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Dedicated to Ludwig Dmitrievich Faddeev
for the 60th birthday

$SU(2)$ WZNW MODEL AT HIGHER GENERA FROM THE GAUGE FIELD FUNCTIONAL INTEGRAL

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Abstract. We compute the gauge field functional integral giving the scalar product of the $SU(2)$ Chern-Simons theory states on a Riemann surface of genus > 1. The result allows us to express the higher genera partition functions of the $SU(2)$ WZNW conformal field theory by explicit finite-dimensional integrals. Our calculation may also shed new light on the functional integral of the Liouville theory.

§1. Introduction

Since the 1967 seminal paper of Faddeev and Popov [1], the functional integral has become the main tool in the treatment of quantum gauge theories. The main breakthrough which this paper has achieved was the realization that the gauge invariance is not an obstruction but an aid in the treatment of quantum gauge fields. Subsequently, this idea has revealed its full power and the gauge invariance has become a cornerstone of modern theoretical physics.

In the present note, we shall discuss how the same idea allows an explicit solution of the WZNW model of conformal field theory on an arbitrary two-dimensional surface. To make things simpler, we shall limit the discussion to the partition functions of the $SU(2)$ WZNW model on closed compact Riemann surfaces Σ of genus > 1. The correlation functions at genus zero were discussed along similar lines in [2] for the $SU(2)$ case and in [3] for general compact groups. Higher genera, however, are more difficult and took some time to understand. The aim of this note is to present the main points of the argument leaving the technical details to the forthcoming publication [4]. The complete work is a small *tour de force*. This fact was hard to hide even in a softened exposition which may only envy the work [1] for its striking simplicity.

Our approach is based on a relation between the WZNW partition functions and the Schrödinger picture states of the Chern-Simons (CS) theory [5, 2]. The partition function of the level k ($k = 1, 2, \dots$) $SU(2)$ WZNW model in an external two-dimensional $su(2)$ gauge field $A \equiv A_z\, dz + A_{\bar z}\, d\bar z$ is formally given by the functional integral

$$Z(A) = \int e^{-kS(g,A)}\, Dg \tag{1.1}$$

over $g : \Sigma \to SU(2)$. On the other hand, the $SU(2)$ CS states are holomorphic functionals Ψ of the gauge field $A_{\bar z}$ s.t. for $h: \Sigma \to SL(2,\mathbb{C})$,

$$e^{-kS(h,A_z\, d\bar z)}\Psi({}^{h^{-1}}A_{\bar z}) = \Psi(A_{\bar z}), \tag{1.2}$$

where ${}^{h^{-1}}A_{\bar z} \equiv h^{-1}A_{\bar z}h + h^{-1}\partial_{\bar z}h$. Above $S(\cdot,\cdot)$ denotes the WZNW action in the external gauge field [6]. The CS states form a finite-dimensional space with the dimension expressed by the Verlinde formula [7]. Their scalar product is formally given by the functional integral

$$\|\Psi\|^2 = \int |\Psi(A_{\bar z})|^2 e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A_z A_{\bar z} d^2 z}\, DA \tag{1.3}$$

(with the convention that $A_z = -A_{\bar z}^\dagger$ for an $su(2)$ gauge field A). The partition function of the WZNW model is [2]

$$Z(A) = \sum_{a,b} H^{ab}\Psi_a(A_{\bar z})\overline{\Psi_b(A_{\bar z})}e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A_z A_{\bar z}\, d^2 z}, \tag{1.4}$$

where (H^{ab}) is the matrix inverse to that of the scalar products (ψ_a, ψ_b), for an arbitrary basis of the CS states. Hence, in order to find the WZNW partition functions, we should be able to construct CS states and to compute their scalar product. The calculation of the latter is another exercise in the functional integration over gauge fields and may be dealt with similarly to the original Faddeev–Popov's geometric argument. The main point is to use the action $A_{\bar z} \mapsto {}^{h^{-1}}A_{\bar z}$ of the group $\mathcal{G}^{\mathbb{C}}$ of the complex gauge transformations in order to reduce the functional integral over the space $\mathcal{A}$ of gauge fields to the orbit space $\mathcal{A}/\mathcal{G}^{\mathbb{C}}$, which is, in fact, finite-dimensional. The main aspect differing our situation from that considered by Faddeev and Popov is that neither the integrated function nor the (formal) integration measure DA are invariant under the complex gauge transformations. Nevertheless, they are transformed in a controllable way. This difference has to be properly accounted for.

The first step in the reduction of the functional integral to the orbit space is to choose a slice $\mathcal{A}/\mathcal{G}^{\mathbb{C}} \ni n \overset{s}{\mapsto} A(n) \in \mathcal{A}$ which cuts every orbit only once (generically) and to change variables by parametrizing

$$A_{\bar z} = {}^{h^{-1}}A_{\bar z}(n). \tag{1.5}$$

The Jacobian of the change of variables $\frac{\partial(A)}{\partial(h,n)}$ plays then the role similar to that of the ghost determinant in the approach of [1] where the slice was fixed by constraining functions. The next step is the calculation of the h-integral. This step is not trivial, in contrast to the gauge invariant situation where the gauge group integral factors out as an overall constant. In fact, in the case at hand, the integration over h, more involved at genera > 1 than for $g = 0$ and $g = 1$, leads to a result which may look surprising at the first sight: it gives not a function but a singular distribution on the orbit space $\mathcal{A}/\mathcal{G}^{\mathbb{C}}$. Its treatment may shed some light on the more complicated case of the Liouville theory functional integral. After the integration over h is done, one is left with an explicit finite-dimensional distributional integral, essentially over the orbit space $\mathcal{A}/\mathcal{G}^{\mathbb{C}}$, which may be further reduced to the standard integral over the support of the distribution.

