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CREATING TRANSVERSE HOMOCLINIC CONNECTIONS IN PLANAR BILLIARDS

ABSTRACT. Given a planar billiard system containing stable and unstable manifolds that intersect non-transversely, we show how to make a local perturbation to the boundary that causes the intersection to become transverse. We apply these ideas to billiards inside an ellipse.

Introduction.

The dynamical system generated by billiard motion is an example of a conservative Hamiltonian system. In the case that the boundary of the billiard region is an ellipse, the system is completely integrable [16]. The Birkhoff Conjecture asserts that the ellipse is the only smooth, strictly convex region for which the billiard system is integrable. Various partial results in support of the conjecture have been obtained by, among others, Bialy [1], Wojtkowski [19], and Innami [10] but at present the complete conjecture has not been proven.

One standard way to show that a system is not integrable is to show the existence of a transverse homoclinic (or heteroclinic) crossing of stable and unstable manifolds which prevents integrability. Such a crossing also generates chaotic dynamics: the crossing produces a smale horseshoe on which the dynamics have positive topological entropy. Typically the set of points lying in the horseshoe have zero measure and hence the chaotic dynamics occurs on a small set.

One can examine a local version of the Birkhoff Conjecture and see whether there are closed curves close to the ellipse that generate integrable billiard systems. M. B. Tabanov [17], A. Delshams and R. Ramírez-Ros [3, 4], and H. E. Lomeli [13] showed that various analytic perturbations of the ellipse are non-integrable. G. Kim [11] showed that a generic, global C^2 perturbation of the ellipse is nonintegrable.

In this note, we make a local perturbation to the ellipse and use elementary methods to prove:

Theorem 1. *There exist C^∞-smooth closed curves that are C^2 close to the ellipse whose billiard systems have a transverse intersection of stable* *and unstable manifolds and hence have positive topological entropy and are nonintegrable.*

The idea of our construction is as follows. We start with the integrable billiards inside an ellipse which has a heteroclinic connection. We make an explicit perturbation to a small interval of the boundary and show that under this perturbation, the heteroclinic connection shifts to a transverse heteroclinic crossing. Because we only perturb a small segment of the boundary, much of the motion in the perturbed system is identical with the motion in the original ellipse which simplifies the analysis of the stable and unstable manifolds. This construction is a discrete analogue of the method we used in [6] to make homoclinic connections transverse for geodesic flows on surfaces.

A straightforward generalization of our proof gives the following result:

Theorem 2. *Given a planar billiard system containing stable and unstable manifolds that intersect nontransversely, there exist C^2 small perturbations of the boundary that will make the intersection transverse.*

Our results might be useful in proving the following conjecture:

Conjecture 3. *Generically, billiards in smooth strictly convex regions are nonintegrable and have positive topological entropy.*

M. Dvorin and V. F. Lazutkin [7] showed that generically billiards in a smooth, strictly convex region have infinitely many hyperbolic periodic orbits and these periodic orbit accumulate near the boundary. One could analyze the stable and unstable manifolds of these hyperbolic periodic points and try to show that they intersect. If the manifolds intersect transversely, then the billiard is nonintegrable and has positive topological entropy. If the manifolds intersect nontransversely, then one can make a small perturbation to the region, using the approach described here, and force the intersection to become transverse. The important results on exponentially small splitting of separatrices initiated by V. F. Lazutkin, and then further developed by his collaborators and others (see, for example, [8, 9], should provide a means of studying these stable and unstable manifolds.

We mention that Poincare in his work on the three-body problem, first noted the existence of transverse heteroclinic connections and the very complicated dynamics that they produce. Our example provides an elementary and geometrically appealing example of such phenomena.

