Алгебра и анализ
Том 30 (2018), №3

Dedicated to the memory of M. Z. Solomyak

BOUND ON THE NUMBER OF NEGATIVE EIGENVALUES OF TWO-DIMENSIONAL SCHRÖDINGER OPERATORS ON DOMAINS

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A fundamental result of Solomyak says that the number of negative eigenvalues of a Schrödinger operator on a two-dimensional domain is bounded from above by a constant times a certain Orlicz norm of the potential. Here it is shown that in the case of Dirichlet boundary conditions the constant in this bound can be chosen independently of the domain.

§1. Introduction and main result

In this paper we are interested in an upper bound on the number of negative eigenvalues of a Schrödinger operator in two space dimensions. In order to state our main result, we introduce the functions

$$\mathcal{A}(t) := e^t - 1 - t \qquad \text{and} \qquad \mathcal{B}(t) := (1+t)\ln(1+t) - t \qquad \text{for } t \geqslant 0.$$

These functions are convex and are Legendre transforms of each other. For a measurable set $\Omega \subset \mathbb{R}^2$ of positive and finite measure we denote by $L^{\mathcal{B}}(\Omega)$ the space of measurable functions g on Ω (identifying functions equal almost everywhere) for which

$$\|g\|_{\mathcal{B},\Omega} := \sup\left\{\left|\int_\Omega fg\,dx\right| :\ f : \Omega \to \mathbb{C} \text{ is measurable and } \int_\Omega \mathcal{A}(|f|)\,dx \leqslant |\Omega|\right\}$$

is finite. Also, $-\Delta_\Omega^D$ denotes the Dirichlet Laplacian in Ω. We will see that for real $V \in L^{\mathcal{B}}(\Omega)$ the operator $-\Delta_\Omega^D + V$, defined in a quadratic form sense,

Key words: Schrödinger operator, Dirichlet Laplacian, Neumann Laplacian, Trudinger inequality.

Partially supported by the U.S. National Science Foundation through grant DMS-1363432 (R.L.F.) and by a grant of the Russian Federation Government under the supervision of a leading scientist at the Siberian Federal University, grant № 14.Y26.31.0006 (A.L.).

is bounded from below and we are interested in the number of its negative eigenvalues. For any selfadjoint lower semibounded operator H we denote by $N(H)$ the number of negative eigenvalues, counting multiplicities.

Our main result is as follows.

Theorem 1.1. *There is a constant C such that for any open set $\Omega \subset \mathbb{R}^2$ of finite measure and any real $V \in L^{\mathcal{B}}(\Omega)$, we have*

$$N(-\Delta_\Omega^D + V) \leqslant C\|V_-\|_{\mathcal{B},\Omega}. \tag{1.1}$$

Here we use the notation $V(x)_- = \max\{0, -V(x)\}$ for the negative part.

The main point of this paper is that the constant C in (1.1) can be chosen independently of Ω. In the fundamental paper [15] Solomyak showed that

$$N(-\Delta_\Omega^N + V) \leqslant 1 + C_\Omega\|V_-\|_{\mathcal{B},\Omega} \tag{1.2}$$

for the Neumann Laplacian $-\Delta_\Omega^N$ and for bounded, connected open sets $\Omega \subset \mathbb{R}^2$ with the extension property. (This is essentially Theorem $4'$ in [15].) The constant C_Ω in (1.2) depends on Ω. From this bound and the ideas in its proof, one can obtain (1.1) for bounded open sets $\Omega \subset \mathbb{R}^2$, but again with a constant that depends on Ω. The proof that this constant can, in fact, be chosen independently of Ω needs additional ingredients, which we provide in this paper. We also mention that conversely, it is easy to deduce (1.2) from (1.1), but then a dependence on Ω enters through the use of an extension operator.

A standard application of Theorem 1.1, even with a domain dependent constant, yields the validity of Weyl asymptotics in the strong coupling limit for rough potentials. It is crucial here that the right side of (1.1) is homogeneous of degree one with respect to V. This is a distinguishing feature of (1.1) and (1.2) compared to other bounds discussed below.

