for all* $P \in \mathbb{E}^s$ *and satisfies the bound*

$$H_L(P) < c_s[\ln(2 + |\operatorname{Nm} P|)]^{s-1}. \tag{3.22}$$

Proof. For $P = (p_1, \dots, p_s) \in \mathbb{E}^s$, we set

$$|p_j| = |\operatorname{Nm} P|^{\frac{1}{s}} 2^{-y_j},$$

where $Y = (y_1, \dots y_s) \in \mathcal{L}$. In this notation we have

$$H(2^{-Q} \cdot P^{-1}) = H(|\operatorname{Nm} P|^{\frac{1}{s}} 2^{Y-Q}) \leqslant \min\{1; |\operatorname{Nm} P|^2 2^{-|Q-Y|}\} \tag{3.23}$$

by inequality (3.18).

Bound (3.23) shows that the summands in (3.21) are exponentially decreasing as $Q \to \infty$ along the hyperplane $\mathcal{L}$. Hence the series (3.21) converges.

To prove (3.22) we set

$$\rho = \log_2(2 + |\operatorname{Nm} P|)^{100} = 100 \log_2(2 + |\operatorname{Nm} P|)$$

and split the sum (3.21) into a sum over $Q \in L$ in the ball $|Q - Y| < \rho$ and a sum over $Q \in L$ satisfying $|Q - Y| \geqslant \rho$. Using (3.23), we replace $H(2^{-Q} \cdot P^{-1})$ by 1 in the first sum and by $|\operatorname{Nm} P|^2 2^{-|Q-Y|}$ in the second. As a result, we obtain

$$\begin{aligned} H_L(P) &\leqslant \sum_{|Q-Y|<\rho} 1 + |\operatorname{Nm} P|^2 \sum_{|Q-Y|\geqslant\rho} 2^{-|Q-Y|} \\ &\leqslant c\rho^{s-1} + |\operatorname{Nm} P|^2 2^{-\frac{1}{2}\rho} \sum_{|Q-Y|\geqslant\rho} 2^{-\frac{1}{2}|Q-Y|} \leqslant c\rho^{s-1} + c' \frac{|\operatorname{Nm} P|^2}{(2 + |\operatorname{Nm} P|)^{50}} \\ &< C_s[\ln(2 + |\operatorname{Nm} P|)]^{s-1}. \end{aligned}$$

This completes the proof of Lemma 3.4.

§4. Auxiliary analytic results

In this section we present necessary facts about the functional spaces $\overset{o}{V}{}^l_q(\mathbb{K}^s)$ and give the bounds for certain integrals that will be needed in our further studies.

The facts about the functional spaces are collected in the following

Lemma 4.1. *Let* $f \in \overset{o}{V}{}^l_q(\mathbb{K}^s)$ *and let* $\|f\|_{l,q}$ *be the norm* (2.13) *and* D *be the differential operator* (2.12). *Then the following asertions are valid*:

1. *For* $l \geqslant 1$ *and* $q \geqslant 1$, *all the functions* $D^j f(X)$, $j = 0, 1, \dots, l-1$, *are essentially continuous* (*i.e., these functions can be corrected on a set of measure zero so as to be continuous*), *and one has the bounds*

$$\operatorname*{ess\,sup}_{X \in \mathbb{K}^s} |D^j f(X)| \leqslant \|f\|_{l,q}, \quad j = 0, 1, \dots, l-1. \tag{4.1}$$

(*Thus, without loss of generality, we assume below that the functions* $D^j f(X)$, $j = 0, 1, \dots, l-1$ *are continuous*).

2. *For* $l > 1$ *and* $q \geqslant 1$, *one has the inequality*

$$|f(X) - f(X')| \leqslant c_s \|f\|_{l,q} |X - X'|, \quad \text{where } X, X' \in \mathbb{R}^s. \tag{4.2}$$

3. *For* $l = 1$ *and* $q > 1$, *one has the inequality*

$$|f(X) - f(X')| \leqslant c_{s,q} \|f\|_{1,q} |X - X'|^{\frac{q-1}{q}}, \quad \text{where } X, X' \in \mathbb{R}^s. \tag{4.3}$$

4. *For* $l = q = 1$, *the function* $f(X)$ *has the bounded variation* $\mathrm{Var}_{HK} f$ *in the sense of Hardy and Krause and the bounded variation* $\mathrm{Var}_V f$ *in the sense of Vitali, and the following relations are valid*:

$$\mathrm{Var}_{HK} f = \mathrm{Var}_V f = \|f\|_{1,1}. \tag{4.4}$$

Proof. To simplify the notation, we shall give the proof only in the dimension 2. Since the function $f \in \overset{o}{V}{}^l_q(\mathbb{K}^s)$ is compactly supported, we have (after a correction of f on a set of measure zero) the following formulas:

$$D^j f(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} D^{j+1}(t_1, t_2)\, dt_1\, dt_2 \tag{4.5}$$

for $j = 0, 1, \dots, l-1$. From (4.5) we obtain the formulas

$$D^j f(x_1, x_2) - D^j f(x_1', x_2') = \Big(\int_{x_1'}^{x_1} \int_{x_2'}^{x_2} + \int_{x_1'}^{x_1} \int_{-\infty}^{x_2'} + \int_{-\infty}^{x_1'} \int_{x_2'}^{x_2} \Big) D^{j+1} f(t_1, t_2)\, dt_1\, dt_2 \tag{4.6}$$

for $k = 0, 1, \dots l-1$ and, in addition,

$$\max\{|x_1 - x_1'|; |x_2 - x_2'|\} \leqslant |X - X'|.$$

1. For $l \geqslant 1$ and $q \geqslant 1$, $D^l f \in L_q(\mathbb{R}^s) \subset L_1(\mathbb{K}^s)$. Using formulas (4.5) and (4.6) for $j = l-1, \dots, 1, 0$, we verify successively that the functions $D^j f(X), j = l-1, \dots, 1, 0$, are continuous and bounds (4.1) are valid.

2. For $l > 1$ and $q \geqslant 1$, the function $Df(X)$ is continuous by assertion 1. Hence inequality (4.2) follows at once from formula (4.6) with $j = 0$.

3. For $l > 1$ and $q \geqslant 1$, $Df \in L_q(\mathbb{K}^s)$. Hence inequality (4.3) follows from formula (4.6) with $j = 0$ by means of Hölder's inequality.

4. By definition (cf. [12, Chapter 2, Section 5, 13]), the difference of the variations $\mathrm{Var}_{HK} f$ and $\mathrm{Var}_V f$ depends upon the behaviour of the function f only on the boundary of the unit cube. But the function f is equal to zero on the boundary. This proves the first equality in (4.4). The second equality in (4.4) is a well known formula (cf. [12, 13]) expressing the variation $\mathrm{Var}_V f$ in terms of the L_1-norm of the mixed derivative Df.

The proof of Lemma 4.1 is completed.

Now we wish to estimate certain integrals. First we introduce the following functions. Let $\omega(t) = \omega(|t|)$, $t \in \mathbb{R}^1$ be an even function of the class C^∞; moreover, let $\omega(t)$ and all the derivatives $\omega^{(k)}(t)$ satisfy the bounds

$$\omega^{(k)}(t) = O(|t|^{-\alpha}), \quad |t| \to \infty, \ k = 0, 1, \dots \tag{4.7}$$

with arbitrary $\alpha > 0$. We set

$$\Omega_k'(\tau) = \tau^k \sup_{1 \leqslant t < \infty} |\omega^{(k)}(\tau t)|, \qquad \Omega_k''(\tau) = \tau^k \int_1^\infty |\omega^{(k)}(\tau t)|\, dt,$$

where $\tau > 0$ and $k = 0, 1, \dots$. One can easily check the bounds

$$\Omega'(\tau) < c_{k,\alpha}(1 + r)^{-\alpha}, \qquad \Omega_k''(\tau) < c_{k,\alpha} \tau^{k-1} (1 + \tau)^{-\alpha}$$

with arbitrary $\alpha > 0$.