There are often similarities between results for billiards and results for geodesic flow on surfaces. The geodesic flow on an ellipsoid is integrable and, in the case that the lengths of the three principal axes are not equal, has a pair of hyperbolic periodic orbits whose stable and unstable manifolds form a nontransverse heteroclinic connection. G. Kneiper and H. Weiss [12], V. Donnay [6], D. Petroll [15], and G. Paternain [14] showed that under various types of perturbations to the ellipsoid, the stable and unstable manifolds intersect transversely. Petroll showed how, given a nontransverse intersection of unstable and unstable manifolds for a geodesic flow on a manifold of arbitrary dimension, to make a perturbation that causes the intersection to become transverse. His result suggests the possibility of generalizing Theorem 2 from planar billiards to billiards of arbitrary dimension.

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1. Background.

We outline the basic definitions of billiards ([2, 18]).

Let Q be a connected domain in the plane with piecewise smooth boundary. By a billiard in Q we mean the dynamical system arising from the uniform motion of a point mass inside Q with elastic reflections at the boundary: the angle of reflection equals the angle of incidence. This motion produces a flow Ψ^t in the space T_1Q of unit tangent vectors of Q with the obvious identifications.

Let $\Phi : M \to M$ be the standard section map, where $M \subset T_1Q$ is the two dimensional manifold consisting of unit vectors attached at the boundary ∂Q and pointing inside Q. For $z \in M$, the point Φz is gotten by following the point z under the billiard flow Ψ^t until its next collision with the boundary.

On M, we introduce coordinates (s, θ) where s is the arc length parameter along ∂Q and $\theta \in [0, \pi]$ is the angle which the unit vector makes with ∂Q (Fig. 1). We denote by $\{X_s, X_\theta\}$ the associated tangent vector; they form a basis for the tangent space TM. Let $k(s)$ denote the curvature of the boundary as a function of arc-length. Φ preserves the measure $\mu = c \sin\theta \, ds \, d\theta$, where c is a normalizing constant.

To a tangent vector $\xi \in T_zM$, $z = (s, \theta)$, we associate the infinitesimal one-parameter family of trajectories $\sigma_\xi(s)$, $s \in (-\epsilon, \epsilon)$ that generates it:

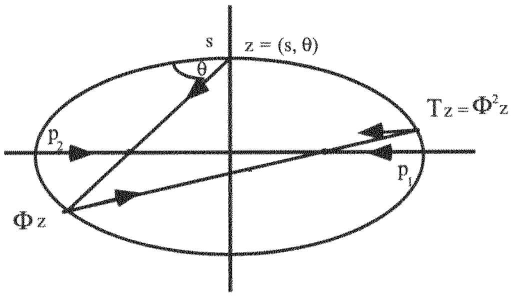

Fig. 1. Billiard coordinates and map.

i.e., $\sigma_\xi(0) = z$, $\sigma'_\xi(0) = \xi$. We say the vector ξ focuses if the associated one parameter family $\sigma_\xi(s)$ focuses in linear approximation under the billiard flow Ψ^t.

Lemma 4 [18, 5]. *If the vector* $\xi = s'X_s + \theta' X_\theta \in T_z M$ *satisfies* $\theta'/s' \in (-k(s), +\infty)$ *then the associated infinitesimal family* $\sigma_\xi(s)$ *will focus at the point* $\Psi^{t^+} z$ *where*

$$t^+ = \frac{\sin\theta}{\frac{\theta'}{s'} + k(s)}. \tag{1a}$$

If $\theta'/s' \in (-\infty, k(s))$ *then, moving backwards under the flow, the associated infinitesimal family will focus at the point* $\Psi^{t^-} z$ *where* $t^- < 0$ *and*

$$t^- = \frac{\sin\theta}{\frac{\theta'}{s'} - k(s)}. \tag{1b}$$

Combing (1a) *and* (1b) *gives the well known "mirror equation"*

$$\frac{1}{t^-} + \frac{1}{t^+} = -\frac{2k}{\sin\theta}. \tag{2}$$

2. Billiards in an Ellipse.

Billiards inside an ellipse γ_E defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are integrable [16]. Let us assume that $a > b$. Then the focal points of the ellipse are the points $\{(\pm c, 0)\}$ where $c = \sqrt{a^2 - b^2}$. The phase space of the billiard map (Fig. 2) decomposes into invariant circles of two types corresponding to trajectories that cross the x axis with $|x| > c$ ($\mathcal{E}$ trajectories) and to trajectories that cross the x axis $-c < x < c$ ($\mathcal{H}$ trajectories). $\mathcal{E}$ trajectories are repeatedly tangent to confocal ellipses; $\mathcal{H}$ trajectories are repeatedly tangent to confocal hyperbolas.