Corollary 1.2. *If $\Omega \subset \mathbb{R}^2$ is an open set of finite measure and $V \in L^{\mathcal{B}}(\Omega)$ is real, then*

$$\lim_{\alpha\to\infty} \alpha^{-1} N(-\Delta_\Omega^D + \alpha V) = \frac{1}{4\pi} \int_\Omega V(x)_- \, dx.$$

The basic strategy in our proof of Theorem 1.1 is similar to that of Solomyak's proof of (1.2), which, in turn, follows the strategy of Rozenblum's proof of the CLR bound [12, 13]. The set Ω is covered by squares chosen in such a way that the Schrödinger operator on each square with Neumann boundary conditions has at most one negative eigenvalue. Therefore, the number of negative eigenvalues is bounded by the number of squares and it remains to bound the latter number. In contrast to earlier works, Rozenblum used overlapping squares and the selection of the squares proceeds by the Besicovitch covering lemma. In order to guarantee that the Schrödinger operator on each square

has at most one negative eigenvalue, the squares are chosen so that the norm of V on each square is equal to a given small constant. This construction is considerably more difficult in the two-dimensional case (where one deals with an Orlicz norm) than in the higher-dimensional case (where one deals with the $L^{d/2}$-norm). In fact, the choice of the (nonstandard) norm $\|\cdot\|_{\mathcal{B},\Omega}$, which has suitable superadditivity properties, was one of the key insights in Solomyak's work.

The new difficulty that we have to face in this work is that intersections of squares with Ω have essentially an arbitrary shape and that our constants must be in a certain sense uniform with respect to this shape. In contrast, Solomyak needed only to consider intersections of squares with each other, that is, rectangles, and the necessary uniformity follows in a rather straightforward manner. In our case we obtain uniformity by carefully reviewing the proof of the Trudinger inequality.

We end this introduction by placing our result in the context of eigenvalue bounds for Schrödinger operators. As is well known, the two-dimensional case is a borderline case and is still not as well understood as the case of three and higher dimensions and the one-dimensional case. Recently, there have been several results on the two-dimensional case, see [4, 8, 14, 3, 6, 7]. Solomyak's pioneering paper [15] had a profound influence on these developments and we would like to dedicate our results here, which are also a variation on the theme in [15], to his memory.

In order to understand the particularity of the two-dimensional case, we recall that in dimensions $d \geqslant 3$ the number $N(-\Delta+V)$ of negative eigenvalues of the Schrödinger operator $-\Delta+V$ in $L^2(\mathbb{R}^d)$ is bounded by the Cwikel–Lieb–Rozenblum inequality as

$$N(-\Delta+V) \leqslant C_d \int\limits_{\mathbb{R}^d} V(x)_-^{d/2}\,dx, \tag{1.3}$$

where C_d is a constant depending only on d. This inequality should be compared with the Weyl asymptotics

$$\lim_{\alpha\to\infty} \alpha^{-d/2} N(-\Delta+\alpha V) = \frac{|\{\xi\in\mathbb{R}^d:\ |\xi|\leqslant 1\}|}{(2\pi)^d} \int\limits_{\mathbb{R}^d} V(x)_-^{d/2}\,dx. \tag{1.4}$$

These asymptotics are initially proved for continuous, compactly supported V and then extended, using (1.3), to any $V\in L^{d/2}(\mathbb{R}^d)$. Moreover, from (1.3) it is easy to deduce that if for $V\leqslant 0$ one has $\limsup_{\alpha\to\infty} \alpha^{-d/2} N(-\Delta+\alpha V)<\infty$, then $V\in L^{d/2}(\mathbb{R}^d)$.