The paper is organized as follows. §2 is devoted to the description of the slice $s: \mathcal{A}/\mathcal{G}^{\mathbb{C}} \to \mathcal{A}$. The use of s allows also a more explicit characterization of the CS states. In §3, we perform the change of variables (1.5) and study its Jacobian. In §4, we describe the calculation of the h-integral over the $\mathcal{G}^{\mathbb{C}}$-orbits in $\mathcal{A}$ which turns out to be iterative Gaussian. Finally, in §5 we discuss the resulting finite-dimensional integral representation for the CS scalar product.

§2. The slice

We would like to describe a surface $\{A(n)\}$ inside $\mathcal{A}$ which cuts each orbit of $\mathcal{G}^{\mathbb{C}}$ once (or a fixed number of times). The orbit space $\mathcal{A}/\mathcal{G}^{\mathbb{C}}$ (after removal of a small subset of bad orbits) is an object well studied in the mathematical literature under the name of the moduli space of (stable) holomorphic $SL(2,\mathbb{C})$-bundles [8, 9]. It has complex dimension $3(g-1)$, where g denotes the genus of the underlying Riemann surface. This should then also be the dimension of our slice of $\mathcal{A}$. Let L_0 be a spin bundle over Σ. L_0 is a holomorphic line bundle with local sections $(dz)^{1/2}$. L_0^{-1} will denote its dual bundle. $L_0^{-1}\oplus L_0$ is a rank two vector bundle over Σ which, as a smooth bundle, is isomorphic to the trivial bundle $\Sigma\times\mathbb{C}^2$. We shall fix a smooth isomorphism

$$U\colon L_0^{-1}\oplus L_0\to\Sigma\times\mathbb{C}^2. \tag{2.1}$$

We may assume that U preserves the length of the vectors calculated in $L_0^{-1}\oplus L_0$ with the help of a fixed Riemannian metric $\gamma=\gamma_{z\bar z}dzd\bar z$ on Σ. U may be used to transport the gauge fields $A_{\bar z}$ to the bundle $L_0^{-1}\oplus L_0$. More exactly, the relation

$$B_{\bar z}=UA_{\bar z}U^{-1}+U\partial_{\bar z}U^{-1} \tag{2.2}$$

establishes a one to one correspondence between $A_{\bar z}d\bar z$ and

$$B_{\bar z}d\bar z=\begin{pmatrix}-a & b\\ c & a\end{pmatrix}, \tag{2.3}$$

where a is a scalar 01-form on Σ ($a\in\Lambda^{01}$), b is an L_0^{-2}-valued one ($b\in\Lambda^{01}(L_0^{-2})$), and $c\in\Lambda^{01}(L_0^2)$.

Let us present the surface Σ as a polygone with sides $a_1,b_1,a_1^{-1},b_1^{-1},\dots,a_g,b_g,a_g^{-1},b_g^{-1}$ given by the basic cycles. We shall fix $x_0\in\Sigma$ in the corner where b_g^{-1} and a_1 meet. Let ω^i, $i=1,\dots,g$, be the standard basis of holomorphic forms with $\int_{a_i}\omega^j=\delta^{ij}$, $\int_{b_i}\omega^j=\tau^{ij}$. We shall take a slice of $\mathcal{A}$ formed by the gauge fields $A^{x,b}$ corresponding to

$$B_{\bar z}d\bar z=\begin{pmatrix}-a^x & b\\ 0 & a^x\end{pmatrix}\equiv B_{\bar z}^{x,b}, \tag{2.4}$$

where

$$a^x=\pi\Big(\int_{x_0}^{x}\omega^i\Big)\Big(\frac{1}{\operatorname{Im}\tau}\Big)_{ij}\bar\omega^j\equiv\pi\Big(\int_{x_0}^{x}\omega\Big)(\operatorname{Im}\tau)^{-1}\bar\omega. \tag{2.5}$$

Let L_x denote the holomorphic line bundle obtained from L_0 by replacing its $\bar\partial$ operator by $\bar\partial+a^x\wedge\equiv\bar\partial_{L_x}$. We shall restrict the forms b further by taking one representative in each class of $\frac{\Lambda^{01}(L_0^{-2})}{(\bar\partial-2a^x\wedge)(C^\infty(L_0^{-2}))}\cong H^1(L_x^{-2})$. This may be done by imposing the condition

$$(\nabla+2\overline{a^x}\wedge)b=0 \tag{2.6}$$

with ∇ standing for the holomorphic covariant derivative of the sections of $L_0^{-2}=T^{10}\Sigma$. Finally, only one b in each complex ray of solutions of (2.6) should be taken since

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}B_{\bar z}^{x,b}\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}=B^{x,\lambda^2 b} \tag{2.7}$$

and, consequently, $A^{x,b}$ and $A^{x,\lambda b}$ are gauge related. It may be shown that the union of the $\mathcal{G}^{\mathbb{C}}$-orbits passing through a slice of $\mathcal{A}$ constructed in this way is dense in $\mathcal{A}$ and that the generic $\mathcal{G}^{\mathbb{C}}$-orbit cuts the slice a fixed number of times [4]. Here, we shall content ourselves with the count of dimensions. By the Riemann–Roch theorem, the dimension of $H^1(L_x^{-2})$ is $3(g-1)$. The projectivization subtracts one dimension which is restored by varying $x \in \Sigma$.

