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ON THE ANALYSIS OF AN ENDOTHERMAL/EXOTHERMAL SATURATION PROBLEM

ABSTRACT. In this paper we consider an endo or exo-thermal saturation problem that corresponds to a parabolic quasi-variational inequality. By using regularity results and inequalities of Lewy–Stampacchia type we prove the solvability of a modified problem (including the Steklov averaging and the mollification of the saturation velocity) for the nonlinear case and also of the exact problem for the linear case with a small coefficient in the temperature equation.

§1. INTRODUCTION

The endo or exo-thermal saturation problem is an interesting example of a nonlinear coupled problem leading to a parabolic quasi-variational inequality. We investigate here the solvability in several special cases of a problem proposed by J. L. Lions [5], which is still an open problem in the general case.

We denote by $u = u(x,t)$ and $\theta = \theta(x,t)$, respectively, the saturation and the temperature of a substance at a point $x \in \Omega \subset \mathbb{R}^N$ and at an instant $t \in [0,T]$. We assume that the saturation lies between two thresholds

$$0 \leqslant u(x,t) \leqslant k(\theta(x,t)), \quad (x,t) \in Q = \Omega \times]0,T[, \tag{1.1}$$

where k is a given nonnegative function, and if $f = f(x,t)$ represents the volume injection of that substance, its saturation satisfies a simple diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{in the region} \quad \{(x,t) : 0 < u < k(\theta)\}, \tag{1.2}$$

while if one of the threshold is attained a sign condition is imposed, namely,

$$\frac{\partial u}{\partial t} - \Delta u - f \geqslant 0 \quad \text{in} \quad \{(x,t) : u = 0\}, \tag{1.3}$$

$$\frac{\partial u}{\partial t} - \Delta u - f \leqslant 0 \quad \text{in} \quad \{(x,t) : u = k(\theta)\}. \tag{1.4}$$

The temperature also solves a heat equation, but we assume the existence of a heating or cooling effect depending on the saturation velocity $\partial u/\partial t$, i.e., saturation is supposed to be endo or exo-thermal:

$$\frac{\partial \theta}{\partial t} - \Delta\theta = g\left(\frac{\partial u}{\partial t}\right) \quad \text{in} \quad Q, \tag{1.5}$$

where g is a (possible nonlinear) given function.

The coupled problem is completed by initial data and boundary conditions on $\Sigma = \partial\Omega\times]0,T[$, for instance, with $\gamma \geqslant 0$:

$$u = 0 \quad \text{on} \quad \Sigma, \quad u\big|_{t=0} = u_0 \quad \text{in} \quad \Omega, \tag{1.6}$$

$$\frac{\partial \theta}{\partial n} + \gamma\theta = h \quad \text{on} \quad \Sigma, \quad \theta\big|_{t=0} = \theta_0 \quad \text{in} \quad \Omega. \tag{1.7}$$

We observe that if $\frac{\partial u}{\partial t}$ is known, we many obtain the temperature θ by solving the linear problem (1.5)–(1.7). Hence, if we denote $\theta = G[\partial u/\partial t]$, being $G = \Gamma \circ g$ the operator composed by the function g and the heat Green operator, from the conditions (1.1)–(1.4), we see that u satisfies the following parabolic quasi-variational inequality (see [1])

$$\int\limits_Q \left(\frac{\partial u}{\partial t} - \Delta u - f\right)(v-u)dx\,dt \geqslant 0, \quad \forall v : 0 \leqslant v \leqslant k\left\{\Gamma\left[\left(\frac{\partial u}{\partial t}\right)\right]\right\}, \tag{1.8}$$

with the conditions (1.6). Of course, if k is constant, the problem reduces to well-known inequalities (see [3], for instance) but in general, even for smooth k and g, the nonlinear dependence of the upper obstacle on $\partial u/\partial t$ creates serious difficulties to the mathematical solvability of the coupled problem. In fact, the nonlinearity of g prevents us the use of weak topologies, while the strong one creates a compactness problem.