The analog of (1.3) is not true in dimension two. Indeed, in two dimensions for any potential $V \not\equiv 0$ with $\int_{\mathbb{R}^2} V(x)\,dx \leqslant 0$ the operator $-\Delta + V$ has a negative eigenvalue. However, not even a modified bound with the right side in (1.3) replaced by $C(1 + \int_{\mathbb{R}^2} V(x)_-\,dx)$ can be fulfilled. In fact, a more subtle failure of (1.3) was discovered in [1]. Namely, for any $q > 1$ there is a potential $V \in L^1(\mathbb{R}^2)$ such that $\lim_{\alpha\to\infty} \alpha^{-q} N(-\Delta + \alpha V)$ exists and is finite and positive. Since a bound like (1.3) even with a constant added to the right side, would imply that $N(-\Delta + \alpha V)$ grows at most linearly with α, no such bound can be true. In fact, the modified asymptotics is valid for any long-range potential that behaves like

$$-|x|^{-2}(\ln|x|)^{-2}(\ln\ln|x|)^{-1/q} \quad \text{as} \quad |x| \to \infty.$$

Moreover, if $q = 1$ in these examples, then $\lim_{\alpha\to\infty} \alpha^{-1} N(-\Delta + \alpha V)$ exists, but is different from the right side of (1.4). We also mention that not only the slow decay at infinity can give rise to modified asymptotics but also strong (but integrable) local singularities. This can be understood via the conformal invariance of the problem. More specifically, the same modified asymptotics hold for any potential behaving like

$$-|x|^{-2}|\ln|x||^{-2}(\ln|\ln|x||)^{-1/q} \quad \text{as} \quad |x| \to 0.$$

Note that the last function belongs to $L^{\mathcal{B}}(\Omega)$ for an open set Ω containing 0 if and only if $q < 1$, so the modified asymptotics for $q \geqslant 1$ do not contradict our Corollary 1.2. We also note that these examples with local singularities show that even for the number of eigenvalues of $-\Delta + V$ below some fixed number $E < 0$, there cannot be a bound of the form $C_E(1 + \int_{\mathbb{R}^2} V(x)_-\,dx)$.

This discussion raises the question of characterizing all $V \in L^1(\mathbb{R}^2)$ (or all $0 \geqslant V \in L^1(\mathbb{R}^2)$) either such that $\limsup_{\alpha\to\infty} \alpha^{-1} N(-\Delta + \alpha V) < \infty$, or such that (1.4) is true. This problem was solved in the radial case in [6], but is still open in general. The eigenvalue bounds in [15, 4, 14, 3, 7] can be understood as sufficient conditions for an asymptotically linear bound. Other, faster growing bounds can be found, for instance, in [8, 17, 19].

This paper is organized as follows. In §2 we explain the strategy of the proof of Theorem 1.1 and Corollary 1.2 and reduce it to two main ingredients, namely a Sobolev-type inequality and a covering argument, which will be discussed in §3 and 4, respectively. We present all facts about Orlicz spaces that are relevant for us in Appendix A and include a proof of the Besicovitch lemma in Appendix B.

Note added in proof. After a first version of this paper was submitted, Grigori Rozenblum suggested to us that the method of his paper [11] might lead to an alternative proof of Theorem 1.1. We provide the details of this

argument in §2. (In contrast to [11] we avoid the use of capacities.) We emphasize that this argument takes Solomyak's inequality (1.2) *for squares* as an input.

Acknowledgments. The authors are grateful to Timo Weidl for extensive discussions related to this material and to Grigori Rozenblum for many helpful remarks on the manuscript.

§2. Strategy of the proof

The proof of Theorem 1.1 is based on two ingredients, namely, Sobolev-type inequalities and a covering argument. We present these ingredients in this section and then explain how to derive Theorem 1.1 from them.

Throughout, by a norm $\|\cdot\|$ without subscript we mean the L^2 norm.

Proposition 2.1. *There is a constant $S_2 > 0$ such that for any open set $\Omega \subset \mathbb{R}^2$ of finite measure and any $u \in H^1_0(\Omega)$, we have*

$$\int_\Omega \mathcal{A}(S_2|u|^2/\|\nabla u\|^2)\,dx \leqslant |\Omega|. \tag{2.1}$$

Moreover, there is a constant $S_2' > 0$ such that for any open set $\Omega \subset \mathbb{R}^2$, any open square $Q \supset \Omega$, and any $u \in H^1(\Omega)$ that vanishes near $Q \cap \partial\Omega$ and satisfies $\int_\Omega u\,dx = 0$, we have

$$\int_\Omega \mathcal{A}(S_2'|u|^2/\|\nabla u\|^2)\,dx \leqslant |\Omega|. \tag{2.2}$$