The CS states Ψ admit an explicit representation when restricted to the slice described above. Let us put

$$\psi(x,b) = e^{-\pi(\int_{x_0}^{x}\omega)(\mathrm{Im}\,\tau)^{-1}(\int_{x_0}^{x}\omega)}\, e^{\frac{k}{\pi}\int_\Sigma \mathrm{tr}\, A_z^0 A_{\bar z}^{x,b} d^2z}\, \Psi(A_{\bar z}^{x,b}), \tag{2.8}$$

where $A_z^0 \equiv U\nabla_z U^{-1}$. ψ is an analytic function of x and b depending only of the class of b modulo $(\bar\partial - 2a^x\wedge)(\mathcal{C}^\infty(L_0^{-2}))$:

$$\psi(x, b + (\bar\partial - 2a^x\wedge)w) = \psi(x,b). \tag{2.9}$$

Under constant complex rescalings of b,

$$\psi(x,\lambda b) = \lambda^{k(g-1)}\psi(x,b). \tag{2.10}$$

Under the action of the fundamental group $\pi_1(\Sigma)$,

$$\psi(a_j x, c_{a_j}^2 b) = \nu(a_j)^k \psi(x,b), \tag{2.11}$$

$$\psi(b_j x, c_{b_j}^2 b) = \nu(b_j)^k e^{-2\pi i k\tau^{jj} - 4\pi i k\int_{x_0}^{x}\omega^j}\, \psi(x,b), \tag{2.12}$$

where, for each $p \in \pi_1(\Sigma)$, c_p is a non-vanishing (univalued) function on Σ,

$$c_p(y) = e^{2\pi i\, \mathrm{Im}((\int_p \omega)(\mathrm{Im}\,\tau)^{-1}(\int_{x_0}^{y}\bar\omega))} \tag{2.13}$$

and ν is a character of $\pi_1(\Sigma)$,

$$\nu(p) = e^{-\frac{i}{2\pi}\int_\Sigma R\ln c_p} \prod_{j=1}^{g} W_{a_j}^{-\frac{1}{\pi}\int_{b_j} c_p^{-1}dc_p}\, W_{b_j}^{\frac{1}{\pi}\int_{a_j} c_p^{-1}dc_p}. \tag{2.14}$$

In the last formula, the integral $\int_\Sigma R\ln c_p$ is over the polygone representing the surface, R is the metric curvature form of Σ normalized so that $\int_\Sigma R = 4\pi i(g-1)$, $\ln c_p(y) = \int_{x_0}^{y} c_p^{-1}dc_p$, and W_p stands for the holonomy of L_0 around the closed curve p. One may identify functions ψ satisfying relations (2.9), (2.10), (2.11) and (2.12) with sections of the kth power of Quillen's determinant bundle [10] of the holomorphic family $(\bar\partial + B^{x,b})$ of operators in $L_0^{-1}\oplus L_0$. The holomorphic ψ's form a finite-dimensional space, and the relation $\Psi \mapsto \psi$ realizes the space of CS states as its, in general proper, subspace.

§3. The change of variables

We apply the change of variables (1.5) with $A_{\bar z}(n) \equiv A_{\bar z}^{x,b}$ in the functional integral (1.3) giving the scalar product of CS states. The Jacobian is

$$\frac{\partial(A)}{\partial(h,n)} = \det(D_{\bar z}(h,n)^\dagger D_{\bar z}(h,n))\det\left(\left(\frac{\partial A_{\bar z}}{\partial n_\alpha}\right)^\perp, \left(\frac{\partial A_{\bar z}}{\partial n_\beta}\right)^\perp\right), \tag{3.1}$$

where $D_{\bar z}(h,n) = h^{-1}(\partial_{\bar z} + [A_{\bar z}(n),\cdot])h$ and $(\frac{\partial A_{\bar z}}{\partial n_\alpha})^\perp$ denotes the component of $h^{-1}\frac{\partial A_{\bar z}(n)}{\partial n_\alpha}h$ perpendicular to the image of $D_{\bar z}(h,n)$. The first determinant should be regularized (what specific regularization is used does not matter as long as it is insensitive to unitary conjugations of the operator, like the zeta-function regularization). The h-dependence of the regularized determinants is given by the global version of the non-abelian chiral anomaly formula [11]

$$\det(D_{\bar z}(h,n)^\dagger D_{\bar z}(h,n))\det((\tfrac{\partial A_{\bar z}}{\partial n_\alpha})^\perp,(\tfrac{\partial A_{\bar z}}{\partial n_\beta})^\perp)$$
$$= e^{4S(hh^\dagger,A(n))}\det(D_{\bar z}(n)^\dagger D_{\bar z}(n))\det((\tfrac{\partial A_{\bar z}(n)}{\partial n_\alpha})^\perp,(\tfrac{\partial A_{\bar z}(n)}{\partial n_\beta})^\perp), \qquad (3.2)$$

where $D_{\bar z}(n)\equiv D_{\bar z}(1,n)$. The covariance property (1.2) implies that

$$|\Psi(A_{\bar z})|^2 e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A_z A_{\bar z} d^2z} = |\Psi(A_{\bar z}(n)|^2 e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A_z(n)A_{\bar z}(n)d^2z} e^{kS(hh^\dagger,A(n))}. \qquad (3.3)$$

Hence, after the change of variables, the functional integral (1.3) becomes

$$\|\Psi\|^2 = \int |\Psi(A_{\bar z}(n)|^2 e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A_z(n)A_{\bar z}(n)d^2z} e^{(k+4)S(hh^\dagger,A(n))}\det(D_{\bar z}(n)^\dagger D_{\bar z}(n))$$
$$\times \det((\frac{\partial A_{\bar z}(n)}{\partial n_\alpha})^\perp,(\frac{\partial A_{\bar z}(n)}{\partial n_\beta})^\perp)D(hh^\dagger)\prod_\alpha d^2 n_\alpha, \qquad (3.4)$$

where we have used the $SU(2)$ gauge invariance of the integral to factor out the integration over the $SU(2)$-valued gauge transformations, like in Faddeev-Popov's case. There remains, however, the integral over the field $hh^\dagger$ effectively taking values in the hyperbolic space $\mathcal{H}\equiv SL(2,\mathbb{C})/SU(2)$. $D(hh^\dagger)$ should be viewed as a formal product of the $SL(2,\mathbb{C})$-invariant measures on $\mathcal{H}$.