Let $\eta(t) = \eta(|t|)$, $t \in \mathbb{R}^1$, be an even function of the class C^∞; moreover, $\eta(t) = 0$ for $0 \leqslant |t| \leqslant 1$ and $\eta(t) = 1$ for $|t| \geqslant 2$. We set

$$\eta_{-1} = \sup_{1 \leqslant t < \infty} |\eta(t)|, \qquad \eta_0 = \sup_{1 \leqslant t < \infty} |t^{-1}\eta(t)|,$$

$$\eta_k = \int_1^\infty |(t^{-1}\eta(t))^{(k)}|\,dt < \infty, \quad k = 1, 2, \ldots$$

Let $f(t)$, $t \in \mathbb{R}^1$, be a function of the class C^∞; moreover, let $f(t)$ and all derivatives $f^{(k)}$ belong to $L_1(\mathbb{R}^1)$. We set

$$f_k = \int_{-\infty}^\infty |(f(t)\eta(t))^{(k)}|\,dt < \infty, \quad k = 0, 1, \ldots$$

We consider the following integrals for $\xi \in \mathbb{R}^1$ and $\tau > 0$:

$$J_f(\tau, \xi) = \int_{-\infty}^\infty dt e^{-2\pi i \xi t} f(t)\eta(t)\omega(\tau t) \tag{4.8}$$

and

$$I(\tau, \xi) = \int_{-\infty}^\infty dt e^{-2\pi i \xi t} \frac{1}{t}\eta(t)\omega(\tau t) = -2i \int_0^\infty dt \frac{\sin 2\pi\xi t}{t}\eta(t)\omega(\tau t). \tag{4.9}$$

Lemma 4.2. *For all $\alpha > 0$ and $\beta > 0$, one has the bounds*

$$|J_f(\tau, \xi)| < c_{\alpha,\beta}(1+\tau)^{-\alpha}(1+|\xi|)^{-\beta}, \tag{4.10}$$

$$|I(\tau, \xi)| < c_{\alpha,\beta}(1+\tau)^{-\alpha}(1+|\xi|)^{-\beta}. \tag{4.11}$$

Proof. Let us prove bound (4.10). From (4.8) we find directly that

$$|J_f(\tau, \xi)| \leqslant 2 f_0 \Omega_0'(\tau) < c(1+\tau)^{-\alpha}. \tag{4.12}$$

Integrating in (4.8) $4m$ times by parts, we find that

$$(2\pi\xi)^{4m} J_f(\tau, \xi) = \int_{-\infty}^\infty dt e^{-2\pi i \xi t} \sum_{j=0}^{4m} b_{4m}^j (f(t)\eta(t))^{(4m-j)} \tau^j \omega^{(j)}(\tau t),$$

where b_k^j are binomial coefficients. Therefore, we have the following estimate

$$|2\pi\xi|^{4m} |J_f(\tau, \xi)| \leqslant 2 \sum_{j=0}^{4m} b_{4m}^j f_{4m-j} \Omega_j'(\tau) < C(1+\tau)^{-\alpha} \tag{4.13}$$

Summing inequalities (4.12) and (4.13), we obtain (4.10).

Now we prove bound (4.11). Integrating in (4.9) $4m > 0$ times by parts, we obtain

$$(2\pi\xi)^{4m} I(\tau, \xi) = \int_{-\infty}^\infty dt e^{-2\pi i \xi t} \sum_{j=0}^{4m} b_{4m}^j (t^{-1}\eta(t))^{(4m-j)} \tau^j \omega^{(j)}(\tau t).$$

Therefore, we have the following estimate:

$$|2\pi\xi|^{4m}|I(\tau,\xi)| \leqslant 2\sum_{j=0}^{4m-1} b_{4m}^{j}\eta_{4m-j}\Omega'_j(\tau)+2\eta_0\Omega''_{4m}(\tau)\leqslant c(1+\tau^{4m-1})(1+\tau)^{-\alpha'}$$
$$\leqslant C(1+\tau)^{-\alpha}, \tag{4.14}$$

where $\alpha=\alpha'-4m+1>0$ is arbitrary, since the number $\alpha'>0$ can be chosen arbitrarily.

Let $\tau \geqslant 1$. From (4.9) we find directly that

$$|I(\tau,\xi)| \leqslant \eta_0\Omega''_0(\tau) < c\tau^{-1}(1+\tau)^{-\alpha} \leqslant c(1+\tau)^{-\alpha}. \tag{4.15}$$

Summing inequalitions (4.14) and (4.15), we obtain bound (4.11) for $\tau \geqslant 1$.

Let $0<\tau<1$. We prove the following bound:

$$|I(\tau,\xi)| < C \tag{4.16}$$

for $\xi \in \mathbb{R}^1$ and $\tau \in (0,1)$. Since $I(\tau,0)=0$ and $I(\tau,-\xi)=-I(\tau,\xi)$, it is sufficient to prove (4.16) only for $\xi>0$.

Let $0<\xi\leqslant\tau<1$. We have

$$|I(\tau,\xi)| \leqslant 2\eta_{-1}\int_1^\infty dt\left|\frac{\sin 2\pi\xi t}{t}\omega(\tau t)\right| \leqslant 4\pi\eta_{-1}\xi\Omega''_0(\tau)\leqslant c\xi\tau^{-1}\leqslant c.$$

If $\tau<\xi<1$, then we have

$$\frac{i}{2}I(\tau,\xi)=\int_0^\infty dt\frac{\sin t}{t}\eta\left(\frac{t}{2\pi\xi}\right)\omega\left(\frac{\tau t}{2\pi\xi}\right)=\int_0^{9\pi/2}+\int_{9\pi/2}^\infty = I_1+I_2.$$

The first integral can be estimated as follows:

$$|I_1(\tau,\xi)| \leqslant \int_{2\pi\xi}^{9\pi/2}\left|\eta\left(\frac{t}{2\pi\xi}\right)\omega\left(\frac{\tau t}{2\pi\xi}\right)\right|dt \leqslant \frac{9\pi}{2}\eta_{-1}\Omega'_0(\tau)\leqslant c.$$

We note that $\eta\left(\frac{t}{2\pi\xi}\right)=1$ in the second integral. The integration by parts gives

$$I_2(\tau,\xi)=\int_{9\pi/2}^\infty dt\frac{\cos t}{t}\left\{-\frac{1}{t}\omega\left(\frac{\tau t}{2\pi\xi}\right)+\frac{\tau}{2\pi\xi}\omega^{(1)}\left(\frac{\tau t}{2\pi\xi}\right)\right\}.$$

Therefore, we have the following estimate:

$$|I_2(\tau,\xi)| \leqslant \Omega'_0\left(\frac{\tau}{2\pi\xi}\right)+\Omega''_1\left(\frac{\tau}{2\pi\xi}\right)\leqslant c.$$

Finally, let $1\leqslant\xi$. Integrating by parts in the original integral (4.9), we obtain

$$\pi i\xi I(\tau,\xi)=\int_0^\infty dt\cos 2\pi\xi t\{(t^{-1}\eta^{(t)})^{(1)}\omega(\tau t)+t^{-1}\eta^{(t)}\tau\omega^{(1)}(\tau t)\}.$$

Therefore, we have the following estimate:

$$\pi\xi|I(\tau,\xi)| \leqslant \eta_1\Omega'_0(\tau)+\eta_0\Omega''_1(\tau)\leqslant c.$$

This proves bound (4.16). Summing inequalities (4.14) and (4.16), we obtain (4.11) for $0<\tau<1$. Thus, (4.11) is proved for all $\tau>0$. This completes the proof of Lemma 4.2.

§5. Dyadic decompositions of periodic functions. The Littlewood{Paley theory

First we recall some auxiliary facts. Let $L_q[\Gamma]$, $q \geqslant 1$, be the L_q-space of periodic functions $\varphi(X)$, $X \in \mathbb{R}^s$, with period lattice $\Gamma \subset \mathbb{R}^s$: $\varphi(X+\gamma) = \varphi(X)$, $\gamma \in \Gamma$. The norm in $L_q[\Gamma]$ is given by

$$\|\varphi\|_q = \|\varphi\|_{L_q[\Gamma]} = \Big[(\det\Gamma)^{-1} \int\limits_{F(\Gamma)} |\varphi(X)|^q \, dX\Big]^{\frac{1}{q}}, \tag{5.1}$$

where $\mathcal{F}(\Gamma)$ is a fundamental set for Γ (cf. Section 1). Let $C^\infty[\Gamma]$ be the space of infinitely diffrentiable periodic functions on $\mathbb{R}^s$ with period lattice $\Gamma \subset \mathbb{R}^s$.