The crucial point for us is that the constants S_2 and S_2' do not depend on Ω. Note that we can take $\Omega = Q$ in the second part of the proposition, and then (2.2) becomes

$$\int_Q \mathcal{A}(S_2'|u|^2/\|\nabla u\|^2)\,dx \leqslant |Q| \tag{2.3}$$

for any $u \in H^1(Q)$ with $\int_Q u\,dx = 0$. This inequality, however, is weaker than the second part of the proposition. Indeed, while it is true that the functions $u \in H^1(\Omega)$ that vanish near $Q \cap \partial\Omega$ and satisfy $\int_\Omega u\,dx = 0$ can be extended by zero to functions in $H^1(Q)$ with mean value zero, applying (2.3) to this extension gives (2.2) only with $|Q|$ on the right side and not with $|\Omega|$. This would not be sufficient for our purposes. The proof of Proposition 2.1 will be discussed in §3.

Remark 2.2. For given $\theta \in (0,1)$, there is a constant S_2' such that (2.2) holds if the assumption $\int_\Omega u\,dx = 0$ is replaced by the assumption $|Q \setminus \Omega| \geqslant \theta|Q|$. This will also be proved in §3.

The second ingredient in the proof of Theorem 1.1 is the following covering result. By a *square* we always mean an open square with edges parallel to the coordinate axes, and by a *covering* of a set $K \subset \mathbb{R}^2$ by squares $Q_1, \dots, Q_M$ we mean that $K \subset \bigcup_m Q_m$. The *multiplicity* of such a covering is

$$\sup_{x \in \mathbb{R}^2} \#\{m : \ x \in Q_m\}.$$

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^2$ be an open set of finite measure, and let $0 \leqslant W \in L^{\mathcal{B}}(\Omega)$ have compact support. Then for any $0 < A \leqslant \|W\|_{\mathcal{B},\Omega}$ there is a covering of Ω by open squares $Q_1, \dots, Q_M \subset \mathbb{R}^2$ of multiplicity at most 4 such that*

$$\|W\|_{\mathcal{B},Q_m \cap \Omega} = A \quad \text{for all} \quad 1 \leqslant m \leqslant M. \tag{2.4}$$

Moreover,

$$M \leqslant 17 A^{-1} \|W\|_{\mathcal{B},\Omega}. \tag{2.5}$$

The proof of this proposition will be discussed in §4.

Having introduced our tools, we are now in a position to prove Theorem 1.1

Proof of Theorem 1.1. First, we assume that $\operatorname{supp} V_-$ is bounded. We denote by S_2 and S_2' the constants from Proposition 2.1. If $\|V_-\|_{\mathcal{B},\Omega} \leqslant S_2$, then we can use the first part of that proposition as well as the definition of $\|\cdot\|_{\mathcal{B},\Omega}$ to show that, for any $u \in H_0^1(\Omega)$,

$$\begin{aligned} \int_\Omega \left(|\nabla u|^2 + V|u|^2\right) dx &\geqslant \|\nabla u\|^2 \left(1 - S_2^{-1} \int_\Omega V_- \frac{S_2|u|^2}{\|\nabla u\|^2}\, dx\right) \\ &\geqslant \|\nabla u\|^2 \left(1 - S_2^{-1}\|V_-\|_{\mathcal{B},\Omega}\right) \geqslant 0. \end{aligned}$$

Thus, $N(-\Delta_\Omega^D - V) = 0$ and the theorem follows in this case.