Working out the explicit form of various terms under the integral in equation (3.4) for $A(n)\equiv A^{x,b}$ is a rather straightforward matter. One obtains

$$|\Psi(A^{x,b}_{\bar z})|^2 e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A^{x,b}_z A^{x,b}_{\bar z} d^2z} = e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A^0_z (A^0_z)^\dagger d^2z} e^{-4\pi k(\int_{x_0}^x \operatorname{Im}\omega)(\operatorname{Im}\tau)^{-1}(\int_{x_0}^x \operatorname{Im}\omega)}$$
$$\times e^{\frac{k}{2\pi i}\int_\Sigma\langle b,\wedge b\rangle}|\psi(x,b)|^2. \qquad (3.5)$$

(Here and below, $\langle\cdot,\cdot\rangle$ denotes the Hermitian structure induced by the Riemannian metric of Σ on the powers of its canonical bundle.) The determinant of the operator $D_{\bar z}(n)^\dagger D_{\bar z}(n)$, unitarily equivalent to the operator $(\partial_{\bar z}+[B^{x,b}_{\bar z},\cdot])^\dagger(\partial_{\bar z}+[B^{x,b}_{\bar z},\cdot])$ acting on smooth endomorphisms of $L_0^{-1}\oplus L_0$, may be found by performing the Gaussian integration

$$\det(\bar D_{\bar z}(n)^\dagger \bar D_{\bar z}(n)) = \int e^{-i\int_\Sigma(2(\overline{\bar\partial X+Zb})\wedge(\bar\partial X+Zb)+\langle(\bar\partial-2a^x)Y-2Xb,\wedge((\bar\partial-2a^x)Y-2Xb)\rangle)}$$
$$\cdot e^{\langle(\bar\partial+2a^x)Z,\wedge(\bar\partial-2a^x)Z\rangle}\,DY\,DX\,DZ \qquad (3.6)$$

over the anticommuting ghost fields: the scalar X, the L_0^{-2}-valued Y, and the L_0^2-valued Z. Formally, the computation may be done iteratively, first over Y, then over X and, at the end, over Z. In this way one obtains the product of determinants

$$\det(\bar\partial^\dagger_{L_x^{-2}}\bar\partial_{L_x^{-2}})\,\det{}'(-\Delta)\det{}'(\bar\partial^\dagger_{L_x^2}\bar\partial_{L_x^2}), \qquad (3.7)$$

where Δ is the scalar Laplacian on Σ, decorated with zero mode terms. Some care should be taken, since the regularization (e.g., by the zeta-function prescription) of the big determinant requires, besides similar regularization of the product determinants, also an additional term which may be found by demanding that the result is transformed as in (3.2) under the complex gauge transformations.

The final expression for the measure

$$du(n)\equiv\det(D_{\bar z}(n)^\dagger D_{\bar z}(n))\det\left(\left(\frac{\partial A_{\bar z}(n)}{\partial n_\alpha}\right)^\perp,\left(\frac{\partial A_{\bar z}(n)}{\partial n_\beta}\right)^\perp\right)\prod_\alpha d^2n_\alpha \qquad (3.8)$$

on the slice becomes rather simple. Fix $x \in \Sigma$. Let $(b^\alpha)_{\alpha=1}^{3(g-1)}$ be an orthonormal basis (in the natural L^2 scalar product) of L_0^{-2}-valued 01-forms b solving equation (2.6). Similarly, let $(\kappa_r)_{r=1}^{g-1}$ be an orthonormal basis of sections of L_0^{-2} annihilated by $\nabla - 2\overline{a^x}\wedge$. Set $M_{ir}^\alpha \equiv \int_\Sigma \omega^i \langle \kappa_r, b^\alpha \rangle$. Denote by (z^α) the complex coordinates w.r.t. the basis (b^α) on the space of b satisfying equation (2.6). One may use $x \in \Sigma$ and the homogeneous coordinates z^α to parametrize the slice of $\mathcal{A}$ and

$$du(x,b) = \text{const}\, \frac{\text{area}(\Sigma)}{\det(\text{Im}\,\tau)} \det(\bar\partial^\dagger_{L_x^{-2}} \bar\partial_{L_x^{-2}})' \det(-\Delta) \det{}'(\bar\partial^\dagger_{L_x^2} \bar\partial_{L_x^2}) e^{\frac{2}{\pi i}\int_\Sigma \langle b, \wedge b\rangle}$$
$$\times\, |\varepsilon_{\alpha_1,\dots,\alpha_{3(x-1)}} z^{\alpha_1} dz^{\alpha_2} \wedge \cdots \wedge dz^{\alpha_{3(x-1)}}|^2 \Big| \sum_{j=1}^{g} \det\Big(\sum_\alpha M_{ir}^\alpha z^\alpha\Big)_{i\neq j} \omega^j(x)\Big|^2. \tag{3.9}$$

Notice that $du(n)$ contains as a factor the natural measure on the projectivization of the space of solutions of equation (2.6).