For any $\varphi \in L_q[\Gamma]$, we consider the Fourier series

$$\varphi(X) = \sum_{\gamma^* \in \Gamma^*} \widehat{\varphi}(\gamma^*) \exp(2\pi i \langle \gamma^*, X \rangle), \tag{5.2}$$

where Γ^* is the lattice dual to Γ (cf. Sect. 3) and the Fourier coefficients $\widehat{\varphi}(\gamma^*)$ are given by the formula

$$\widehat{\varphi}(\gamma^*) = (\det\Gamma)^{-1} \int\limits_{\mathcal{F}(\Gamma)} \varphi(X) \exp(-2\pi i \langle \gamma^*, X \rangle) \, dX. \tag{5.3}$$

The series (5.2) converges in the norm (5.1). It is obvious that for $\varphi \in C^\infty[\Gamma]$, the series (5.2) converges absolutely and uniformly in $X \in \mathbb{R}^s$.

Further, we shall need the Poisson summation formula for an arbitrary lattice $\Gamma \subset \mathbb{R}^s$ (cf. Stein and Weiss [29, Chap. VII, Sect. 2]):

$$\det\Gamma \sum_{\gamma \in \Gamma} f(\gamma - X) = \sum_{\gamma^* \in \Gamma^*} \widetilde{f}(\gamma^*) \exp(2\pi i \langle \gamma^*, X \rangle), \tag{5.4}$$

where

$$\widetilde{f}(Y) = \int\limits_{\mathbb{R}^s} f(X) \exp(2\pi i \langle Y, X \rangle) dX \tag{5.5}$$

is the Fourier transform of $f(X)$, $X \in \mathbb{R}^s$. Formula (5.4) holds for functions $f(X)$ decreasing together with all their partial derivatives faster then $|X|^{-\alpha}$ for arbitrary $\alpha > 0$. We note that in (5.4) the sum over $\gamma \in \Gamma$ represents a periodic function of the class $C^\infty[\Gamma]$ and the sum over $\gamma^* \in \Gamma^*$ gives the Fourier series (5.2) for this function.

Let $L_q^0[\Gamma] \subset L_q[\Gamma]$ and $C_0^\infty[\Gamma] \subset C^\infty[\Gamma]$ be subspaces consisting of all functions $\varphi \in L_q[\Gamma]$ and, respectively, $\varphi \in C^\infty[\Gamma]$ with Fourier coefficients $\widehat{\varphi}(\gamma^*)$ satistying the following additional conditions:

$$\widehat{\varphi}(\gamma^*) = 0 \quad \text{if } \operatorname{Nm}\gamma^* = 0,\ \gamma^* \in \Gamma^*. \tag{5.6}$$

Note that if the period lattice Γ is admissible, then the dual lattice Γ^* is also admissible (cf. Lemma 3.1) and $\gamma^* = 0$ is a unique point of Γ^* with $\operatorname{Nm}\gamma^* = 0$. As a result, the additional conditions (5.6) are reduced to a single condition

$$\widehat{\varphi}(0) = \int\limits_{\mathcal{F}(\Gamma)} \varphi(X) \, dX = 0. \tag{5.7}$$

Diadic decompositions of periodic functions can be described as follows. We split $\mathbb{R}^s$ into the orthogonal sum

$$\begin{gathered} \mathbb{R}^s = \mathbb{R}^1 \dot{+} \mathbb{R}^{s-1}, \\ \mathbb{R}^s \ni X = (x_1, x_2, \ldots, x_s) = (x_1, x), \quad x_1 \in \mathbb{R}^1,\ x = (x_2, \ldots, x_s) \in \mathbb{R}^{s-1}; \end{gathered} \tag{5.8}$$

moreover,

$$\langle X,Y\rangle = x_1y_1 + \langle x,y\rangle, \quad \operatorname{Nm} X = x_1 \operatorname{Nm} x.$$

Let $m(t)$, $t \in \mathbb{R}^1$, be a non negative function of the class C^∞ vanishing in neighborhoods of $t = 0$ and $t = \pm\infty$. We set

$$M(X) = M(x) = \prod_{j=2}^{s} m(x_j), \tag{5.9}$$

so as a function of $X = (x_1, x)$ $M(X)$ does not depend on the coordinate x_1. We set

$$M_Q(X) = M(2^{-Q}\cdot X) = \prod_{j=1}^{s} m(2^{-q_j}x_j), \tag{5.10}$$

where $Q = (q_1, \dots, q_s) \in L$ (cf. (3.15), (3.16)). In the notation (5.8) $Q = (q_1, q)$, where $q \in \mathbb{Z}^{s-1}$ is arbitrary and $q_1 = -(q_2 + \cdots + q_s)$.

We introduce the following multipliers:

$$(\mathfrak{M}_Q\varphi)\widehat{\ }(\gamma^*) = M_Q(\gamma^*)\widehat{\varphi}(\gamma^*), \tag{5.11}$$

where $\widehat{\varphi}(\gamma^*)$ and $(\mathfrak{M}_Q\varphi)\widehat{\ }(\gamma^*)$ are the Fourier coefficients (5.3) of the functions $\varphi(X)$ and $\mathfrak{M}_Q\varphi(X)$, respectively (cf. [30, Sect. 1.5]). The multipliers $\mathfrak{M}_Q$ are linear operators in $L_q^0[\Gamma]$ defined at least on the dense subset $C_0^\infty[\Gamma] \subset L_q^0[\Gamma]$.

Suppose that the functions (5.10) satisfy the relation

$$\sum_{Q\in L} M_Q(X) = 1 \tag{5.12}$$

for all $X \in \mathbb{R}^s$ with $\operatorname{Nm} \neq 0$. Then, obviously, we have

$$\sum_{Q\in L} \mathfrak{M}_q\varphi = \varphi \tag{5.13}$$

for any function $\varphi \in C_0^\infty[\Gamma]$. Similarly, if the functions (5.10) satisfy the relation

$$\sum_{Q\in L} M_Q^2(X) = 1 \tag{5.14}$$

for all $X \in \mathbb{R}^s$ with $\operatorname{Nm} x \neq 0$, then we have

$$\sum_{Q\in L} \|\mathfrak{M}_Q\varphi\|_2^2 = \|\varphi\|_2^2 \tag{5.15}$$

for any function $\varphi \in L_2^0[\Gamma]$. It is obvious that (5.15) follows from Plancherel's formula for the L_2-norm (5.1).

Let us give examples of functions (5.9) and (5.10) satisfying relations (5.12) and (5.14). We define an even nonnegative function of $t \in \mathbb{R}^1$ by the formula

$$\alpha(t) = \begin{cases} 0 & \text{for } 0 \leqslant |t| \leqslant 1, \\ \beta(t) & \text{for } 1 \leqslant |t| \leqslant 2, \\ 1 & \text{for } |t| = 2, \\ 1-\beta(\frac{t}{2}) & \text{for } 2 < |t| < 4, \\ 0 & \text{for } |t| \geqslant 4, \end{cases} \tag{5.16}$$

where

$$\beta(t) = \exp\left[\frac{1-\frac{|t|}{2}}{1-|t|}\right]\left\{1 - \exp\left[\frac{|t|}{2} - 1\right]^{-1}\right\}.$$

It is easy to check that $\alpha(t)$ and $\alpha^{\frac{1}{2}}(t)$ are of the class C^∞ and vanish in neighbourhoods of $t = 0$ and $t = \pm\infty$ and, in addition, we have the relation

$$\sum_{q=-\infty}^{\infty} \alpha(2^{-q}t) = 1$$

for all $t \neq 0$. Now we see that the functions (5.10) with $m(t) = \alpha(t)$ and with $m(t) = \alpha^{\frac{1}{2}}(t)$ satisfy relations (5.12), and (5.14), respectively.

From the dyadic decomposition (5.13) we obtain immediatly the following result.

Lemma 5.1. *Suppose that the functions* (5.10) *satisfy relation* (5.12). *Then we have the inequality*

$$\sup_X |\varphi(X)| \leqslant \sum_{Q\in L} \sup_X |\mathfrak{M}_Q\varphi(X)|$$

for any function $\varphi \in C_0^\infty[\Gamma]$.

In order to formulate a similar result for the L_q-norms, we introduse the following function:

$$\Phi[\varphi(X)] = \Big(\sum_{Q\in L} |\mathfrak{M}_Q\varphi(X)|^2\Big)^{\frac{1}{2}} \tag{5.17}$$

for $\varphi \in L_q[\Gamma]$.