There are two important pairs of periodic orbits:

$$\mathcal{O}_h = \{p_1 = ((x = a, y = 0), \theta = \pi/2), p_2 = ((x = -a, y = 0), \theta = \pi/2)\},$$
$$\mathcal{O}_e = \{p_3 = ((x = 0, y = b), \theta = \pi/2), p_4 = ((x = 0, y = -b), \theta = \pi/2)\}.$$

$\mathcal{O}_h$ is an hyperbolic period-two orbit, $\mathcal{O}_e$ is elliptic.

The stable (unstable) manifold of the hyperbolic periodic orbit $\mathcal{O}_h$ consists of points that are forward (backward) asymptotic to the orbit. These are trajectories that pass through a focal point. By the properties of ellipses, a trajectory that once passes through a focal point will pass through a focal point on all future reflections. Note that if the trajectory starting at (s, θ) passes through a focal point, then so will the trajectory starting at $(s, \pi - \theta)$. Thus if a trajectory is forward asymptotic to $\mathcal{O}_h$, it will also be backwards asymptotic to it. Thus the stable manifold and unstable manifold of $\mathcal{O}_h$ coincide. These stable and unstable manifolds form a separatrix between the homotopically trivial invariant curves surrounding the elliptic periodic points $\{p_3, p_4\}$ and the homotopically nontrivial invariant curves.

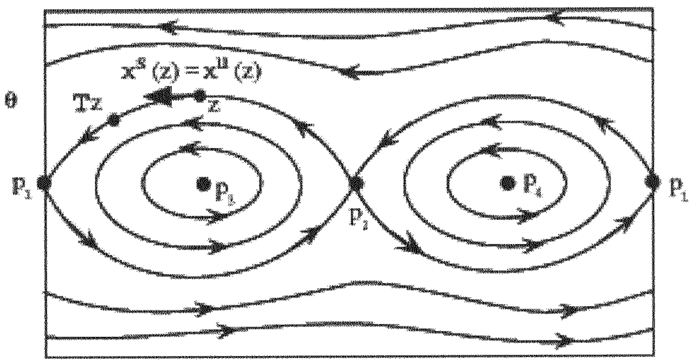

Fig. 2. Integrable phase space for billiard map of ellipse.

Our goal will be to perturb the boundary of the ellipse so that the stable and unstable manifolds cross transversely rather than coinciding.

Because the hyperbolic orbit is of period-two, rather than a fixed point, it is slightly confusing to analyze the dynamics of the map Φ. To simplify matters, we will look at the time-two map $\mathcal{T} = \Phi^2$. For this map, the points p_i, $i = 1, 2$ are hyperbolic fixed points: $Tp_i = p_i$. The stable and unstable manifolds are defined respectively by

$$W^s(p_i) = \{z \in M : \lim_{n\to+\infty} \mathcal{T}^n z = p_i\},$$
$$W^u(p_i) = \{z \in M : \lim_{n\to-\infty} \mathcal{T}^n z = p_i\}. \tag{3}$$

For the map T, we have a heteroclinic connection:

$$W^s(p_1) = W^u(p_2), \quad W^u(p_1) = W^s(p_2). \tag{4}$$

For each point $x \in W^s(p_i)$, the stable vector $\xi^s(z) \in T_z M$ is the vector tangent to the stable manifold. Similarly for the unstable vector $\xi^u(z)$. The existence of an heteroclinic connection implies that

$$\xi^s(z) = \xi^u(z)$$

for every $z \in W^s(p_1) \cap W^u(p_2)$. The equality of these vectors is implies that their forward and backward focusing distances (see Fig. 3) match:

$$t^+(\xi^s) = t^+(\xi^u), \quad t^-(\xi^s) = t^-(\xi^u). \tag{5}$$

The focusings occur at the focal points of the ellipse.