§4. Integral over the gauge orbits

It remains to compute the functional integral

$$Z_{\mathcal{H}}(A^{x,b}) \equiv \int e^{(k+4)S(hh^\dagger, A^{x,b})} D(hh^\dagger) \tag{4.1}$$

giving the genus g partition function of a (nonunitary) WZNW model with fields taking values in the hyperbolic space $\mathcal{H}$, first considered in [12]. The calculation of the functional integral (4.1) is the crucial step of the argument. The fields $hh^\dagger$ may be uniquely parametrized by real functions φ and sections w of $L_0^{-2} = T^{10}\Sigma$ by writing

$$hh^\dagger = U \begin{pmatrix} 1 & e^\varphi w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\varphi & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^\varphi w^\dagger & 1 \end{pmatrix} U^{-1}, \tag{4.2}$$

where $w^\dagger$ is the section of L_0^2 obtained by contracting the vector field $\bar w$ with the Riemannian metric. Rewritten in terms of φ and w, the functional integral (4.1) becomes

$$Z_{\mathcal{H}}(A^{x,b}) = e^{-\frac{k+4}{2\pi i}\int_\Sigma \langle b, \wedge b\rangle}$$
$$\times \int e^{\frac{k+4}{2\pi i}\int_\Sigma [-\varphi(\partial\bar\partial\varphi + R) + \langle e^{-\varphi} b + (\bar\partial + (\bar\partial\varphi))w, \wedge(e^{-\varphi} b + (\bar\partial + (\bar\partial\varphi))w)\rangle]} Dw\, D\varphi, \tag{4.3}$$

where $\bar\partial \equiv \bar\partial_{L_x^2}$ when acting on w. The w-integral is Gaussian and may be easily performed:

$$\mathcal{C}_w \equiv \int e^{\frac{k+4}{2\pi i}\int_\Sigma \langle e^{-\varphi} b + (\bar\partial + (\bar\partial\varphi))w, \wedge(e^{-\varphi} b + (\bar\partial + \bar\partial(\varphi))w)\rangle} Dw$$
$$= e^{\frac{k+4}{2\pi i}\int_\Sigma \langle P_\varphi(e^{-\varphi} b), \wedge P_\varphi(e^{-\varphi} b)\rangle} \det\left((\bar\partial_{L_x^2} + (\bar\partial\varphi))^\dagger (\bar\partial_{L_x^2} + (\bar\partial\varphi))\right)^{-1}, \tag{4.4}$$

where P_φ denotes the orthogonal projection on the kernel of $(\bar\partial_{L_x^2} + (\bar\partial\varphi))^\dagger$. Since the latter is spanned by the vectors $e^\varphi b^\alpha$,

$$\int_\Sigma \langle P_\varphi(e^{-\varphi} b), \wedge P_\varphi(e^{-\varphi} b)\rangle \equiv \|P_\varphi(e^{-\varphi} b)\|^2$$
$$= \overline{(b^\alpha, b)} (H_\varphi^{-1})_{\alpha\beta} (b^\beta, b) = \bar z^\alpha (H_\varphi^{-1})_{\alpha\beta} z^\beta, \tag{4.5}$$

where H_φ is the matrix of scalar products $(e^\varphi b^\alpha, e^\varphi b^\beta)$. The factor $e^{-\frac{k+4}{2\pi}\|P_\varphi(e^{-\varphi}b)\|^2}$ is the classical value of the w-integral. One may rewrite it as a finite-dimensional integral

$$e^{-\frac{k+4}{2\pi}\|P_\varphi(e^{-\varphi}b)\|^2} = \text{const}\det(H_\varphi)\int e^{-\frac{2\pi}{k+4}\bar{c}_\alpha(H_\varphi)^{\alpha\beta}c_\beta+i\bar{c}_\alpha z^\alpha+ic_\alpha\bar{z}^\alpha}\prod_\alpha d^2c_\alpha. \tag{4.6}$$

By the global version of the abelian chiral anomaly formula,

$$\det(H_\varphi)\det\left((\bar{\partial}_{L_x^{-2}}+(\bar{\partial}\varphi))^\dagger(\bar{\partial}_{L_x^{-2}}+(\bar{\partial}\varphi))\right)^{-1}$$
$$= e^{-\frac{1}{\pi i}\int_\Sigma\partial\varphi\wedge\bar{\partial}\varphi+\frac{3}{2\pi i}\int_\Sigma\varphi R}\det(\bar{\partial}^\dagger_{L_x^{-2}}\bar{\partial}_{L_x^{-2}})^{-1} \tag{4.7}$$

(recall that H_0 is the unit matrix). Gathering equations (4.4), (4.6), and (4.7), one obtains

$$\mathcal{C}_w = \text{const}\det(\bar{\partial}^\dagger_{L_x^{-2}}\bar{\partial}_{L_x^{-2}})^{-1}e^{-\frac{1}{\pi i}\int_\Sigma\partial\varphi\wedge\bar{\partial}\varphi+\frac{3}{2\pi i}\int_\Sigma\varphi R}$$
$$\cdot\int e^{-\frac{2\pi}{k+4}\bar{c}_\alpha(H_\varphi)^{\alpha\beta}c_\beta+i\bar{c}_\alpha z^\alpha+ic_\alpha\bar{z}^\alpha}\prod_\alpha d^2c_\alpha. \tag{4.8}$$

Note that the right hand side which, together with the φ-terms left over in (4.3), has to be integrated over φ contains a Liouville-type term

$$e^{-\frac{2\pi}{k+4}\bar{c}_\alpha(H_\varphi)^{\alpha\beta}c_\beta} = e^{-\frac{2\pi i}{k+4}\int_\Sigma e^{2\varphi}\langle c_\alpha b^\alpha, c_\beta b^\beta\rangle} \tag{4.9}$$

notorious for causing problems in the functional integration. Indeed, with the w-integral done, the φ-integral takes the form

$$\int e^{\frac{k+2}{2\pi i}\int_\Sigma\partial\varphi\wedge\bar{\partial}\varphi-\frac{k+1}{2\pi i}\int_\Sigma\varphi R-\frac{2\pi}{k+4}\bar{c}_\alpha(H_\varphi)^{\alpha\beta}c_\beta}D\varphi \equiv \mathcal{C}_\varphi \tag{4.10}$$