Lemma 5.2. *Suppose that the functions* (5.10) *satisfy relation* (5.14) *and* $q > 1$. *Then for any* $\varphi \in L_q^0[\Gamma]$, *the function* $\Phi[\varphi] \in L_q[\Gamma]$ *and we have the inequality*

$$\|\varphi\|_q < c_{q,s}\|\Phi[\varphi]\|_q \tag{5.18}$$

with the constant independent of the lattice Γ.

Lemma 5.2 represents a modification of one of the main results in the Littlewood–Paley theory for the multiple Fourier series. See, for example, Nikol'skii [30], Sect. 1.5 where inequalities of the type (5.18) are proved for the case of the period lattice $\Gamma = \mathbb{Z}^s$. For the sake of completeness, we give a sketch of the proof of inequalities (5.18) for arbitrary period lattices. Here we follow the approach to the Littlewood–Paley theory developed by Stein, see [31, Chap. IV].

Proof of Lemma 5.2 can be given in three stages.

Step 1. Well-known arguments (cf. [31, p. 105]) show that it is sufficient to prove the following inequality:

$$\|\Phi[\varphi]\|_q \leqslant C_{q,s}\|\varphi\|_q, \quad 1 < q < \infty, \tag{5.19}$$

with the constant independent of the lattice Γ. It is worth noting that relation (5.14) in Lemma 5.2 is needed only at this point.

Step 2. Sinse the operator $\varphi \to \Phi[\varphi]$ (see (5.17)) is nonlinear, it is more convenient to deal with multipliers. The corresponding construction can be given as follows (cf. [31, Chap. IV, Sect. 5]). Let $r_k(t) = \operatorname{sign}\sin 2^{k+1}\pi t$, $t \in \mathbb{R}^1$, $k = 0, 1, \ldots$ be the Rademacher functions. We consider the following function of $x \in \mathbb{R}^1$ depending on the parameter $t \in \mathbb{R}^1$:

$$m_t(x) = \sum_{k=0}^{\infty} r_{2k}(t)m(2^{-k}x) + \sum_{k=0}^{\infty} r_{2k+1}(t)m(2^k x), \tag{5.20}$$

where $m(x)$, $x \in \mathbb{R}^1$, is the function from (5.9) and (5.10). From the definition of $m(x)$ it is clear that for any x, at most finitely (and independent of x) many terms in the sums (5.20) can be nonzero. Moreover, we also easily see that

$$|m_t(x)| < c, \qquad \left|\frac{d}{dx} m_t(x)\right| < \frac{c}{|x|}, \tag{5.21}$$

where the constant is independent of t.

Now we consider the following function of $X = (x_1, \dots, x_s) \in \mathbb{E}^s$ depending on the parameter $T = (t_1, \dots t_s) \in \mathbb{R}^s$:

$$\Psi_T(X) = \prod_{j=2}^{s} m_{t_j}(x_j), \tag{5.22}$$

and we introduce the multipliers Ψ_T^Γ, $T \in \mathbb{R}^s$, in the space $L_q[\Gamma]$ by setting

$$(\Psi_T^\Gamma \varphi)\widehat{\ }(\gamma^*) = \Psi_T(\gamma^*)\widehat{\varphi}(\gamma^*), \quad \gamma^* \in \Gamma^* \tag{5.23}$$

(cf. (5.11)).

Well-known arguments (cf. [31, p. 105-108]) show that inequality (5.19) holds if the following bound is valid

$$\|\Psi_T^\Gamma \varphi\|_q \leqslant c_{q,s} \|\varphi\|_q, \quad \varphi \in L_q[\Gamma], \ 1 < q < \infty, \tag{5.24}$$

with the constant independent of $\Gamma \subset \mathbb{R}^s$ and $T \in \mathbb{R}^s$.

Step 3. In order to prove bound (5.24) we introduce nonperiodic multipliers Ψ_T, $T \in \mathbb{R}^s$, in the space $L_q(\mathbb{R}^s)$ by setting

$$(\Psi_T f)^{\sim}(Y) = \Psi_T(Y)\widetilde{f}(Y), \quad Y \in \mathbb{R}^s, \ f \in L_q(\mathbb{R}^s) \cap L_2(\mathbb{R}^s), \tag{5.25}$$

where $\widetilde{f}(Y)$ and $(\Psi_T f)^{\sim}(Y)$ are the Fourier transforms (5.5) of the functions $f(X)$ and $\Psi_T f(X)$, respectively (see [31, Chap. IV, Sect. 3 for details]).

Using the definition (5.22) and estimates (5.21), from the Marcinkiewicz multiplier theorem (see [31, Chap. IV, Sect. 6, Theorem 6′]) we derive the following bound:

$$\|\Psi_T f\|_{L_q(\mathbb{R}^s)} \leqslant c_{q,s} \|f\|_{L_q(\mathbb{R}^s)}, \quad f \in L_q(\mathbb{R}^s) \cap L_2(\mathbb{R}^s), \ 1 < q < \infty, \tag{5.26}$$

with the constant independent of $T \in \mathbb{R}^s$.

Now we are interested in the relationship between the periodic and nonperiodic multipliers (5.23) and (5.25). A well-known theorem by Stein and Weiss (see [29, Chap. VII, Theorem 3.8]) on periodization of multipliers says that bound (5.26) implies (5.24), and moreover, with the same constant. More precisely, here we use a simple modification of this theorem adapted for the space $L_q^0[\Gamma]$.

Thus, bound (5.24) is proved. This completes the proof of Lemma 5.2.

A more detailed discussion of the Littlewood–Paley theory in the context of number theory will be given on a suitable occasion.

§6. Bounds for sums over admissible lattices

In this section we study the following sum:

$$W^l(\Gamma, P, X) = \sum_{\gamma^* \in \Gamma^* \setminus \{0\}} (\operatorname{Nm} \Gamma^*)^{-l} \Omega(P^{-1} \cdot \gamma^*) \exp(2\pi i \langle \gamma^*, X \rangle), \tag{6.1}$$

where the lattice Γ^* is dual to a given lattice Γ, $l \geqslant 1$ is an integer, $P = (p_1, \dots, p_s) \in \mathbb{E}^s$, and the function $\Omega(X)$, $X \in \mathbb{R}^s$, is given by the relation

$$\Omega(X) = \prod_{j=1}^{s} \omega(x_j), \tag{6.2}$$

where $\omega(t) = \omega(|t|)$, $t \in \mathbb{R}^1$, is an even function of the class C^∞ satisfying bounds (4.7). Hence

$$\Omega(X) = O(|X|^{-\alpha}) \tag{6.3}$$

with arbitrary $\alpha > 0$.

Lemma 6.1. *If* Γ *is an admissible lattice, then the series* (6.1) *converges absolutely and*

$$W^l(\Gamma, P, \cdot) \in C_0^\infty[\Gamma]. \tag{6.4}$$

Proof. Since the lattice Γ^* is also admissible (cf. Lemma 3.1), one can estimate the denominator in (6.1) from below: $|(\mathrm{Nm}\,\gamma^*)^l| \geqslant (\mathrm{Nm}\,\Gamma^*)^l > 0$. This estimate, together with bound (6.3) for the function (6.2), provides the absolute convergence of the series (6.1). Moreover, the periodic function (6.1) belongs to the class $C^\infty[\Gamma]$. Since the term with $\gamma^* = 0$ is absent in the sum (6.1), we see that the condition (5.7) defining the subspace $C_0^\infty[\Gamma]$ for admissible lattices is satisfied. This proves (6.4). The proof is completed.

We wish to evaluate the multipliers $\mathfrak{M}_Q$, $Q \in L$, given by (5.11) on the periodic functions (6.1).

Lemma 6.2. *If* Γ *is an admissible lattice, then for all* $Q \in L$ *we have the bounds*

$$\sup_X |\mathfrak{M}_Q W^l|(\Gamma, P, X)| < c_l(\Gamma) H(P^{-1} \cdot 2^{-Q}), \tag{6.5}$$

where $H(\cdot)$ *is given by* (3.7). *The constant in* (6.5) *depends upon the lattice* Γ *only by means of the invariants* $\det\Gamma$ *and* $\mathrm{Nm}\,\Gamma$.