Now we assume that $\|V_-\|_{\mathcal{B},\Omega} > S_2$. We apply Proposition 2.3 with $W = V_-$ and $A = \min\{4^{-1}S_2', S_2\}$. We obtain a covering of Ω by open squares $Q_1, \dots, Q_M$ of multiplicity at most 4 such that

$$\|V_-\|_{\mathcal{B},Q_m \cap \Omega} \leqslant 4^{-1} S_2' \quad \text{for all} \quad m = 1, \dots, M$$

and

$$M \leqslant 17 \max\left\{4(S_2')^{-1}, S_2^{-1}\right\} \|V_-\|_{\mathcal{B},\Omega}.$$

Thus, for any $u \in H_0^1(\Omega)$ satisfying the orthogonality conditions

$$\int_{Q_m} u\,dx = 0 \quad \text{for all} \quad m = 1, \dots, M,$$

we can bound, using the second part of Proposition 2.1,

$$\begin{aligned}\int_\Omega \big(|\nabla u|^2 + V|u|^2\big)dx &\geqslant \sum_{m=1}^M \int_{Q_m} \Big(\frac{1}{4}|\nabla u|^2 - V_-|u|^2\Big)dx \\ &= \frac{1}{4}\sum_{m=1}^M \|\nabla u\|^2_{L^2(Q_m)}\Big(1 - \frac{4}{S_2'}\int_{Q_m\cap\Omega} V_- \frac{S_2'|u|^2}{\|\nabla u\|^2_{L^2(Q_m)}}\,dx\Big) \\ &\geqslant \frac{1}{4}\sum_{m=1}^M \|\nabla u\|^2_{L^2(Q_m)}\Big(1 - \frac{4}{S_2'}\|V_-\|_{\mathcal{B},Q_m\cap\Omega}\Big) \geqslant 0.\end{aligned}$$

By the variational principle, this implies that

$$N(-\Delta^D_\Omega - V) \leqslant M,$$

and the upper bound on M from the covering proves the theorem in the case where $\operatorname{supp} V_-$ is bounded.

In the general case, we fix $\varepsilon > 0$ and, by an argument similar to that in Lemma A.4, we choose $R > 0$ such that

$$W := \mathbb{1}_{\{|x|<R\}}V_-$$

satisfies

$$\|W - V_-\|_{\mathcal{B},\Omega} \leqslant \varepsilon S_2.$$

Then for any $u \in H^1_0(\Omega)$ we have

$$\begin{aligned}\int_\Omega \big(|\nabla u|^2 + V|u|^2\big)\,dx &\geqslant (1-\varepsilon)\int_\Omega \big(|\nabla u|^2 - (1-\varepsilon)^{-1}W|u|^2\big)\,dx \\ &\quad + \varepsilon\int_\Omega \big(|\nabla u|^2 - \varepsilon^{-1}(V_- - W)|u|^2\big)\,dx \\ &\geqslant (1-\varepsilon)\int_\Omega \big(|\nabla u|^2 - (1-\varepsilon)^{-1}W|u|^2\big)\,dx \\ &\quad + \varepsilon\int_\Omega |\nabla u|^2\,dx\,\big(1 - (\varepsilon S_2)^{-1}\|V_- - W\|_{\mathcal{B},\Omega}\big) \\ &\geqslant (1-\varepsilon)\int_\Omega \big(|\nabla u|^2 - (1-\varepsilon)^{-1}W|u|^2\big)\,dx.\end{aligned}$$

By the variational principle, this implies

$$N(-\Delta^D_\Omega + V) \leqslant N(-\Delta^D_\Omega - (1-\varepsilon)^{-1}W),$$

and by the first part of the proof this can be bounded by

$$C(1-\varepsilon)^{-1}\|W\|_{\mathcal{B},\Omega} \leqslant C(1-\varepsilon)^{-1}\|V_-\|_{\mathcal{B},\Omega},$$

where the last inequality follows easily from the definition of the $\|\cdot\|_{\mathcal{B},\Omega}$ norm (see also the proof of Lemma A.2). The proof of Theorem 1.1 is complete. □

Alternative proof of Theorem 1.1. In accordance with [11, Lemma 1], there are pairwise disjoint open squares Q_m whose closures cover Ω and such that

$$\frac{1}{5}|Q_m| \leqslant |Q_m \cap \Omega| \leqslant \frac{4}{5}|Q_m| \qquad \text{for all } m.$$