which, unlike at low genera, is of the Liouville theory type, not a Gaussian one. A possible approach to such an integral is to try to get rid of the Liouville term by integrating out the zero mode $\varphi_0 \equiv (\text{area}(\Sigma))^{-\frac{1}{2}}\int_\Sigma\varphi da$ of φ (da denotes the metric volume measure on Σ). This was tried in the Liouville theory in [13] and, supplemented with rather poorly understood formal tricks, has led in [14] to the functional integral calculation of three-point functions for the minimal models coupled to gravity. Multiplying $\mathcal{C}_\varphi$ by $1 = (\text{area})^{\frac{1}{2}}\int\delta(\varphi_0-u(\text{area})^{\frac{1}{2}})du$, changing the order of integration, shifting φ_0 to φ_0+u and setting $M \equiv (k+1)(g-1)$, one obtains

$$\mathcal{C}_\varphi = (\text{area})^{1/2}\int e^{-2uM}e^{\frac{k+2}{2\pi i}\int_\Sigma\partial\varphi\wedge\bar{\partial}\varphi-\frac{k+1}{2\pi i}\int_\Sigma\varphi R-\frac{2\pi}{k+4}e^{2u}\bar{c}_\alpha(H_\varphi)^{\alpha\beta}c_\beta}du\,\delta(\varphi_0)D\varphi$$
$$= \tfrac{1}{2}(\text{area})^{1/2}\Gamma(-M)(\tfrac{2\pi}{k+4})^M\int e^{\frac{k+2}{2\pi i}\int_\Sigma\partial\varphi\wedge\bar{\partial}\varphi-\frac{k+1}{2\pi i}\int_\Sigma\varphi R}(\bar{c}_\alpha(H_\varphi)^{\alpha\beta}c_\beta)^M\delta(\varphi_0)D\varphi. \tag{4.11}$$

Hence the integration over the zero mode of φ diverges but may be easily (multiplicatively) renormalized by removing the overall divergent factor $\Gamma(-M)$. Now, the c-integral is easy to perform:

$$\int(\bar{c}_\alpha(H_\varphi)_{\alpha\beta}c_\beta)^M e^{i\bar{c}_\alpha z^\alpha+ic_\alpha\bar{z}^\alpha}\prod_\alpha d^2c_\alpha$$
$$= (2\pi)^{6(g-1)}(-(H_\varphi)^{\alpha\beta}\partial_{z^\alpha}\partial_{\bar{z}^\beta})^M\prod_\alpha\delta(z^\alpha). \tag{4.12}$$

Gathering the above results, we obtain the following "Coulomb gas representation" for the higher genus partition function of the $\mathcal{H}$-valued WZNW model:

$$\int e^{(k+4)S(hh^\dagger,A^{x,b})}D(hh^\dagger) = \mathrm{const}(\mathrm{area})^{1/2}\det(\bar\partial^\dagger_{L_x^{-2}}\bar\partial_{L_x^{-2}})^{-1}e^{-\frac{k+4}{2\pi i}\int_\Sigma\langle b,\wedge b\rangle}$$
$$\times\left(\int e^{\frac{k+2}{2\pi i}\int_\Sigma\partial\varphi\wedge\bar\partial\varphi-\frac{k+1}{2\pi i}\int_\Sigma\varphi R}(-(H_\varphi)^{\alpha\beta}\partial_{z^\alpha}\partial_{\bar z^\beta})^M\delta(\varphi_0)D\varphi\right)\prod_\alpha\delta(z^\alpha)$$
$$= \mathrm{const}(\mathrm{area})^{1/2}\det(\bar\partial^\dagger_{L_x^{-2}}\bar\partial_{L_x^{-2}})^{-1}e^{-\frac{k+4}{2\pi i}\int_\Sigma\langle b,\wedge b\rangle}\left(\prod_m\tfrac{1}{i}\partial_{z^{\alpha_m}}\partial_{\bar z^{\beta_m}}\prod_\alpha\delta(z^\alpha)\right)$$
$$\times\int\left(\int e^{\frac{k+2}{2\pi i}\int_\Sigma\partial\varphi\wedge\bar\partial\varphi-\frac{k+1}{2\pi i}\int_\Sigma\varphi R+2\sum_m\varphi(x_m)}\delta(\varphi_0)D\varphi\right)\prod_m\langle b^{\alpha_m},b^{\beta_m}\rangle(x_m), \tag{4.13}$$

where m runs from 1 to M. The φ integral is now purely Gaussian and easily doable, provided that we Wick order the "screening charge" insertions $e^{2\varphi(x_m)}$ (this is, again, a multiplicative renormalization). In the end, we obtain

$$Z_{\mathcal{H}}(A^{x,b}) = \mathrm{const}$$
$$\times\det(\bar\partial^\dagger_{L_x^{-2}}\bar\partial_{L_x^{-2}})^{-1}\left(\tfrac{\det'(-\Delta)}{\mathrm{area}(\Sigma)}\right)^{-1/2}e^{-\frac{k+4}{2\pi i}\int_\Sigma\langle b,\wedge b\rangle}\left(\prod_m\tfrac{1}{i}\partial_{z^{\alpha_m}}\partial_{\bar z^{\beta_m}}\prod_\alpha\delta(z^\alpha)\right)$$
$$\times\int\left(\prod_{m_1\neq m_2}e^{-\frac{4\pi}{k+2}G(x_{m_1},x_{m_2})}\right)\left(\prod_m e^{-\frac{4\pi}{k+2}:G(x_m,x_m):}\langle b^{\alpha_m},b^{\beta_m}\rangle(x_\alpha)\right), \tag{4.14}$$

where $G(\cdot,\cdot)$ is the Green function of the scalar Laplacian Δ on Σ chosen so that $\int_\Sigma G(\cdot,y)R(y) = 0.$: $G(y,y) :\equiv \lim_{\varepsilon\to 0}(G(y,y')-\frac{1}{2\pi}\ln\varepsilon)$ where $\varepsilon = d(y,y')$ is the distance between y and y'.