Proof. From the definitions (5.10) and (5.11) we obtain

$$\mathfrak{M}_Q W^l(\Gamma, P, X) = \sum_{\gamma^* \in \Gamma^* \setminus \{0\}} (\mathrm{Nm}\,\gamma^*)^{-l} \Omega(P^{-1} \cdot \gamma^*) M(2^{-Q} \cdot \gamma^*) \exp(2\pi i \langle \gamma^*, X \rangle). \tag{6.6}$$

In particular,

$$\mathfrak{M}_0 W^l(\Gamma, P, X) = \sum_{\gamma^* \in \Gamma^* \setminus \{0\}} (\mathrm{Nm}\,\gamma^*)^{-l} \Omega(P^{-1} \cdot \gamma^*) M(\gamma^*) \exp(2\pi i \langle \gamma^*, X \rangle). \tag{6.7}$$

Comparing (6.6) and (6.7) we find that

$$\mathfrak{M}_Q W^l(\Gamma, P, X) = \mathfrak{M}_0 W^l(2^Q \cdot \Gamma, 2^{-Q} \cdot P, 2^Q \cdot X). \tag{6.8}$$

Since the latticies Γ and $2^Q \cdot \Gamma$ have the same invariants $\det\Gamma$ and $\mathrm{Nm}\,\Gamma$, we see from (6.8) that it is sufficient to prove Lemma 6.2 only for $Q = 0$.

Let us prove bound (6.5) for $Q = 0$. We wish to apply the Poisson summation formula (5.4) to the series (6.7). However, we can not do this directly because of the singularity of the factor $(\mathrm{Nm}\,X)^{-l}$ at $x_1 = 0$. In order to overcome this difficulty, we shall replace the singular function in the sum (6.7) by a smooth one with the same values at the points $\gamma^* \in \Gamma^*$. To be definite, we suppose, in accordance with the example (5.16), that the function $m(t)$ in (5.3)-(5.15) satisfies the conditions: $m(t) = 0$ for $|t| \leqslant 1$ and for $|t| \geqslant 4$.

Let $\eta(t) = \eta(|t|)$, $t \in \mathbb{R}^1$ be an even function of the class C^∞; moreover, let $\eta(t) = 0$ for $0 \leqslant |t| \leqslant 1$ and $\eta(t) = 1$ for $|t| \geqslant 2$. We extend this function on $X = (x_1, x) \in \mathbb{R}^s$ by setting $\eta(X) = \eta(x_1)$, i.e., $\eta(X)$ does not depend on $x \in \mathbb{R}^{s-1}$. It is easy to check the relation

$$M(X)\eta(aX) = M(X), \quad \text{where } a = 2^{2s-1}(\mathrm{Nm}\,\Gamma)^{-1}, \tag{6.9}$$

for all $X \in \mathbb{R}^s$ with $|\mathrm{Nm}\,X| \geqslant \mathrm{Nm}\,\Gamma$. Indeed, the parameter a in (6.9) is chosen so that either $\eta(aX) = 1$ or $M(X) = 0$ for all X in the indeicated region.

Taking (6.9) into account, we represent the series (6.1) in the following form:

$$\mathfrak{M}_0 W^l(\Gamma, P, X) = \sum_{\gamma^* \in \Gamma^*} \widetilde{W}^l(P, \gamma^*) \exp(2\pi i \langle \gamma^*, X \rangle), \tag{6.10}$$

where $\widetilde{W}^l(P, Y)$ is a smooth function given by the formula

$$\begin{aligned}\widetilde{W}^l(P, Y) &= (\mathrm{Nm}\,Y)^{-l}\eta(aY)\Omega(P^{-1} \cdot Y)M(Y) \\ &= \frac{\omega(p_1^{-1}y_1)\eta(ay_1)}{y_1^l} \prod_{j=2}^{s} \frac{\omega(p_j^{-1}y_j)m(y_j)}{y_j^l}.\end{aligned} \tag{6.11}$$

Now we can apply the Poisson summation formula (5.4) to the series (6.10). As a result, we get

$$\mathfrak{M}_0 W^l(\Gamma, P, X) = \sum_{\gamma \in \Gamma} W^l(P, \gamma - X), \tag{6.12}$$

where the functions $\widetilde{W}^l(P, Y)$ in (6.10) and $W^l(P, X)$ in (6.12) are related by the Fourier transform (5.5). Using (6.11) we find that

$$W^l(P, X) = \prod_{j=1}^{s} w_j^l(|p_j|^{-1}, x_j), \tag{6.13}$$

where co-factors can be described as follows:

If $j = 1$ and $l = 1$, then

$$w_1^1(\tau, \xi) = \int_{-\infty}^{\infty} dt e^{-2\pi i \xi t} \frac{1}{t} \eta(at)\omega(\tau t) = I(a\tau, a^{-1}\xi). \tag{6.14}$$

Note that here we have used formula (4.9).

If $j = 1$ and $l > 1$, then

$$w_1^l(\tau, \xi) = \int_{-\infty}^{\infty} dt e^{-2\pi i \xi t} \frac{1}{t^l} \eta(at)\omega(\tau t) = a^{l-1} J_{f_1}(a\tau, a^{-1}\xi). \tag{6.15}$$

Here we have used formula (4.8) with $f_1(t) = t^{-l}\eta(t)$.

If $j = 2, \ldots, s$ and $l \geqslant 1$, then

$$w_j^l(\tau, \xi) = \int_{-\infty}^{\infty} dt e^{-2\pi i \xi t} \frac{1}{t^l} m(t)\omega(\tau t) = J_{f_2}(\tau, \xi). \tag{6.16}$$

Here we have used formula (4.8) with $f_2(t) = t^{-l}m(t)$.

Let us estimate the co-factors (6.14)-(6.16) by means of Lemma 4.2. As a result, for all $j = 1, \ldots s$ and $l \geqslant 1$ we obtain the bounds

$$|w_j^l(\tau, \xi)| < c_{\alpha,\beta}^l (1 + \tau)^{-\alpha} (1 + |\xi|)^{-\beta} \tag{6.17}$$

with any $\alpha > 0$ and $\beta > 0$. If here we choose $\alpha = \beta = 2s$ and substitute (6.17) in the product (6.13), then we get

$$|W^l(P, X)| < c^l H(P^{-1}) H(X), \tag{6.18}$$

where $H(\cdot)$ is given by (3.7). From (6.18) we obtain the following bound for the series (6.12):

$$\sup_X |\mathfrak{M}_0 W^l(\Gamma, P, X)| \leqslant c^l H(P^{-1}) \sup_X H(\Gamma, X), \tag{6.19}$$

where $H(\Gamma, X)$ is given by (3.10). Now bound (6.5) for $Q = 0$ follows from (6.19) with the help of Lemma 3.2. This completes the proof of Lemma 6.2.

From Lemma 6.2 we can derive very sharp bounds for the norms of the sum (6.1). We set

$$V^l(\Gamma, P) = \sup_{X \in \mathcal{F}(\Gamma)} |W^l(\Gamma, P, X)|, \tag{6.20}$$

$$V_q^l(\Gamma, P) = \left[(\det \Gamma)^{-1} \int_{\mathcal{F}(\Gamma)} |W^l(\Gamma, P, X)|^q \, dX \right]^{\frac{1}{q}}, \tag{6.21}$$

where $q > 0$ is a real number.

Lemma 6.3. *Let $\Gamma \subset \mathbb{R}^s$ be an admissible lattice. Then for all $P \in \mathbb{E}^s$, we have the bounds*

$$V^l(\Gamma, P) < c^l(\Gamma)[\ln(2 + |\operatorname{Nm} P|)]^{s-1}, \tag{6.22}$$

$$V_q^l(\Gamma, P) < c_q^l(\Gamma)[\ln(2 + |\operatorname{Nm} P|)]^{\frac{s-1}{2}}. \tag{6.23}$$

The constants in (6.22) *and* (6.23) *depend upon the lattice* Γ *only by means of the invariants* $\det \Gamma$ *and* $\operatorname{Nm} \Gamma$.

Proof. Let us prove bound (6.22). Assume that the functions (5.9), (5.10) satisfy relation (5.12) and let $\mathfrak{M}_Q$, $Q \in L$ be the corresponding multipliers (5.11). Using Lemmas 3.4, 5.1, 6.1, and 6.2 we obtain the following inequalities for the norm (6.20):

$$\begin{aligned} V^l(\Gamma, P) &\leqslant \sum_{Q \in L} \sup_X |\mathfrak{M}_Q W^l(\Gamma, P, X)| \leqslant c^l(\Gamma) \sum_{Q \in L} H(P^{-1} \cdot 2^{-Q}) \\ &\leqslant C^l(\Gamma)[\ln(2 + |\operatorname{Nm} P|)]^{s-1}, \end{aligned}$$

where the constants depend upon the lattice Γ only by means of the invariants $\det \Gamma$ and $\operatorname{Nm} \Gamma$. This proves bound (6.22).