In order to overcome this last difficulty, it was suggested in [5] the approximation of $\partial u/\partial t$ by the differential quotient $[u(t)-u(t-\tau)]/\tau, \tau > 0$, but we may also consider, more generality, any other "regularization" of the saturation. We shall solve here the τ-modified problem, where (1.5) is replaced by

$$\frac{\partial \theta}{\partial t} - \Delta\theta = g(\delta_\tau, u) \quad \text{in} \quad Q, \tag{1.9$_\tau$}$$

where δ_τ is a suitable operator, but we shall also solve the exact linearized problem

$$\frac{\partial \theta}{\partial t} - \Delta\theta = \eta\frac{\partial u}{\partial t} \quad \text{in} \quad Q, \tag{1.10$_\eta$}$$

for small values of the parameter $|\eta|$.

In the next section we give the assumptions and the main new results, extending the existence theorem for the differential quotient case $(1.9)_\tau$ from [5], without the restriction $N \leqslant 2$ and allowing a sublinear of linear growth of g, in the case of a concave k, as well as, by letting $\tau \to 0$, obtaining the solvability of the exact problem $(1.10)_\eta$ for small $|\eta|$. The proofs are given in Sec. 4 and are based in regularity results for the two obstacles problem that are recalled in Sec. 3.

We ignore if the exact problem is solvable for nonlinear g or for arbitrary values of η in the linear case, but even the regularized problem for unbounded g may present serious difficulties as the one-dimensional numerical experiments of [6] have suggested.

§2. Existence results

In order to formulate rigorously our problems we suppose Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 1$, with a C^2-boundary $\partial\Omega$, and we shall use the Sobolev spaces $W_p^s(\Omega)$ and $W_p^{s,r}(Q)$, in $Q = \Omega \times]0,T[$, following the notations of [4], as well as their trace spaces and $H_0^1(\Omega) = \overset{\circ}{W}{}_2^1(\Omega)$, the closure in $W_2^1(\Omega)$ of the infinitely differentiable functions with compact support in Ω. We also consider $L^2(0,T;H_0^1(\Omega))$ as a Hilbert subspace of $W_2^{1,0}(Q)$, and the following convex subset

$$\mathbb{K}[\theta] = \{v \in L^2(0,T;H_0^1(\Omega)) : 0 \leqslant v(x,t) \leqslant k(\theta(x,t)), \quad \text{a.e.} \quad (x,t) \in Q\}, \tag{2.1}$$

for given functions $\theta = \theta(x,t)$ and $k = k(\tau)$.

For convenience we take $p, q \geqslant 2$ and we prescribe

$$k, g : \mathbb{R} \to \mathbb{R} \quad \text{are continuous functions, with} \quad k \geqslant 0, \tag{2.2}$$

$$f \in L^p(Q), \quad u_0 \in W_p^{2-2/p}(\Omega) \cap H_0^1(\Omega), \quad u_0 \geqslant 0 \quad \text{in} \quad \Omega, \tag{2.3}$$

$$\theta_0 \in W_q^{2-2/q}(\Omega), \quad k(\theta_0) \geqslant u_0 \quad \text{in} \quad \Omega, \tag{2.4}$$

$$h \in W_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{q}}(\Sigma) \text{ with } h\big|_{t=0} = \frac{\partial\theta_0}{\partial n} + \gamma\theta_0\big|_{\partial\Omega} \text{ if } q > 3, \tag{2.5}$$

where $q \neq 3$, to avoid additional technical assumptions.

For a given θ, we consider the variational inequality

$$u \in W_p^{2,1}(Q) \cap \mathbb{K}[\theta], \quad u\big|_{t=0} = u_0 \quad \text{in} \quad \Omega, \tag{2.6}$$

$$\int_Q \left(\frac{\partial u}{\partial t} - \Delta u - f\right)(v - u)\,dx\,dt \geqslant 0, \quad \forall v \in \mathbb{K}[\theta], \tag{2.7}$$

and we introduce the following two problems.

τ-Modified Problem $(M\tau)$. *: Find a pair* $\{u,\theta\} \in W_p^{2,1}(Q) \times W_q^{2,1}(Q)$, *where u solves* (2.6)–(2.7) *and* θ *solves* $(1.9)_\tau$–(1.7).