By the variational principle,

$$N(-\Delta_\Omega^D + V) \leqslant \sum_m N(-\Delta_{Q_m\cap\Omega}^{DN} + V), \tag{2.6}$$

where $-\Delta_{Q_m\cap\Omega}^{DN}$ has Dirichlet boundary conditions on $Q_m \cap \partial\Omega$ and Neumann boundary conditions on $\Omega \cap \partial Q_m$. Arguing as at the beginning of the first proof of Theorem 1.1, we deduce that

$$N(-\Delta_{Q_m\cap\Omega}^{DN} + V) = 0 \qquad \text{if} \quad \|V_-\|_{\mathcal{B},Q_m} \leqslant S_2',$$

where S_2' comes from Remark 2.2 with $\theta = 1/5$. Here we use the fact that by the choice of Q_m we have $|Q_m \setminus \Omega| \geqslant (1/5)|Q_m|$. On the other hand, by the variational principle (extending V by zero to Q_m) and Solomyak's bound (1.2),

$$N(-\Delta_{Q_m\cap\Omega}^{DN} + V) \leqslant N(-\Delta_{Q_m}^{N} + V) \leqslant 1 + C\|V_-\|_{\mathcal{B},Q_m}.$$

We emphasize that, by scaling, the constant C here is independent of m. By Lemma A.2 and the choice of Q_m,

$$\|V_-\|_{\mathcal{B},Q_m} \leqslant \frac{|Q_m|}{|Q_m\cap\Omega|}\|V_-\|_{\mathcal{B},Q_m\cap\Omega} \leqslant 5\|V_-\|_{\mathcal{B},Q_m\cap\Omega}.$$

Combining these two bounds on $N(-\Delta_{Q_m\cap\Omega}^{DN} + V)$, we conclude that

$$N(-\Delta_{Q_m\cap\Omega}^{DN} + V) \leqslant ((S_2')^{-1} + 5C)\|V_-\|_{\mathcal{B},Q_m\cap\Omega}.$$

Inserting this into (2.6) and using the superadditivity from Lemma A.1, we obtain the claim of the theorem. □

Proof of Corollary 1.2. We use an approximation argument similar to that at the end of the proof of Theorem 1.1. For fixed $\varepsilon > 0$ and continuous and compactly supported W we write

$$-\Delta_\Omega^D + \alpha V = (1-\varepsilon)\left(-\Delta_\Omega^D + \alpha(1-\varepsilon)^{-1}W\right) + \varepsilon\left(-\Delta_\Omega^D + \alpha\varepsilon^{-1}(V-W)\right)$$

and bound, using the variational principle and Theorem 1.1,

$$\begin{aligned} N(-\Delta_\Omega^D + \alpha V) &\leqslant N\big(-\Delta_\Omega^D + \alpha(1-\varepsilon)^{-1}W\big) + N\big(-\Delta_\Omega^D + \alpha\varepsilon^{-1}(V-W)\big) \\ &\leqslant N\big(-\Delta_\Omega^D + \alpha(1-\varepsilon)^{-1}W\big) + C\alpha\varepsilon^{-1}\|V-W\|_{\mathcal{B},\Omega}. \end{aligned}$$

Using the Weyl asymptotics for continuous and compactly supported potentials, we obtain

$$\begin{aligned} &\limsup_{\alpha\to\infty} \alpha^{-1} N(-\Delta_\Omega^D + \alpha V) \\ &\leqslant (1-\varepsilon)^{-1} \frac{1}{4\pi} \int_\Omega W(x)_-\, dx + C\varepsilon^{-1}\|V-W\|_{\mathcal{B},\Omega}. \end{aligned} \tag{2.7}$$

Similarly, we write

$$-\Delta_\Omega^D + \alpha(1-\varepsilon)W = (1-\varepsilon)\big(-\Delta_\Omega^D + \alpha V\big) + \varepsilon\big(-\Delta_\Omega^D + \alpha\varepsilon^{-1}(1-\varepsilon)(W-V)\big)$$

and bound

$$\begin{aligned} N(-\Delta_\Omega^D + \alpha(1-\varepsilon)W) &\leqslant N(-\Delta_\Omega^D + \alpha V) + N\big(-\Delta_\Omega^D + \alpha\varepsilon^{-1}(1-\varepsilon)(W-V)\big) \\ &\leqslant N(-\Delta_\Omega^D + \alpha V) + C\alpha\varepsilon^{-1}(1-\varepsilon)\|W-V\|_{\mathcal{B},\Omega}. \end{aligned}$$