Let us prove bound (6.23). Assume that the functions (5.9), (5.10) satisfy relation (5.14) and let $\mathfrak{M}_Q$, $Q \in L$, be the corresponding multipliers (5.11). We evaluate the function (5.17) for $\varphi(X) = W^l(\Gamma, P, X)$. Using Lemmas 3.4, 6.2, and the second bound in (3.8), we obtain the following inequalities:

$$\begin{aligned} \Phi[W^l(\Gamma, P, X)] &= \left(\sum_{Q \in L} |\mathfrak{M}_Q W^l(\Gamma, P, X)|^2 \right)^{\frac{1}{2}} \\ &\leqslant \left(c^l(\Gamma) \sum_{Q \in L} H^2(P^{-1} \cdot 2^{-Q}) \right)^{\frac{1}{2}} \leqslant \left(c^l(\Gamma) \sum_{Q \in L} H(P^{-1} \cdot 2^{-Q}) \right)^{\frac{1}{2}} \\ &\leqslant C^l(\Gamma)[\ln(2 + |\operatorname{Nm} P|)]^{\frac{s-1}{2}}, \end{aligned} \tag{6.24}$$

where the constants depend upon the lattice Γ only by means of the invariants $\det\Gamma$ and $\operatorname{Nm}\Gamma$.

If $q > 1$, then bound (6.23) follows from (6.24) with the help of Lemma 5.2. It remains to note that the quantity $\|\cdot\|_q$ is a nondecreasing function of $q > 0$. This proves bound (6.23) for all $q > 0$.

§7. Proof of Theorem 1.1

First we give general bounds for the number of lattice points inside an arbitrary compact body. We recall the following definition (cf. [5]): given a compact body $\mathcal{O}$ and a number $\tau > 0$, we say that the compact bodies $\mathcal{O}_\tau^\pm$ are a τ-coapproximation of $\mathcal{O}$ if $\mathcal{O}_\tau^- \subset \mathcal{O} \subset \mathcal{O}_\tau^+$ and the points of the boundaries $\partial\mathcal{O}_\tau^\pm$ are at a distance $\geqslant \tau$ from the boundary $\partial\mathcal{O}$.

Let $\chi(\mathcal{E}) = \chi(\mathcal{E}, X)$ denote the characteristic function of a set $\mathcal{E} \subset \mathbb{R}^s$ and let $\widetilde{\chi}(\mathcal{E}) = \widetilde{\chi}(\mathcal{E}, Y)$ denote the Fourier transform (5.5) of $\chi(\mathcal{E})$. If $\mathcal{E}$ is the empty set, then we take $\chi(\mathcal{E}) = \widetilde{\chi}(\mathcal{E}) = 0$. Fix a nonnegative function $G(X)$, $X \in \mathbb{R}^s$, of the class C^∞, with a support inside the ball $|X| \leqslant 1$ and assume that $\int_{\mathbb{R}^s} G(X)dx = 1$. We set $G_\tau(X) = \tau^{-s}G(\tau^{-1}X), \tau > 0$. Taking into account that the Fourier transform $\widetilde{G}_\tau(Y) = \widetilde{G}(\tau Y)$, we obtain the estimate

$$|\widetilde{G}_\tau(Y)| < c_\alpha(1 + \tau|Y|)^{-\alpha} \tag{7.1}$$

for any $\alpha > 0$.

We consider the convolutions of the functions $\chi(\mathcal{O}_\tau^\pm, X)$ and $G_\tau(X)$:

$$(G_\tau * \chi(\mathcal{O}_\tau^\pm))(X) = \int_{\mathbb{R}^s} dY G_\tau(X - Y)\chi(\mathcal{O}_\tau^\pm, Y). \tag{7.2}$$

It is obvious that the nonnegative functions (7.2) are of the class C^∞ and are compactly supported in τ-neighborhoods of the bodies $\mathcal{O}_\tau^\pm$, respectively. From the definition of τ-coapproximation we obtain the inequalities

$$(G_\tau * \chi(\mathcal{O}_\tau^-))(X) \leqslant \chi(\mathcal{O}, X) \leqslant (G_\tau * \chi(\mathcal{O}_\tau^+))(X). \tag{7.3}$$

Replacing X by $\gamma - X$ in (7.3) and summing these inequalities over $\gamma \in \Gamma$, we find that

$$N_\tau^-(X) \leqslant N(\mathcal{O} + X, \Gamma) \leqslant N_\tau^+(X), \tag{7.4}$$

where

$$N_\tau^\pm(X) = \sum_{\gamma\in\Gamma}(G_\tau * \chi(\mathcal{O}_\tau^\pm))(\gamma - X). \tag{7.5}$$

Here we have used formula (1.5).

We can apply the Poisson summation formula (5.4) to the series (7.5) (unlike the series (1.5) for $N(\mathcal{O} + X, \Gamma)$), which gives

$$N_\tau^\pm(X) = \frac{\operatorname{vol}\mathcal{O}_\tau^\pm}{\det\Gamma} + R_\tau^\pm(X), \tag{7.6}$$

where

$$R_\tau^\pm(X) = (\det\Gamma)^{-1} \sum_{\gamma^*\in\Gamma^*\setminus\{0\}} \widetilde{\chi}(\mathcal{O}_\tau^\pm, \gamma^*)\widetilde{G}_\tau(\gamma^*)\exp(2\pi i\langle\gamma^*, X\rangle). \tag{7.7}$$

Note that estimate (7.1) with $\alpha > s$ ensures the convergence of the series (7.7) over $\gamma^* \in \Gamma^* \setminus \{0\}$.

Comparing relations (7.4)-(7.7) with the definitions (1.7), (1.9), (1.10) of the remainder terms $R(\mathcal{O}+X,\Gamma)$, $r(\mathcal{O},\Gamma)$, $r_q(\mathcal{O},\Gamma)$, we obtain (cf. [5, Lemma 3.3]) the following.

Lemma 7.1. *Let a compact body $\mathcal{O}\subset\mathbb{R}^s$ and a lattice $\Gamma\subset\mathbb{R}^s$ be given. Then for any τ-coapproximation $\mathcal{O}_\tau^\pm$ of $\mathcal{O}$ we have the bounds*

$$|R(\mathcal{O}+X,\Gamma)|\leqslant\frac{\operatorname{vol}\mathcal{O}_\tau^+-\operatorname{vol}\mathcal{O}_\tau^-}{\det\Gamma}+|R_\tau^-(X)|+|R_\tau^+(X)|, \tag{7.8}$$

$$r(\mathcal{O},\Gamma)\leqslant\frac{\operatorname{vol}\mathcal{O}_\tau^+-\operatorname{vol}\mathcal{O}_\tau^-}{\det\Gamma}+\sup_{X\in\mathcal{F}(\Gamma)}(|R_\tau^-(X)|+|R_\tau^+(X)|), \tag{7.9}$$

$$r_q(\mathcal{O},\Gamma)\leqslant\frac{\operatorname{vol}\mathcal{O}_\tau^+-\operatorname{vol}\mathcal{O}_\tau^-}{\det\Gamma}+\|R_\tau^-(\cdot)\|_q+\|R_\tau^+(\cdot)\|_q, \tag{7.10}$$

where $\|\cdot\|_q$ is the norm (5.1).

Now we can prove Theorem 1.1. Let $\Gamma\subset\mathbb{R}^s$ be an admissible lattice. If $\operatorname{Nm}T=0$, then the parallelepiped $T\cdot\mathbb{K}^s+X$ has a dimension less than s and is entirely contained in a hyperplane given by the equation $x_j=a$. By assertion 1 of Lemma 3.1, we have $0\leqslant N(T\cdot\mathbb{K}^s+X,\Gamma)\leqslant1$. Thus Theorem 1.1 holds for T with $\operatorname{Nm}T=0$.

If $T\in\mathbb{E}^s$, then without loss of generality we can assume that $T=t\mathbb{I}=(t,\dots,t)$, $t>0$. Indeed, for any $T\in\mathbb{E}^s$ we have $T=t\mathbb{I}\cdot U$ where $t>0$ and $U\in\operatorname{Um}$ (cf. (1.2), (1.3)). From (1.6)-(1.8) we obtain the following relations for the norms (1.9) and (1.10):

$$r(T\cdot\mathbb{K}^s,\Gamma)=r(t\mathbb{K}^s,U^{-1}\cdot\Gamma),\qquad r_q(T\cdot\mathbb{K}^s,\Gamma)=r_q(t\mathbb{K}^s,U^{-1}\cdot\Gamma).$$

It remains to note that the latticies Γ and $U^{-1}\cdot\Gamma$, $U\in\operatorname{Um}$, have the same invariants $\det\Gamma$ and $\operatorname{Nm}\Gamma$.