Here we suppose that in $(1.9)_\tau$ the operator δ_τ is defined, such that, for each parameter $\tau > 0$,

$$\delta_\tau : L^p(Q) \to L^p(Q) \quad \text{is continuous.} \tag{2.8}$$

Although δ_s does not need to be linear, we may consider, as examples, the Steklov averaing or the mollification of $\partial u/\partial t$:

$$\delta_\tau u(x,t) = \frac{u(,t-\tau) - u(x,t)}{\tau}, \tag{2.9}$$

$$\delta u(x,t) = \left(\rho_\tau * \frac{\partial u}{\partial t}\right)(x,t) = -\int_Q \frac{\partial}{\partial t}\rho_\tau(x-y,t-s)u(y,s)dy\,ds, \tag{2.10}$$

respectively, with $u(t) = u_0$ if $t \in [-\tau, 0]$ and $u = 0$ in $\mathbb{R} \setminus Q$, being ρ_τ a smooth nonnegative averaging kernel with compact support in a ball of radius $\tau > 0$, such that $\int_{\mathbb{R}^{N+1}} = 1$.

Theorem 1. *In addition to the assumptions* (2.2)–(2.5) *and* (2.8), *suppose* $k \in C^2$, $q > \frac{N+2}{2}$ *and that g is bounded function. Then there exist at least a solution to the tau-monifold problem* (M_τ)

$$\{u,\theta\} \in W_r^{2,1}(Q) \times W_q^{2,1}(Q),$$

$$\textit{with}\, r = \min\left(p, \frac{N+2}{2}\frac{q}{(N+2)-q}\right) \quad \textit{if} \quad q < N+2. \quad \bullet$$

In order to consider also the

Exact Problem (E_η). *: Find a pair* $(u,\theta) \in W_p^{2,1}(Q)$ *where u solves (2.6)–(2.7) and* θ *solves* $(1.10)_\eta$–(1.7) *, with* $\eta \in \mathbb{R}$.

we shall consider also the unbounded case for g under the subliner or linear growth conditions, $0 < \alpha \leqslant 1$:

$$\exists \eta_\alpha > 0 : |g(v)| \leqslant \eta_\alpha |v|^\alpha, \quad \forall v \in \mathbb{R}, \tag{2.11$_\alpha$}$$

with the additional condition on the operator δ_τ,

$$\begin{aligned} \delta_\tau : W_p^{2,1}(Q) \to L^p(Q) \quad &\text{is completely continuous} \\ \text{and} \exists d_\tau > 0 : \|\delta_\tau v\|_{L^p(Q)} &\leqslant d_\tau \|v\|_{W_p^{2,1}(Q)}, \end{aligned} \tag{2.12}$$

and the following assumptions on k:

$$|k'(v)| \leqslant k_1, \quad k''(v) \leqslant 0, \quad \forall v \in \mathbb{R}, \tag{2.13}$$

i.e., k is a uniformly Lipshitz continuous and concave function.

Theorem 2 (Sublinear growth). *Under the assumptions* (2.)–(2.5) *with* $p = q > 2(N+2)/(N+4)$, (2.8), $(2.11)_\alpha$, *with* $0 < \alpha < 1$, *and* (2.12)–(2.13) *there exists a solution* $\{u,\theta\} \in [W_p^{2,1}(Q)]^2$ *to the* τ*-modified problem* (M_τ). •

Finally in the linear case we need the additional assumption on the constant η_1 of $(2.11)_1$

$$\eta_1 < \frac{1}{k_1 C_p d_\tau}, \tag{2.14}$$

where d_τ and k_1 are the constants given respectively in (2.12) and (2.13) and $C_p > 0$ is the smallest constant in the estimate (see [4]], p. 621)

$$||v||_{W_p^{2,1}(q)} \leqslant C_p \left\{ ||F||_{L^p(Q)} + ||u_0||_{W_p^{2-2/p}(Q)} \right\} \tag{2.15}$$

for the solution v of the linear problem $\partial v/\partial t - \Delta v = F$ in Q with conditions (1.6), where C_p is independent of F and $u_0 (u_0|_{\partial\Omega} = 0)$.

We also note that in both cases (2.9) and (2.10) the constant $d_\tau = 1$.