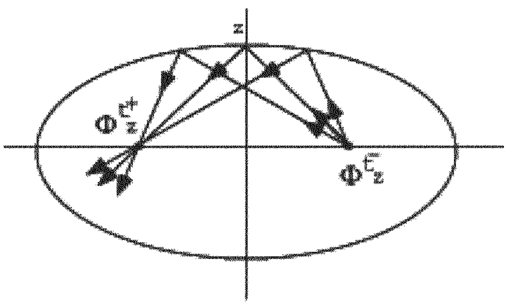

Fig. 3. Infinitesimal family focusing.

3. Perturbations of ellipse.

Theorem 1 follows from the following result.

Theorem 5. *There exists C^∞ smooth perturbations $\tilde{\gamma}$ of the ellipse γ_E, with $\tilde{\gamma}$ C^2 close to γ_E, such that the system $(\tilde{M}, \tilde{T})$ generated by the* *billiard inside of $\tilde{\gamma}$ has hyperbolic fixed points $\{\tilde{p}_1, \tilde{p}_2\}$ whose stable and unstable manifolds have points at which they intersect transversely.*

If we were trying to achieve such a result simply by perturbing the map T within the class of diffeomorphisms, the result would be well known. However producing the desired map $\tilde{T}$ as a result of perturbing the boundary γ_E is more challenging.

We will perturb the ellipse γ_E only in a small neighborhood U of the point $(x = 0, y = b)$. Henceforth we normalize our arc-length parametrizations of curves so that $\gamma(s = 0)$ is always the point $(x = 0, y = b)$.

To give an explicit description of the size of the neighborhood U, note that there is a unique angle θ_0 such that the point $z_0 = (s_0 = 0, \theta_0)$ will be in $W^s(p_1)$. This angle can be determined by requirement that the corresponding trajectory $\Psi^t z_0$ pass through the focal point $(-c, 0)$. Now locate the points $\mathcal{T} z_0 = (s_1, \theta_1)$ and $\mathcal{T}^{-1} z_0 = (s_{-1}, \theta_{-1})$ and let $\Delta s = \frac{1}{2}\min\{|s_1 - s_0|, |s_{-1} - s_0|\}$. We then set $U = \{\gamma_E(s) : |s - s_0| < \Delta s\}$.

We perturb γ_E in such a way that the resulting curve $\tilde{\gamma}$ will be C^∞ smooth, strictly convex and outside of the set U will equal γ_E. Furthermore, we require that $\tilde{\gamma}(0) = \gamma_E(0), \tilde{\gamma}'(0) = \gamma_E'(0)$ and that $\tilde{k}(0) \neq k_E(0)$.

Lemma 6. *There exist C^∞ smooth strictly convex closed curves $\tilde{\gamma}$ that are C^2 close to γ_E, that satisfy $\tilde{\gamma}(0) = \gamma_E(0)$, $\tilde{\gamma}'(0) = \gamma_E'(0)$, $\tilde{k}(0) \neq k_E(0)$ and for which $\tilde{\gamma} = \gamma_E$ outside of U.*

Proof. We define the curve $\tilde{\gamma}$ in terms of polar coordinates $\tilde{r}(\psi)$, $\psi \in [-\pi/2, 3\pi/2)$, with the normalization that $\psi = 0$ corresponds to the point $\tilde{\gamma}(0) = (0, b)$. The ellipse is defined by $r_E(\psi)$. We define a family of curves $\tilde{\gamma}_\epsilon$ in terms of functions $\tilde{r}_\epsilon$ given by

$$\tilde{r}_\epsilon(\psi) = r_E(\psi) + P(\psi)\epsilon\psi^2. \tag{6}$$

$P(\psi)$ is a partition of unity function satisfying $P(\psi) \equiv 1$ for $|\psi| \leq L/2$ and $P(\psi) \equiv 0$ for $|\psi| \geq L$. We choose the value L small enough so that $\tilde{r}_\epsilon(\psi) = r_E(\psi)$ outside the neighborhood U.