If $T=t\mathbb{I}$ and $0\leqslant t\leqslant10$, then Theorem 1.1 holds by assertion 2 of Lemma 3.2. Thus, in proving we shall assume that $T=t\mathbb{I}$ and $t>10$.

We make use of Lemma 7.1, taking

$$\mathcal{O}=t\mathbb{K}^s,\qquad \mathcal{O}_\tau^\pm=(t\pm\tau)\mathbb{K}^s,\qquad \tau=t^{-101s}. \tag{7.11}$$

From (7.11) we obtain

$$0<\operatorname{vol}\mathcal{O}_\tau^+-\operatorname{vol}\mathcal{O}_\tau^-=(t+\tau)^s-(t-\tau)^s<C_st^{s-1}\tau<C_st^{-100s}. \tag{7.12}$$

The Fourier transform (5.5) of the characteristic function $\chi(t\mathbb{K}^s,X)$ can be easily computed:

$$\widetilde{\chi}(t\mathbb{R}^s,Y)=\prod_{j=1}^{s}\frac{e^{i\pi ty_j}-e^{-i\pi ty_j}}{2\pi iy_j}. \tag{7.13}$$

Further, we assume that the function $G_\tau(X)$ in (7.2) is given by the formula

$$G_\tau(X)=\tau^{-s}\prod_{j=1}^{s}g(\tau^{-1}x_j), \tag{7.14}$$

where $g(t)=g(|t|)$, $t\in\mathbb{R}^1$, is a nonnegative even function of class C^∞ with a support inside the segment $\left[-\frac12;\frac12\right]$ and satisfying the condition $\int_{-\infty}^{\infty}g(t)dt=1$.

If we substitute (7.13) and (7.14) in (7.7), we can express the series (7.7) in terms of the series (6.1) with $l=1$, $P=\tau^{-1}\mathbb{I}=(\tau^{-1},\dots,\tau^{-1})$, $\Omega(X)=\widetilde{G}(X)$. We find that

$$\det\Gamma\quad R_\tau^\pm(X)=\left(\frac{1}{2\pi i}\right)^s\sum_{j=1}^{2^s}\pm W^1(\Gamma,\tau^{-1}\mathbb{I},X+Z_j), \tag{7.15}$$

where 2^s vectors $Z_j \in \mathbb{R}^s$ are independent of X. An explicit description of vectors Z_j and an arangement of signs in (7.15) is not needed for us.

Using the norms (6.20) and (6.21) we derive from formula (7.15) the following bounds:

$$\det\Gamma \sup_X |R_\tau^\pm(X)| \leqslant \pi^{-s} V^1(\Gamma, \tau^{-1}\mathrm{II}), \tag{7.16}$$

$$\det\Gamma \, \|R_\tau^\pm(\cdot)\|_q \leqslant \pi^{-s} V_q^1(\Gamma, \tau^{-1}\mathrm{II}) \tag{7.17}$$

where $\|\cdot\|_q$ is the norm (5.3).

Substituting (7.12), (7.16), and (7.17) in (7.9) and (7.10), we obtain the bounds

$$\det\Gamma\, r(t\mathbb{K}^s, \Gamma) < c_s t^{-100s} + \pi^{-s} V^1(\Gamma, t^{100s}\mathrm{II}),$$

$$\det\Gamma\, r_q(t\mathbb{K}^s, \Gamma) < c_s t^{-100s} + \pi^{-s} V_q^1(\Gamma, t^{100s}\mathrm{II}).$$

Now the reference to Lemma 6.3 completes the proof of Theorem 1.1.

§8. Proof of Theorem 2.1

Let Γ be an admissible lattice and let $f \in \overset{0}{V}{}_q^l(\mathbb{K}^s)$ be normalized by the condition $\|f\|_{l,q} = 1$. For $l = q = 1$, Theorem 2.1 follows directly from Corollary 2.1 and from assertion 4 of Lemma 4.1. Thus, we can assume further that $lq > 1$.

Without loss of generality, we can assume also that $T = t\mathrm{II} = (t, \dots, t)$, $t > 0$. Indeed, for any $T \in \mathbb{E}^s$ we have $T = t\mathrm{II} \cdot U$, where $t > 0$ and $U \in \mathrm{Um}$ (cf. (1.2), (1.3)). Therefore $\delta(f, T^{-1}\cdot\Gamma) = \delta(f, t^{-1}U^{-1}\cdot\Gamma)$. It remains to note that the lattices Γ and $U^{-1}\cdot\Gamma$, $U \in \mathrm{Um}$, have the same invariants $\det\Gamma$ and $\mathrm{Nm}\,\Gamma$. Thus, in proving we shall assume that $T = t\mathrm{II}$, $t \to \infty$.

Let $G_\tau(X)$, $X \in \mathbb{R}^s$, $\tau > 0$, be the function (7.14). We consider the convolution

$$f_\tau(X) = (G_\tau * f)(X) = \int_{\mathbb{R}^s} G_\tau(X - Y) f(Y)\, dY. \tag{8.1}$$

It is obvious that the function (8.1) is of the class C^∞ and is compactly supported in the cube $\mathbb{K}_\tau^s = (1 + \tau)\mathbb{K}^s$; moreover, we have

$$\int_{\mathbb{K}_\tau^s} f_\tau(X)\, dX = \int_{\mathbb{K}^s} f(X)\, dX. \tag{8.2}$$

Let us consider the following quadrature formula for the function (8.1):

$$\int_{\mathbb{K}_\tau^s} f_\tau(X)\, dX = \det\Gamma \sum_{\gamma\in\Gamma} f_\tau(\gamma) + \delta(f_\tau, \Gamma). \tag{8.3}$$

Comparing (8.3) with (2.14) and taking (8.2) into account, we obtain the formula

$$\delta(f, \Gamma) = \delta(f_\tau, \Gamma) + D_\tau(f, \Gamma), \tag{8.4}$$

where

$$D_\tau(f,\Gamma) = -\det\Gamma \sum_{\gamma\in\Gamma} [f(\gamma) - f_\tau(\gamma)] = -\det\Gamma \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^s} G_\tau(\gamma - Y)[f(\gamma) - f(Y)]\, dY. \tag{8.5}$$

Let us estimate the sum (8.5). Since the function $G_\tau(X)$ is concentrated in the ball $|X| \leqslant \tau$, from assertions 2 and 3 of Lemma 4.1 we derive the following bound:

$$\int_{\mathbb{R}^s} G_\tau(\gamma - Y)|f(\gamma) - f(Y)|\, dY \leqslant c_{l,q} \int_{\mathbb{R}^s} G_\tau(\gamma - Y)|\gamma - Y|^B\, dY \leqslant c_{l,q}\tau^B, \qquad (8.6)$$

where $B = 1$ for $l > 1$, $q \geqslant 1$ and $B = \frac{q-1}{q}$ for $l = 1$, $q > 1$. Let us substitute bound (8.6) in (8.5) and note that the sum (8.5) is extended over $\gamma \in \Gamma \cap \mathbb{R}^s_\tau$. As a result, we find

$$|D_\tau(f, \Gamma)| \leqslant c_{l,q}\tau^B \det\Gamma\, N(\mathbb{K}^s_\tau, \Gamma) \leqslant c_{l,q}\tau^B[(1+\tau)^s + \det\Gamma\, R(\mathbb{K}^s_\tau, \Gamma)]. \qquad (8.7)$$

Here we have used formulas (1.5) and (1.7).

Further, we assume that t and τ are related as follows:

$$r = t^{-\frac{100ls}{B}}. \qquad (8.8)$$

Now in (8.7) we replace the lattice Γ by $t^{-1}\Gamma$, $t \to \infty$ Using (1.8), (8.8), and bound (1.11) in Theorem 1.1, we can continue estimates (8.7) as follows:

$$|D_\tau(f, t^{-1}\Gamma)| \leqslant c_{l,q}t^{-100ls}\left[1 + c_s(\Gamma)\frac{\ln^{s-1} t}{t^s}\right] \leqslant C_{l,q}(\Gamma)t^{-100ls}. \qquad (8.9)$$

The constants in (8.9) depend upon the lattice Γ only by means of the invariants $\det\Gamma$ and $\operatorname{Nm}\Gamma$.