Theorem 3 (Linear case). *Supposing* (2.13) *in addition to the assumption* (2.2)–(2.5), *with* $p = q > 2(N+2)/(N+4)$, (2.8), $(2.11)_1$ *and* (2.12)–(2.14), *there exists a solution* $\{u_\tau, \theta_\tau\} \in [W_p^{2,1}(Q)]^2$ *to the* τ*-modified problem* (M_τ). *Furthermore, if* δ_τ *is given by* (2.9) *or* (2.10) *and*

$$g(v) = \eta v, \quad \text{with} \quad |\eta| \leqslant \eta_1 < \frac{1}{k_1 C_p}, \tag{2.16}$$

as $\tau \to 0$, *tehre exists a sequence of solutions* $\{u_\tau, \theta_\tau\}$ *weakly converging in* $[W_p^{2,1}(Q)]^2$ *to* $\{u,\theta\}$, *which solves the Exact Problem* (E_η). •

Remark 1. The uniqueness in these problems is an question, even for small data. •

Remark 2. These results still for a more general nonlinearity $g = g(x,t,v) : Q \times \mathbb{R} \to \mathbb{R}$, provided g is a Carathéodory function satisfying, for instance, the assumption

$$|g(x,t,v)| \leqslant g_0(x,t) + \eta_\alpha |v|^\alpha, \quad \text{a.e.} \quad (x,t) \in Q, \quad v \in \mathbb{R},$$

with a prescribed $g_0 \in L^q(Q)$, or in the case (2.16) with $\eta = \eta(x,t)$ such that $||\eta||_{L^\infty(Q)} \leqslant \eta_1$. •

§3. AN AUXILIARY TWO-OBSTACLES PROBLEM

In this section we consider regular solutions to the following variational inequality for the heat operator $E(u) = \partial u/\partial t - \Delta u$:

$$u \in W_p^{2,1}(Q) \cap \mathbb{K}_0^\varphi, \quad u\big|_{t=0} = u_0 : \int_Q (Eu - f)(v-u)dx\,dt \geqslant 0, \forall v \in \mathbb{K}_0^\varphi, \tag{3.1}$$

where $f \in L^p(Q)$, $u_0 \in W_p^{2-2/p}(\Omega) \cap H_0^1(\Omega)$, $0 \leqslant u_0 \leqslant \varphi\big|_{t=0}$, $\varphi \in W_p^{2,1}(Q)$, $\varphi \geqslant 0$ in Q, are given functions. We denote $a \wedge b = \inf(a,b)$, $a^- = -(a \wedge 0)$ $a^+ = a + a^-$, and the convex set $\mathbb{K}_0^\varphi$ is defined by

$$\mathbb{K}_0^\varphi = \{v \in L^2(0,T; H_0^1(\Omega)) : 0 \leqslant v \leqslant \varphi \text{ a.e. in } Q\}.$$

Proposition 1. *There exists a unique solution u to (3.1) which, in addition, satisfies the following a.e. pointwise estimate*

$$f \wedge E\varphi \leqslant Eu \leqslant f^+, \quad a.e. \quad Q. \tag{3.2}$$

Proof. Set $f_0 = f^{-1}$ and $f_1(f - E\varphi)^+$ and, $\varepsilon > 0$, consider the following approximation H_ε of the Heaviside function:

$$H_\varepsilon(v) = 0 \quad \text{if} \quad v \leqslant 0, \quad H_\varepsilon(v) = \frac{v}{\varepsilon} \quad \text{if} \quad 0 \leqslant v \leqslant \varepsilon,$$
$$H_\varepsilon(v) = 1 \quad \text{if} \quad v \geqslant \varepsilon.$$

Then, by the Schauder fixed point theorem, and the linear theory for parabolic equations (see [4]), we can easily show that there exists a $u_\varepsilon \in W_p^{2,1}(Q) \cap L^2(0,T; H_0^1(\Omega))$, $u_\varepsilon\big|_{t=0} = u_0$ solving the semilinear parabolic equation

$$Eu_\varepsilon = f + F_\varepsilon u_\varepsilon \equiv f + f_0[1 - H_\varepsilon(u_\varepsilon)] - f_1[1 - H_\varepsilon(\varphi - u_\varepsilon)] \quad \text{in} \quad Q_T. \tag{3.3}$$