With this definition, we have that $\tilde{r}_\epsilon(0) = r_E(0), \tilde{r}_\epsilon'(0) = r_E'(0)$ and $\tilde{r}_\epsilon''(0) = r_E''(0) + 2\epsilon$. In polar coordinates, the curvature of a curve is given by

$$k = \frac{-r''r + 2r'^2 + r^2}{(r'^2 + r^2)^{3/2}}.$$

Hence we conclude that $\tilde{\gamma}_\epsilon(0) = \gamma_E(0)$, $\tilde{\gamma}_\epsilon'(0) = \gamma_E'(0)$, $\tilde{k}_\epsilon(0) \neq k_E(0)$ and outside of U, $\tilde{\gamma}_\epsilon = \gamma_E$.

Finally to show that $\tilde{\gamma}_\epsilon$ and γ_E are C^2 close, we observe that once the partition function P is fixed, the differences $|\tilde{r}_\epsilon^{(k)}(\psi) - r_E^{(k)}(\psi)|$, $k = 0, 1, 2$, are uniformly bounded for all $\psi \in [-\pi/2, 3\pi/2)$ and go to zero uniformly as ϵ goes to zero. ▲

For a curve $\tilde{\gamma}$ satisfying the conclusions of Lemma 6, the perturbed billiard system $(\tilde{M}, \tilde{T})$ has hyperbolic fixed points $\{\tilde{p}_1, \tilde{p}_2\}$ which are exactly equal to the fixed points of the original system: $\tilde{p}_i = p_i$, $i = 1, 2$, since the support of the perturbation was away from the periodic points. We denote by $\widetilde{W}^s(\tilde{p}_i)$, $\widetilde{W}^u(\tilde{p}_i)$ the stable and unstable manifolds of these fixed points.

Lemma 7.

$$z_0 \in \widetilde{W}^s(p_1) \cap \widetilde{W}^u(p_2).$$

Proof. The billiard trajectory $\widetilde{\Psi}^t z_0$ will pass through the focal point $(-c, 0)$. It will then hit the part of $\tilde{\gamma}$ which is identical with γ_E. Hence after its next reflection, it will again pass through a focal point. By making U sufficiently small, we insure that at its second reflection and then for all future reflections, the trajectory will only hit that part of $\tilde{\gamma}$ that is identical to γ_E. It will continue to pass through the focal points and will converge to p_1: $\lim_{n\to\infty} \widetilde{T}^n z_0 = p_1$. The analogous argument applies that $\lim_{n\to-\infty} \widetilde{T}^n z_0 = p_2$. ▲

Proof of Theorem 5. We now claim that at z_0, the manifolds $\widetilde{W}^s(p_1)$ and $\widetilde{W}^u(p_2)$ intersect transversely.

In this example we can explicitly construct these manifolds. To find the stable manifold in a neighborhood of z_0, flow the trajectory through z_0 forward until it reaches the focal point $(-c, 0)$. Take a one-parameter family of trajectories that have this focal point as base point and have varying angles and which includes the trajectory through z_0. Providing that the angle of inclination of these trajectories is not too different from that of the trajectory through z_0, then every trajectory in this family will, under the flow $\widetilde{\Psi}^t$, accumulate on the periodic orbit $\{p_1, p_2\}$; these trajectories will stay away from the perturbed section U in the future and hence will collide with the boundary only at places where $\tilde{\gamma} = \gamma_E$. Hence they will keep passing through the focal points.

If we now flow this family of trajectories backwards until it intersects the boundary near $\tilde{\gamma}(0)$, we will get a subset of the stable manifold

$\widetilde{W}^s_{loc}(p_1) \subset \widetilde{W}^s(p_1)$ that includes the point z_0. This curve is smooth; we denote its tangent vector at z_0 by $\tilde{\xi}^s(z_0) = (s'_s, \theta'_s)$.