Formula (4.14) reduces the functional integral over $hh^\dagger$ to a finite-dimensional integral over positions $x_m\in\Sigma$ of M screening charges. The integrand is a smooth function except for $\mathcal{O}(d(x_{m_1},x_{m_2})^{-\frac{4}{k+2}})$ singularities at coinciding points. Power counting shows that the integral converges for $g=2$ but for higher genera it diverges unless special combinations of forms $\langle b^{\alpha_m},b^{\beta_m}\rangle(x_m)$ are integrated. Another feature of the right-hand side of equation (4.14) is even more surprising as a candidate for a partition function: its dependence on the external field $A^{x,b}$ is not functional but distributional! The entire dependence on b resides in the term $\prod_\alpha\partial_{z^{\alpha_m}}\partial_{\bar z^{\beta_m}}\prod_\alpha\delta(z^\alpha)$ (recall that $z^\alpha\equiv(b^\alpha,b)$). This fact is not so astonishing, since the partition function of the $\mathcal{H}$-valued WZNW model may be expected, by formal arguments similar to the ones used in [15], to be the hermitian square of a holomorphic section of a negative power of the determinant bundle. But there are no such global sections but only distributional solutions of the corresponding Ward identities. The right-hand side of (4.14) is one of them.

§5. The scalar product formula

In view of the results (3.9) giving the integration measure on the slice and (4.14) computing the integral along the $\mathcal{G}^{\mathbb{C}}$-orbits, the functional integral expression (3.4) for the scalar product of the CS states is reduced to

$$\begin{aligned}\|\Psi\|^2 &= \text{const}\,\det(\operatorname{Im}\tau)^{-1}\left(\frac{\det'(-\Delta)}{\operatorname{area}(\Sigma)}\right)^{1/2} e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A_z^0 (A_z^0)^\dagger d^2 z} \\ &\times \int e^{-4\pi k(\int_{x_0}^x \operatorname{Im}\omega)(\operatorname{Im}\tau)^{-1}(\int_{x_0}^x \operatorname{Im}\omega)} |\psi(x,b)|^2 \Big|\sum_{j=1}^g \det\Big(\sum_\alpha M_{ir}^\alpha z^\alpha\Big)_{i\neq j}\omega^j(x)\Big|^2 \\ &\times \det{}'(\bar\partial_{L_x^2}^\dagger \bar\partial_{L_x^2})\Big(\prod_m \tfrac{1}{i}\partial_{z^{\alpha_m}}\partial_{\bar z^{\beta_m}} \prod_\alpha \delta(z^\alpha)\Big)|\varepsilon_{\alpha_1,\dots,\alpha_{3(g-1)}} z^{\alpha_1} dz^{\alpha_2}\wedge\cdots\wedge dz^{\alpha_{3(g-1)}}|^2 \\ &\times \Big(\prod_{m_1\neq m_2} e^{-\frac{4\pi}{k+2}G(x_{m_1},x_{m_2})}\Big)\Big(\prod_m e^{-\frac{4\pi}{k+2}:G(x_m,x_m):}\langle b^{\alpha_m}, b^{\beta_m}\rangle(x_m)\Big). \end{aligned} \tag{5.1}$$

The integral is, for fixed $x \in \Sigma$, over the $(3g-4)$-dimensional projective space with homogeneous coordinates (z^α) and over the Cartesian product of $M \equiv (k+1)(g-1)$ copies of Σ (positions x_m of the screening charges). Finally, one should integrate over $x \in \Sigma$.

The (z^α)-integral should be interpreted as

$$\int_{\mathbb{C}^{3(g-1)}} |P(z)|^2 \Big(\prod_m \tfrac{1}{i}\partial_{z^{\alpha_m}}\partial_{\bar z^{\beta_m}} \prod_\alpha \delta(z^\alpha)\Big) d^{6(g-1)}z = \prod_m \big(\tfrac{1}{i}\partial_{z^{\alpha_m}}\partial_{\bar z^{\beta_m}}\big)\Big|_{z=0} |P(z)|^2, \tag{5.2}$$

where $P(z) = \psi(x,b)\sum_{j=1}^g \det(\sum_\alpha M_{ir}^\alpha z^\alpha)_{i\neq j}\omega^j(x)$ is a homogeneous polynomial in (z^α) of degree M. Formally, the latter integral differs from the one involving the volume form on $P\mathbb{C}^{3g-4}$ by an infinite constant, which may be interpreted as the factor $\Gamma(-M)$ which appeared in the integral over the zero mode of the field φ. With the z-integration given by equation (5.2), one obtains the following formula for the scalar product of CS states:

$$\begin{aligned}\|\Psi\|^2 &= \text{const}\,\det(\operatorname{Im}\tau)^{-1}\left(\frac{\det'(-\Delta)}{\operatorname{area}(\Sigma)}\right)^{1/2} e^{\frac{k}{\pi}\int_\Sigma \operatorname{tr} A_z^0 (A_z^0)^\dagger d^2 z} \\ &\times \int \prod_m \Big(\frac{1}{i}\partial_{z^{\alpha_m}}\partial_{\bar z^{\beta_m}}\Big)\Big|_{z=0}\Big(|\psi(x,b)|^2 \Big|\sum_{j=1}^g \det\Big(\sum_\alpha M_{ir}^\alpha z^\alpha\Big)_{i\neq j}\omega^j(x)\Big|^2\Big) \\ &\times \Big(\prod_{m_1\neq m_2} e^{-\frac{4\pi}{k+2}G(x_{m_1},x_{m_2})}\Big)\Big(\prod_m e^{-\frac{4\pi}{k+2}:G(x_m,x_m):}\langle b^{\alpha_m}, b^{\beta_m}\rangle(x_m)\Big) \\ &\times e^{-4\pi k(\int_{x_0}^x \operatorname{Im}\omega)(\operatorname{Im}\tau)^{-1}(\int_{x_0}^x \operatorname{Im}\omega)} \det{}'(\bar\partial_{L_x^2}^\dagger \bar\partial_{L_x^2}), \end{aligned} \tag{5.3}$$