The solution u_ε unique, since F_ε is a monotone decreasing function. Also, by the weak maximum principle, we easily conclude that $-\varepsilon$ and $\varphi + \varepsilon$ are, respectively, a subsolution and a supersolution to (3.3), since we have $F_\varepsilon(-\varepsilon) = f_0 = f^{-1} \geqslant -f$ and $F_\varepsilon(\varphi + \varepsilon) = -f_1 = -(f - E\varphi)^+ \leqslant E\varphi - f$. Hence

$$-\varepsilon \leqslant u_\varepsilon \leqslant \varphi + \varepsilon, \quad \text{a.e. in} \quad Q. \tag{3.4}$$

Since the u_ε are bounded in $W_p^{2,1}(Q)$ independently of $\varepsilon \to 0$, we can suppose $u_\varepsilon \to u$ weakly in that space and strongly in $L^{p'}(Q)$ where, by (3.4), we have $u \in W_p^{2,1}(Q) \cap \mathbb{K}_0^\varphi$ and $u\big|_{t=0} = u_0$. Hence, taking the limit for any $v \in \mathbb{K}_0^\varphi$, in

$$\int_Q (Eu_\varepsilon - f)(v - u_\varepsilon) = \int_Q F_\varepsilon u_\varepsilon (v - u_\varepsilon) \geqslant -\varepsilon \int_Q (f_0 + f_1),$$

we conclude that u solves (3.1), whose uniqueness of its solution is immediate. Finally, from (3.3) we obtain the pointwise inequalities

$$(f \wedge E\varphi) - f = -(f - E\varphi)^+ \leqslant Eu_\varepsilon - f = F_\varepsilon u_\varepsilon \leqslant f^- = f^+ - f \quad \text{in} \quad Q,$$

from which (3.2) follows easily. •

Remark 3. Notice that the integral inequality of (3.1) is equivalent to the following one

$$\int_\Omega (Eu-f)(w-u)dx \geqslant 0, \quad \forall w \in H_0^1(\Omega), \quad 0 \leqslant w \leqslant \varphi(t), \quad \text{a.e.} \quad t \in]0,T[. \tag{3.1'}$$

Indeed, taking in (3.1) $v \in \mathbb{K}_0^\varphi$ defined by

$$v(\tau) = \begin{cases} w & \tau \in]t-\delta, t+\delta[, \\ u(\tau) & \tau \notin]t-\delta, t+\delta[, \end{cases}$$

with $0 < \delta < t$, and letting $\delta \to 0$ after multiplication by 2δ, we obtain (3.1′) a.e. $t \in]0,T[$. •

Remark 4. The solvability of (3.1) was first shown, in the case $p = 2$, by Brézis [2], without the inequality (3.2) of the Lewy–Stampacchia's type. The dirct approach of Proposition 1 follows [8] (see also [7], for references) and has the advantage to hold for every $p > 2(N+2)/(N+4)$, by the compact emdedding $W_p^{2,1}(Q) \subset L^{p'}(Q)$. Of course, in that case the assumption on the initial condition should be adjusted (see [4]). •

Corollary 1. *Denote by u_n and u the solutions of (3.1) corresponding to φ_n and φ, respectively, under the assumptions of Proposition 1. Then, the following weak continuous dependence result holds: if $\|f - E\varphi_n)^+\|_{L^p(Q)} \leqslant C$ independently of n, then*

$$\varphi_n \to \varphi \quad \text{in} \quad L^{p'}(Q) - \text{strong implies} \quad u_n \rightharpoonup u \quad \text{in} \quad W_p^{2,1}(Q) - \text{weak}. \tag{3.5}$$

Proof. Since (3.2) holds for every pair φ_n, u_n, we have u_n bounded in $W_p^{2,1}(Q)$ independently of n, and we may suppose, for a subsequence, that

$$u_n \rightharpoonup u^* \quad \text{in} \quad W_p^{2,1}(Q) - \text{weak and} \quad L^{p'}(Q) - \text{strong}.$$

Since $\varphi_n \to \varphi$ in $L^{p'}(Q)$-strong, we deduce $u^* \in \mathbb{K}_0^\varphi$, and for any $v \in \mathbb{K}_0^\varphi$, also $v_n = v \wedge \varphi_n = v - (v - \varphi_n)^+ \to v$ in $L^{p'}(Q)$. Then, taking these $v_n \in \mathbb{K}_0^{\varphi_n}$ in

$$\int_Q (Eu_n - f)(v_n - u_n)dx\,dt \geqslant 0,$$

and, taking the limit, we conclude that u^* also solves (3.1). Then, by uniqueness, we have $u^* = u$ and (3.5) follows. •