We construct a subset $\widetilde{W}^u_{loc}(p_2)$ of the unstable manifold $\widetilde{W}^u(p_2)$ in an analogous fashion. We follow the trajectory through z_0 backwards until it reaches the focal point $(c, 0)$. There we take a one-parameter family of trajectories whose direction changes but whose base point is always $(c, 0)$. Flowing backwards, these rays accumulate on the orbit $\{p_1, p_2\}$. Therefore if we flow these points forward until their first intersection with the boundary, we get a subset of the unstable manifold $\widetilde{W}^u_{loc}(p_2) \subset \widetilde{W}^u(p_2)$ whose tangent vector at z_0 we denote by $\tilde{\xi}^u(z_0) = (s'_u, \theta'_u)$.

Since there is a one-to-one correspondence between the slope of a vector and its time to focusing (Lemma 4), to prove that

$$\tilde{\xi}^s(z_0) \neq \tilde{\xi}^u(z_0),$$

and hence that the stable and unstable manifolds intersect transversely, it suffices to show that

$$t^+(\tilde{\xi}^s) \neq t^+(\tilde{\xi}^u). \tag{7}$$

We know that

$$t^+(\tilde{\xi}^s) = t^+(\xi^s)$$

since the stable vectors, for both the perturbed and unperturbed systems, focus at the focal point $(-c, 0)$. Similarly,

$$t^-(\xi^u) = t^-(\tilde{\xi}^u).$$

Furthermore, for the ellipse $\xi^s(z_0) = \xi^u(z_0)$ and hence

$$t^+(\xi^s) = t^+(\xi^u).$$

Thus by the mirror equation (2), we have that

$$\begin{aligned} \frac{2k}{\sin\theta_0} &= \frac{1}{t^-(\xi^u(z_0))} + \frac{1}{t^+(\xi^u(z_0))} \\ &= \frac{1}{t^-(\xi^u(z_0))} + \frac{1}{t^+(\xi^s(z_0))} \\ &= \frac{1}{t^-(\tilde{\xi}^u(z_0))} + \frac{1}{t^+(\tilde{\xi}^s(z_0))}. \end{aligned} \tag{8}$$

Applying the mirror equation to the perturbed system gives

$$\frac{2\tilde{k}}{\sin\theta_0} = \frac{1}{t^-(\tilde{\xi}^u(z_0))} + \frac{1}{t^+(\tilde{\xi}^u(z_0))}, \tag{9}$$

with $\tilde{k} \neq k$, from which we can conclude that

$$\frac{1}{t^+(\widetilde{\xi}^s(z_0))} \neq \frac{1}{t^+(\widetilde{\xi}^u(z_0))},$$

and hence that $\widetilde{\xi}^s(z_0) \neq \widetilde{\xi}^u(z_0)$. ▲

The choice of the point z_0 was arbitrary. We could repeat this argument in the neighborhood of any point $z \in W^s(p_1), z \neq p_1, i = 1, 2$.

Observe that the perturbation we have used creates an unusual topology for $\widetilde{W}^s(p_1) \cap \widetilde{W}^u(p_1)$. There will be points $z \in \widetilde{W}^s(p_1) \cap \widetilde{W}^u(p_1)$ for which $\widetilde{\Psi}^t z$ never intersects the perturbed region U. These trajectories only hit the boundary at points for which $\tilde{\gamma} = \gamma_E$ and hence at such points, and in neighborhoods of such points, the stable and unstable manifolds coincide. On the other hand, at those points $z \in \widetilde{W}^s(p_1) \cap \widetilde{W}^u(p_1)$ whose orbit $\widetilde{\Psi}^t z$ does intersect the perturbed boundary near $\tilde{\gamma}(0)$, the stable and unstable manifolds intersect transversely (Fig. 4). This unusual behaviour can happen because our perturbation produces a C^∞ curve. If our perturbation produced an analytic curve, such behaviour could not happen.