where the numerical constant depends on the genus g and on the level k. The integration in (5.3) is over x_m, $m = 1,\dots,(k+1)(g-1)$, and over x, all in Σ. It is not difficult to check that upon the Weyl rescalings $\gamma \mapsto e^\sigma\gamma$ of the Riemannian metric on Σ, the right hand side of (5.3) (with the zeta-function regularized determinants) changes by the factor $e^{\frac{k}{8\pi i(k+2)}\int_\Sigma(\frac{1}{2}\partial\sigma\wedge\bar\partial\sigma+\sigma R)}$. This produces the correct value of the Virasoro central charge of the WZNW partition functions given by equation (1.4).

The arguments which led to equation (5.3) were clearly formal and the treatment of the Liouville integral might have looked particularly suspicious. Fortunately, one may do the calculation in a different, more satisfying manner. If the z-integral over $P\mathbb{C}^{3g-4}$ is done just after the w-integration and before the one over φ, then the final result is exactly as above but no infinite constants (apart from the Wick ordering ones) appear in the intermediate steps. Thus, it is rather the convergent integration over the (part of) the modular degrees of freedom, not the divergent φ_0 integral, which removes the cumbersome Liouville-type terms from the effective action for φ. This is an important lesson to learn from the above calculation. It is plausible that similar arguments may be used to substantiate the Goulian-Li trick [14] in the gravity case.

Similarly to the genus zero case discussed in [2], the natural conjecture is that the integral on the right hand side converges if and only if the function ψ defines a global non-singular CS state Ψ. It is clear from the form (5.3) of the scalar product that finiteness of the screening charge integral in (5.3) imposes, in general, conditions for the Taylor coefficients of $\psi(x,b)$ at $b=0$ if $g>2$. We shall postpone the study of these "fusion rule conditions" to a future work. The case $g=2$ is specially accessible, since there exists a simple global picture of the moduli space of $SL(2,\mathbb{C})$-bundles (it is the projectivization of the 4-dimensional space of degree 2 theta-functions)[1] [9] and of the space of CS states (homogeneous polynomials of order k on the same space).

If our conjecture is true, then formula (5.3) defines a Hermitian structure on the holomorphic vector bundle with the fibers given by the spaces of the CS states and the base by the moduli space of complex curves. Such a Hermitian structure induces a holomorphic Hermitian connection. This connection should coincide with the generalization to higher genera of the Knizhnik-Zamolodchikov connection studied in [16–18]. The latter has been constructed in [19] in geometric terms, and it is challenging to find an interpretation for equation (5.3) in terms of the moduli space geometry.

References

[1] Faddeev L. D., Popov V. N., *Feynman Diagrams for the Yang-Mills Field*, Phys. Lett. **B 26** (1967), 29–30.

[2] Gawędzki K., *Quadrature of Conformal Field Theories*, Nucl. Phys. **B 328** (1989), 733–752.

[3] Falceto F., Gawędzki K., Kupiainen A., *Scalar Product of Current Blocks in WZW Theory*, Phys. Lett. **B 260** (1991), 101–108.

[4] Gawędzki K., *Free field representation of the $SU(2)$ WZNW model at higher genera*, In preparation.

[5] Witten E., *Quantum Field Theory and Jones Polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

[6] Witten E., *Non-abelian bosonization in two dimensions*, Comm. Math. Phys. **92** (1984), 455–472.

[7] Verlinde E., *Fusion rules and modular transformations in 2-d conformal field theory*, Nucl. Phys. **B 300** (1988), 360–376.

[8] Seshadri C. S., *Space of Unitary Vector Bundles on a Compact Riemann Surface*, Ann. Math. **85** (1967), 303–336.

[9] Narasimhan M. S., Ramanan S., *Moduli of vector bundles on a compact Riemann surface*, Ann. Math. **89** (1969), 15–51.

[10] Quillen D., *Determinants of Cauchy-Riemann Operators over a Riemann surface*, Funct. Anal. Appl. **19** (1985), 31–34.

[11] Polyakov A. M., Wiegmann P. B., *Theory of non-Abelian Goldstone bosons in two dimensions*, Phys. Lett. **B 131** (1983), 121–126.

[12] Gawędzki K., Kupiainen A., *Coset Construction from Functional Integrals*, Nucl. Phys. **B 320** (1989), 625–668.

[13] Gupta A., Trivedi S. P., Wise M. B., *Random Surfaces in Conformal Gauge*, Nucl. Phys. **B 340** (1990), 475–490.

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[14] Goulian M., Li M., *Correlation Functions in Liouville Theory*, Phys. Rev. Lett. **66** (1991), 2051–2055.
[15] Witten E., *On Holomorphic Factorization of WZW and Coset Models*, Comm. Math. Phys. **144** (1992), 189–212.
[16] Bernard D., *On the Wess–Zumino–Witten Models on Riemann Surfaces*, Nucl. Phys. **B 309** (1988), 145–174.
[17] Gawędzki K., *Constructive Conformal Field Theory*, Functional Integration, Geometry and Strings (Z. Haba et J. Sobczyk, eds.), Birkhäuser, Basel, etc., 1989, pp. 277–302.
[18] Axelrod S., Della Pietra S., Witten E., *Geometric Quantization of Chern–Simons Gauge Theory*, J. Diff. Geom. **33** (1991), 787–902.
[19] Hitchin N., *Flat Connections and Geometric Quantization*, Comm. Math. Phys. **131** (1990), 347–380.

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