Remark 5. As in Remark 4 the compact embedding $W_p^{2,1}(Q) \subset L^{p'}(Q)$, for $p > 2(N+2)/(N+4)$, plays an essential role in the continuous dependence property (3.5). Clearly this result also holds in the case $\psi_n \to \psi$ in $W_p^{2,1}(Q)$-weak, $f_n \rightharpoonup f$ in $L^p(Q)$-weak and $u_{0n} \rightharpoonup u_0$ in $\overset{\circ}{W}{}^{2-2/p}(\Omega)$-weak.

Remark 6. As in Theorem 5.4.3 of [7], the regular solution u of (3.1) solves also the nonlinear equation

$$Eu = f + f^{-1}\chi_{\{u=0\}} - f(f - E\varphi)^+\chi_{\{u=\varphi\}} \quad \text{a.e. in} \quad Q,$$

where $\chi_{\{u=0\}}$ and $\chi_{\{u=\varphi\}}$ denote the characteristic function of the coincidence sets $\{u = 0\}$ and $\{u = \varphi\}$, respectively. Also, analogously to Theorem 5.4.5 of [7], the continuous dependence (3.5) also holds for the strong topologies. •

§4. Proofs

The existence proof is based in well-known fixed point theorems (see [1], for instance) and in the auxiliary two-obstacle problem (3.1) where we set $\varphi = k(\theta)$. We observe that the *a priori* estimate (3.2), for a $k \in C^2$, implies in particular, the a.e. pointwise lower bound in Q:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u \geqslant -f^{-1} - \left[\frac{\partial k(\theta)}{\partial t} - \Delta k(\theta)\right]^- \geqslant \\ &\geqslant -f^{-1} - \left[k'(\theta)\left(\frac{\partial \theta}{\partial t} - \Delta\theta\right) - k''(\theta)|\nabla\theta|^2\right]^- . \end{aligned} \tag{4.1}$$

4.1. Proof of Theorem 1.

We consider the nonlinear mapping $S : L^r(Q) \to L^r(Q)$ defined as follows: for an arbitrary $w \in L^r(Q)$ we denote by θ_w the unique solution of the linear heat equation with nonhomogeneous term $g_w = g(\delta_\tau w)$ and boundary initial conditions (1.7); then we consider the unique solution u_w to the parabolic variational inequality (3.1) with $\varphi = k(\theta_w)$. If we set $u_w = Sw$, the fixed point u of S with $\theta = \theta_u$, clearly solve the τ-modified problem.

The mapping S is well defined since $\theta_w \in W_q^{2,1}(Q)$ with $q > \frac{N+2}{2}$ yields also $k(\theta_w) \in W_q^{2,1}(Q)$. Indeed, by Sobolev imbeddings $W_q^{2,1}(Q) \subset C^0(\bar{Q})$, implying that $k'(\theta_w)$ and $k''(\theta_w)$ are also bounded functions in Q, and $W_q^{2,1}(Q) \subset W_s^{1,0}(Q)$, for any $s < \infty$ if $q > N+2$ or $s \leqslant q(N+2)/[(N+2) - q]$ otherwise; consequently we have also

$$\frac{\partial^2 k(\theta_w)}{\partial x_i \partial x_j} = k'(\theta_w)\frac{\partial^2 \theta_w}{\partial x_i \partial x_j} + k''(\theta_w)\frac{\partial \theta_w}{\partial x_i}\frac{\partial \theta_w}{\partial x_j} \in L^q(Q),$$

since $q \geqslant (N+2)/2$. The complete continuity of S follows then easily, by the continuity of δ_τ and of the Nemytskii operators associated with g, k, k' and k'', and the results of Corollary 1 combined with the compactness of the embedding $W_r^{2,1}(Q) \subset L^r(Q)$. In fact, using the estimates (3.2) and (4.1) we find

$$\left\| \frac{\partial u_w}{\partial t} - \Delta u_w \right\|_{L^r(Q)} \leqslant \|f\|_{L^r(Q)} + \|k'(\theta_w)\|_{L^\infty(Q)} \|g(\delta_\tau w)\|_{L^r(Q)} + \\ + \|k''(\theta_w)\|_{L^\infty(Q)} \| \, \||\nabla \theta_w|^2\|_{L^r(Q)}, \tag{4.2}$$

with $r = \min\left(p, \frac{N+2}{2} \frac{q}{(N+2)-q}\right)$ if $q < N+2$, by the Sobolev embedding $W_q^{2,1}(Q) \subset W_{2r}^{1,0}(Q)$.