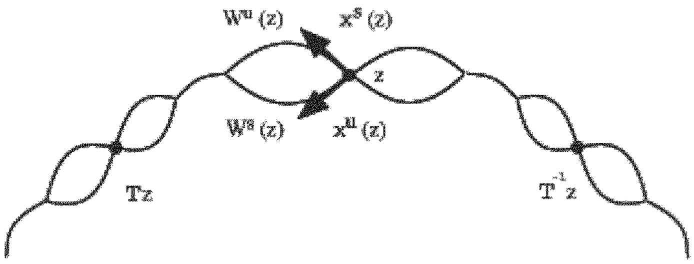

Fig. 4. Co-existence of transverse and nontransverse behaviour.

4. Perturbations of more general billiards.

Consider any planar billiard system that has a periodic orbit whose stable and unstable manifolds intersect non-transversely at a point $z_0 = (s_0 = 0, \theta_0)$. Such a system could be bounded by a closed convex curve, by a closed curve with both convex and concave components or by several disjoint closed curves. An example of the latter system would be the region generated by removing a disk from the interior of an ellipse. Assume that the boundary curve(s) are smooth.

With very minor modifications, the proof for the case of the ellipse can apply to yield the existence of a smooth curve(s) C^2 close to the original boundary curve(s) for which the manifolds intersect transversely at z_0. To avoid possible pathologies, we require that the billiard orbit through z_0 never become tangent to the boundary of the region; such tangencies can happen for a nonconvex region and would produce discontinuities for the billiard map.

Denoting the original curve by γ, we produce a perturbed curve $\tilde{\gamma}$ that agrees with γ except in a neighborhood U of the point $\gamma(s_0)$. The neighborhood is chosen small enough that no iterate $\Phi^n z_0$, $n = \pm 1, \pm 2, \pm 3, \ldots$, will hit the boundary in the region U. Since the orbit accumulates, both in forward and backwards time, on a periodic orbit, only finitely many iterates $\Phi^n z_0$ will be away from an ϵ-neighborhood of the periodic orbit and hence such a neighborhood U exists.

In this neighborhood U, we perturb the original curve in the same manner as given by equation (6) producing a curve $\tilde{\gamma}$ that satisfies $\tilde{\gamma}(s_0) = \gamma(s_0)$, $\tilde{\gamma}'(s_0) = \gamma'(s_0)$, and $\tilde{k}(s_0) \neq k_0(s_0)$. For $n = \pm 1, \pm 2, \pm 3, \ldots$, $\widetilde{\Phi}^n z_0 = \Phi^n z_0 = z_n$, since these points never hit U, and hence z_0 lies in the intersection of the stable and unstable manifold for the perturbed system.

The same logic also implies that the stable vector at z_1 and the unstable vector at z_{-1} are equal for the perturbed and unperturbed systems: $\widetilde{\xi}^s(z_1) = \xi^s(z_1)$ and $\widetilde{\xi}^u(z_{-1}) = \xi^u(z_{-1})$.

The stable and unstable vectors at z_0 are given by: $\widetilde{\xi}^s(z_0) = \widetilde{\Phi}^{-1}\widetilde{\xi}^s(z_1)$, $\widetilde{\xi}^u(z_0) = \widetilde{\Phi}\widetilde{\xi}^u(z_{-1})$. We will show that $\widetilde{\xi}^s(z_0) \neq \widetilde{\xi}^u(z_0)$ by showing that their associated focusing times $t^+(\widetilde{\xi}^s(z_0)) \neq t^+(\widetilde{\xi}^u(z_0))$.

In the case of the ellipse, the stable and unstable vectors focus inside the billiard region at the focal points of the ellipse. Wojtkowski (see [18, Lemma 1]) has generalized the notion of focusing point and focusing distance for a family of rays to all families of rays, even those that focus outside the billiard region. The mirror equation (2) still holds for this generalized focusing distance although now we need to use a signed distance that takes into account whether the boundary is convex or concave.