We wish to express the error $\delta(f_\tau, \Gamma)$ in terms of the series (6.1). For this purpose we apply the Poisson summation formula (5.4) to the relation (8.3). Taking the definitions (2.12) and (8.1) into account, we obtain successively

$$\begin{aligned}\delta(f_\tau, \Gamma) &= -\sum_{\gamma^* \in \Gamma^* \setminus \{0\}} \widetilde{f}_\tau(\gamma^*) = -\sum_{\gamma^* \in \Gamma^* \setminus \{0\}} \widetilde{G}(\tau\gamma^*)\widetilde{f}(\gamma^*) \\ &= -\left(\frac{i}{2\pi}\right)^{ls} \sum_{\gamma^* \in \Gamma^* \setminus \{0\}} (\operatorname{Nm}\gamma^*)^{-l}\widetilde{G}(\tau\gamma^*) \int_{\mathbb{R}^s} D^l f(X) \exp(2\pi i\langle\gamma^*, X\rangle)\, dX \\ &= -\left(\frac{i}{2\pi}\right)^{ls} \int_{\mathbb{R}^s} dX D^l f(X)\left[\sum_{\gamma^* \in \Gamma^* \setminus \{0\}} (\operatorname{Nm}\gamma^*)^{-l}\widetilde{G}(\tau\gamma^*)\exp(2\pi i\langle\gamma^*, X\rangle)\right]. \end{aligned} \qquad (8.10)$$

Now in (8.10) we replace the lalttice Γ by $t^{-1}\Gamma$ and, respectively, the dual lattic Γ^* by $t\Gamma^*$. Using the definition of the seeirs (6.1) with $\Omega(X) = \widetilde{G}(X)$, we derive from (8.10) the following integral representation for the error:

$$\delta(f_\tau, t^{-1}\Gamma) = -\left(\frac{i}{2\pi t}\right)^{ls} \int_{\mathbb{R}^s} dX W^l(\Gamma, P_0, tX) D^l f(X), \qquad (8.11)$$

where

$$P_0 = (\tau t)^{-1}\mathrm{I\!I}, \qquad \operatorname{Nm} P_0 = (\tau t)^{-s} = t^b, \qquad b = s\left(\frac{100ls}{B} - 1\right) > 0. \qquad (8.12)$$

Let us estimate the integral (8.11) for $q = 1$. Using the definition (6.20), bound (6.22) and condition (8.12), we find

$$\begin{aligned}|\delta(f_\tau, t^{-1}\Gamma)| &\leqslant \left(\frac{1}{2\pi t}\right)^{ls} \int_{\mathbb{K}^s} |D^l f(X)| dX \sup_X |W^l(\Gamma,, P_0, X)| = \left(\frac{1}{2\pi t}\right)^{ls} V^l(\Gamma, P_0) \\ &\leqslant c_l(\Gamma) t^{-ls} \ln^{s-1} t. \end{aligned} \qquad (8.13)$$

The constant in (8.13) depends upon the lattice Γ only by means of the invariants $\det\Gamma$ and $\operatorname{Nm}\Gamma$. We recall that $\|f\|_{l,1} = 1$.

Now we can prove bound (2.15) in Theorem 2.1. Substituting bounds (8.13) and (8.9) in formula (8.4), we obtain

$$|\delta(f, t^{-1}\Gamma)| \leqslant c_{l,1}(\Gamma)t^{-100ls} + c_l(\Gamma)t^{-ls}\ln^{s-1} t \leqslant c_l(\Gamma)t^{-ls}\ln^{s-1} t.$$

This proves bound (2.15).

Here we postpone our consideration in order to state a result from geometry of numbers. Certainly, it would be better to give this result earlier, say in Section 3. However, its proof contains a reference to Theorem 1.1, which was proved in Section 7.

Let a compact body $\mathcal{O}$ and a lattice Γ be given in $\mathbb{R}^s$. We define $\nu(\mathcal{O}, \Gamma)$ to be the minimal number of fundamental sets of the lattice Γ needed to cover entirely the body $\mathcal{O}$.

Lemma 8.1. *If $\Gamma \subset \mathbb{R}^s$ is an admissible lattice, then we have the bound*

$$\nu(t\mathbb{K}^s, \Gamma) < c(\Gamma)(1+t)^s, \quad t > 0, \tag{8.14}$$

where the constant depends upon the lattice Γ only by means of the invariants $\det\Gamma$ and $\operatorname{Nm}\Gamma$.

Proof. Let $\mathcal{F}(\Gamma)$ be the fundamental set indicated in assertion 3 of Lemma 3.1. The translations $\mathcal{F}(\Gamma)+\gamma$, $\gamma \in \Gamma$, cover the whole space $\mathbb{R}^s$. Consider the translations $\mathcal{F}(\Gamma)+\gamma_j$, $j = 1, \dots, \nu_t$, which cover entirely the cube $t\mathbb{K}^s$. All of them are contained in the interior of the cube $(t + r_0)\mathbb{K}^s$, where $r_0 = r_0(\Gamma)$ is the radius indicated in assertion 3 of Lemma 3.1. On the other hand, each fundamental set $\mathcal{F}(\Gamma) + \gamma$ contains a single point of the lattice Γ. As a result, we have

$$\nu(t\mathbb{K}^s, \Gamma) \leqslant \nu_t \leqslant N((t + r_0)\mathbb{K}^s, \Gamma) = \frac{(t + r_0)^s}{\det\Gamma} + R((t + r_0)\mathbb{K}^s, \Gamma).$$

It remains to refer to Theorem 1.1. to complete the proof of Lemma 8.1.

Returning to our consideration, let us estimate the integral (8.11) for $q > 1$. Using Hölder's inequality, we obtain

$$\begin{aligned}|\delta(f_\tau, t^{-1}\Gamma)| &\leqslant \left(\frac{1}{2\pi t}\right)^{ls}\left[\int_{\mathbb{K}^s} |D^l f(X)|^q\,dX\right]^{\frac{1}{q}}\left[\int_{\mathbb{K}^s} |W^l(\Gamma, P_0, tX)|^k\,dX\right]^{\frac{1}{k}} \\ &= \left(\frac{1}{2\pi t}\right)^{ls}\left[t^{-s}\int_{t\mathbb{K}^s} W^l(\Gamma, P_0, X)|^k\,dX\right]^{\frac{1}{k}},\end{aligned} \tag{8.15}$$

where $k = \frac{q-1}{q}$ is the conjugate exponent. We recall that $\|f\|_{l,q} = 1$.

To evaluate the integral in (8.15), we cover entirely the cube $t\mathbb{K}^s$ by the minimal number $\nu(t\mathbb{K}^s, \Gamma)$ of fundamental sets of the lattice Γ, and we estimate the integral in (8.15) by the sum of integrals over these fundamental sets. Using the definition of the norm (6.21) we obtain

$$\left[t^{-s}\int_{t\mathbb{K}^s} |W^l(\Gamma, P_0, X)|^k dX\right]^{\frac{1}{k}} \leqslant (\det\Gamma)^{\frac{1}{k}}\left[\frac{\nu(t\mathbb{K}^s, \Gamma)}{t^s}\right]^{\frac{1}{k}} V_k^l(\Gamma, P_0). \tag{8.16}$$

If we take bound (6.23) in Lemma 6.3, bound (8.14) in Lemma 8.1, and condition (8.12) into account, we derive from (8.15) and (8.16) the bound

$$|\delta(f_\tau, t^{-1}\Gamma)| < c_{l,q}(\Gamma)t^{-ls}\ln^{\frac{s-1}{2}} t. \tag{8.17}$$

The constant in (8.17) depends upon the latice Γ only by means of the invariants $\det\Gamma$ and $\mathrm{Nm}\,\Gamma$.

Now we can prove bound (2.16) in Theorem 2.1. Substituting bounds (8.17) and (8.9) in (8.4) we obtain

$$|\delta(f,t^{-1}\Gamma)| \leqslant c'_{l,q}(\Gamma)t^{-100ls} + c''_{l,q}(\Gamma)t^{-ls}\ln^{\frac{s-1}{2}} t \leqslant C_{l,q}(\Gamma)t^{-ls}\ln^{\frac{s-1}{2}} t.$$

This proves bound (2.16).

The proof of Theorem 2.1 is completed.

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