On the other hand, being $|g|$ bounded, say by a constant η_0, not only $\|g(\delta_\tau w)\|_{L^r(Q)}$ but also θ_w is, by the linear parabolic estimates (see [4], p. 621), *a priori* bounded in $W_q^{2,1}(Q)$ independently of $w \in L^r(Q)$. Consequently, by (4.2) and Proposition 1, u_w is also bounded in $W_r^{2,1}(Q)$ independently of w, and we conclude that $S(L^r(Q)) \subset B_R = \{v \in L^p(Q) : \|v\|_{L^r(Q)} \leqslant R\}$ for some $R > 0$. The conclusion follows then by the Schauder fixed point theorem.

4.2. Proof of Theorem 2.

Suppose firsty that $p = q > \frac{N+2}{2}$ and $k \in C^2$. As in the proof of the preceding case we construct a nonlinear mapping $\mathcal{T} : W_p^{2,1}(Q) \to W_p^{2,1}(Q)$ by solving the linear heat equation for θ_w with $g = g(\delta_\tau w)$ and then we set $u_w = \mathcal{T} w \in W_p^{2,1}(Q) \cap \mathbb{K}_0^{k(\theta_w)}$ as the unique solution of (3.1). The conclusion of Theorem 2 will be a consequence of Tychonov's fixed point theorem applied to $\mathcal{T}$ in the weakly compact convex set

$$C_R = \{v \in W_p^{2,1}(Q) : \|v\|_{W_p^{2,1}(Q)} \leqslant R\} \tag{4.3}$$

for a suitable $R > 0$.

Using the assumptions $(2.11)_\alpha$–(2.13) in the estimate (4.1), as in (4.2) we find

$$\left\| \frac{\partial u_w}{\partial t} - \Delta u_w \right\|_{L^p(Q)} \leqslant \|f\|_{L^p(Q)} + k_1 \|g(\delta_\tau w)\|_{L^p(Q)} \leqslant \\ \leqslant \|f\|_{L^p(Q)} + k_1 d_\tau \eta_\alpha \|w\|^\alpha_{W_p^{2,1}(Q)}. \tag{4.4}$$

Then, recalling the estimate (2.25), from (4.4) we obtain

$$\|u_w\|_{W_p^{2,1}(Q)} \leqslant C + k_1 d_\tau \eta_\alpha C_p \|w\|^\alpha_{W_p^{2,1}(Q)}, \tag{4.5}$$

where we set $C = C_p\|f\|_{L^p(Q)} + C_p\|u_0\|_{W_p^{2-2/p}(\Omega)}$. From (4.5), since $0 < \alpha < 1$, by choosing $R \geqslant R_\alpha \equiv \frac{C}{1-\alpha} + (k_1 d_\tau \eta_\alpha C_p)^{\frac{1}{1-\alpha}}$, for $w \in \mathcal{C}_R$ we immediately conclude that $u_w \in \mathcal{C}_R$.

Hence $\mathcal{T}(\mathcal{C}_R \subset \mathcal{C}_R$ and the continuity of $\mathcal{T}$ for the weak topology of $W_p^{2,1}(Q)$ follows easily from the compactness of δ_τ and the Corollary 1.

In order to drop the initial restrictions on k and p we just recall the Remark 5 and the estimate (4.5) for each solution $\{u_n, \theta_n\}$ corresponding to each $k_n \in C^2$ such that $k_n \underset{n}{\longrightarrow} k$ uniformly in compact sets and satisfying the assumption (2.13),

$$\|u_n\|_{W_p^{2,1}(Q)} \leqslant \frac{C}{1-\alpha} + (k_1 d_\tau \eta_\alpha C_p)^{\frac{1}{1-\alpha}}$$

which is, of course, independent of n.