With this generalized notion of focusing time, we have that $t^-(\widetilde{\xi}^s(z_1)) = t^-(\xi^s(z_1))$, since the vectors are equal and locally the boundary curves are the same. This in turn implies that $t^+(\widetilde{\xi}^s(z_0)) = t^+(\xi^s(z_0))$. Similarly, $t^+(\widetilde{\xi}^u(z_{-1})) = t^+(\xi^u(z_{-1}))$ which implies that $t^-(\widetilde{\xi}^u(z_0)) = t^-(\xi^u(z_0))$. Now the proof that $t^+(\widetilde{\xi}^s(z_0)) \neq t^+(\widetilde{\xi}^u(z_0))$ goes through exactly as in the case of the ellipse (see Eqs. (8) and (9)).

References

1. M. Bialy, *Convex billiards and a theorem by E. Hopf.* — Math. Z. **214**, No. 1 (1993), 147–154.
2. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, Springer Verlag (1982).
3. A. Delshams and R. Ramírez-Ros, *Poincare–Melnikov–Arnold method for analytic planar maps.* — Nonlinearity **9**, No. 1 (1996), 1–26.
4. A. Delshams and R. Ramírez-Ros, *On Birkhoff's conjecture about convex billiards*, in: *Proceedings of the 2nd Catalan Days on Applied Mathematics* (Odeillo, 1995), Collect. Etudes, Presses Univ. Perpignan, Perpignan (1995), pp. 85–94.
5. V. J. Donnay, *Using integrability to produce chaos: billiards with positive entropy.* — Comm. Math. Phys. **141** (1991), 225–257.
6. V. J. Donnay, *Transverse homoclinic connections for geodesic flows, Hamiltonian dynamical systems*, Cincinnati, OH (1992), 115–125; IMA Vol. Math. Appl. **63**, Springer, New York (1995).
7. M. M. Dvorin and V. F. Lazutkin, *Existence of an infinite number of elliptic and hyperbolic periodic trajectories for convex billiards.* — Funk. Anal. Priloz. **7**, No. 2 (1973), 20–27; translation in Funct. Anal. Appl. **7** (1973), 103–109.
8. V. Gelfreich, *A century of separatrices splitting in Hamiltonian dynamical systems: perturbation theory, exponential smallness*, in: XIII*th International Congress on Mathematical Physics*, Int. Press, Boston, MA, London (2000), pp. 73–86.
9. V. Gelfreich and V. F. Lazutkin, *Splitting of separatrices: perturbation theory and exponential smallness.* — *Uspekhi Mat. Nauk* **56**, No. 3(339) (2001), 79–142; translation in Russian Math. Surveys 56 (2001), no. 3, 499–558.
10. N. Innami, *Geometry of geodesics for convex billiards and circular billiards*, Nihonkai Math. J. — **13**, No. 1 (2002), 73–120.
11. Gwang Il Kim, *Elliptic Birkhoff's billiards with C^2-generic global perturbations.* — Bull. Korean Math. Soc. **36**, No. 1 (1999),147–159.
12. G. Knieper and H. Weiss, *A surface with positive curvature and positive topological entropy.* — J. Differential Geom. **39**, No. 2 (1994), 229–249.
13. H. E. Lomeli, *Perturbations of elliptic billiards.* — Phys. D 99, No. 1 (1996),59–80.
14. G. Paternain, *Real analytic convex surfaces with positive topological entropy and rigid body dynamics.* — Manuscripta Math. **78**, No. 4 (1993), 397–402.
15. D. Petroll, *Existenz und Transversalitt von Homoklinen und Heteroklinen Orbits beim Geodtischen Fluss* Thesis, Universitt Freiburg (1996).
16. Ya. G. Sinai, *Introduction to Ergodic Theory*, Princeton University Press: Princeton (1976).
17. M. B. Tabanov, *Separatrices splitting for Birkhoff's billiard in symmetric convex domain, closed to an ellipse.* — Chaos **4**, No. 4 (1994), 595–606.
18. M. Wojtkowski, *Principles for the design of billiards with nonvanishing Lyapunov exponent.* — Commun. Math. Phys. **105** (1986), 319–414.

19. M. Wojtkowski, *Two applications of Jacobi fields to the billiard ball problem.* — J. Differential Geom. **40**, No. 1 (1994), 155–164.

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