Hence, for a subsequence, we have

$$\{u_n, \theta_n\} \rightharpoonup \{u, \theta\} \quad \text{in} \quad W_p^{2,1}(Q) - \text{weak and} \quad L^{p'}(Q) - \text{strong}, \tag{4.6}$$

and applying Corollary 1, with $\varphi_n = k_n(\theta_n)$ and $\varphi = k(\theta)$, and observing that, by assumptions $(2.11)_\alpha$–(2.13)

$$\|[f - E(k_n(\theta_n))]^+\|_{L^p(Q)} \leqslant \|f\|_{L^p(Q)} + k_1\|g(\delta_\tau u_n)\|_{L^p(Q)} \leqslant C, \tag{4.7}$$

where $C > 0$ is independent of n, we conclude that the weak limit $\{u, \theta\}$ solves the problem corresponding to k.

4.3. Proof of Theorem 3.

The existence of a solution $\{u_\tau, \theta_\tau\}$ to the τ-modified problem is similar to the previous one. It is enough to choose now $R \geqslant R_1 \equiv C/(1 - k_1 d_\tau \eta_1 C_p) > 0$, which is possible by the assumption (2.14) and the estimate (4.5) for $\alpha = 1$.

In particular, for the case (2.16) with δ_τ given by (2.9) or (2.10), $d_\tau = 1$, and we have

$$\|u_\tau\|_{W_p^{2,1}(Q)} \leqslant C' = \frac{C}{1 - k_1\eta_1 C_p} \quad \text{and} \quad \|\theta_\tau\|_{W_r^{2,1}(Q)} \leqslant C'', \tag{4.8}$$

where the constants C' and C'' do not depend on τ.

Hence we can extract a subsequence $\tau_n \to 0$, such that (4.6) and the equivalent estimates to (4.7) for $\{u_n, \theta_n\} = \{u_{\tau_n}, \theta_{\tau_n}\}$ hold, uniformly in τ_n by the estimate (4.8).

Thenapplying again Corollary 1 and using Lemma 1 below we conclude the convergence towards a solution to the Exact Problem. •

Lemma 1. *Let $\delta_\tau u^\tau$ be defined by (2.9) or (2.10) for a sequence of functions such that, for some constant $C > 0$,*

$$\|\delta_\tau u^\tau\|_{L^p(Q)} \leqslant C \quad \text{and} \quad u^\tau \underset{\tau\to 0}{\rightharpoonup} u \quad in \quad W_p^{0,1}(Q) - \text{weak}.$$

Then, we have $\delta_\tau u^\tau \underset{\tau\to 0}{\rightharpoonup} \frac{\partial u}{\partial t}$ in $L^p(Q)$-weak ($1 < p < \infty$).

Proof. For a subsequence we may suppose that

$$\delta_\tau u^\tau \rightharpoonup u^* \quad \text{in} \quad L^p(Q) - \text{weak}.$$

We conclude that $u^* = \frac{\partial u}{\partial t}$, by considering an arbitrary $\varphi \in \mathcal{D}(Q)$ and, respectively for (2.9) and (2.10), passing to the limit in

$$\int_Q \delta_\tau u^\tau \varphi = -\int_Q u^\tau \delta_{-\tau} \underset{\tau\to 0}{\rightharpoonup} -\int_Q u\frac{\partial\varphi}{\partial t} = \int_Q \frac{\partial u}{\partial t}\varphi$$

and

$$\int_Q \delta_\tau u^\tau \varphi = \int_Q \left(\frac{\partial u^\tau}{\partial t} * \rho_\tau\right)\varphi = \int_Q \frac{\partial u^\tau}{\partial t}(\varphi * \rho_\tau) \underset{\tau\to 0}{\rightharpoonup} \int_Q \frac{\partial u}{\partial t}\varphi. \quad \bullet$$

Remark 7. Arguing as in the proofs of Theorems 2 and 3 we can solve directly the exact promlem by applying the Tychonov's fixed point theorem to $\mathcal{T} : C_{R_1} \ni\mapsto u_w \in C_{R_1}$ constructed directly with $g = \eta\frac{\partial w}{\partial t}$ in the equation for θ_w, under the restriction (2.16). •

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