We obtain

$$\begin{aligned} &\liminf_{\alpha\to\infty} \alpha^{-1} N(-\Delta_\Omega^D + \alpha V) \\ &\geqslant (1-\varepsilon) \frac{1}{4\pi} \int_\Omega W(x)_-\, dx - C\varepsilon^{-1}(1-\varepsilon)\|W-V\|_{\mathcal{B},\Omega}. \end{aligned} \tag{2.8}$$

By an argument as in Lemma A.4, there is a sequence of continuous W_n with compact support such that $\|W_n - V\|_{\mathcal{B},\Omega} \to 0$. We also note that with C' such that $\mathcal{A}(1/C') = 1$, for any $g \in L^{\mathcal{B}}(\Omega)$ we have

$$\|g\|_{L^1(\Omega)} = C' \int_\Omega \frac{\overline{\operatorname{sgn} g}}{C'} g\, dx \leqslant C'\|g\|_{\mathcal{B},\Omega},$$

and therefore

$$\|W_n - V\|_{L^1(\Omega)} \leqslant C'\|W_n - V\|_{\mathcal{B},\Omega} \to 0.$$

Replacing W by W_n in (2.7) and (2.8) and letting $n \to \infty$, we obtain

$$\begin{aligned} (1-\varepsilon)\frac{1}{4\pi}\int_\Omega V(x)_-\, dx &\leqslant \liminf_{\alpha\to\infty} \alpha^{-1} N(-\Delta_\Omega^D + \alpha V) \\ &\leqslant \limsup_{\alpha\to\infty} \alpha^{-1} N(-\Delta_\Omega^D + \alpha V) \leqslant (1-\varepsilon)^{-1}\frac{1}{4\pi}\int_\Omega V(x)_-\, dx. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the corollary follows. □

§3. Trudinger's inequality

Our goal in this section is to prove Proposition 2.1. The first part of this proposition is well known and goes back to [20, 10, 18]. Since we will need some intermediate result from this proof for the proof of the second part, we recall it here, following [9].

Lemma 3.1. *There is a constant $\beta_0 > 0$ and a continuous function*

$$[0, \beta_0) \ni \beta \to C_\beta$$

with $C_0 = 0$ such that for any open set $\Omega \subset \mathbb{R}^2$ of finite measure, any $u \in H_0^1(\Omega)$, and any $0 \leqslant \beta < \beta_0$ we have

$$\int_\Omega \mathcal{A}\left(\frac{\beta|u|^2}{\|\nabla u\|^2}\right) dx \leqslant C_\beta |\Omega|.$$

Proof. We begin with showing that for $q > 2$ and $u \in H^1(\mathbb{R}^2)$,

$$\left(\int_{\mathbb{R}^2} |\nabla u|^2\, dx\right)^{1-2/q} \left(\int_{\mathbb{R}^2} |u|^2\, dx\right)^{2/q} \geqslant \frac{2}{q}(4\pi)^{(q-2)/q} \left(\int_{\mathbb{R}^2} |u|^q\, dx\right)^{2/q}. \quad (3.1)$$

The point here is the explicit expression of the constant on the right side and, in particular, its behavior as $q \to \infty$.

In order to prove this inequality we apply, with a parameter $\kappa > 0$ to be determined, the Hausdorff–Young and the Hölder inequalities to get

$$\|u\|_q \leqslant (2\pi)^{-(q-2)/q} \|\widehat{u}\|_{q'} \leqslant (2\pi)^{-(q-2)/q} \left\|(|\xi|^2+\kappa^2)^{1/2}\widehat{u}\right\| \left\|(|\xi|^2+\kappa^2)^{-1/2}\right\|_{2q/(q-2)}.$$

Since

$$\left\|(|\xi|^2 + \kappa^2)^{-1/2}\right\|_{2q/(q-2)}^{2q/(q-2)} = 2\pi \int_0^\infty \frac{k\, dk}{(k^2+\kappa^2)^{q/(q-2)}} = \frac{q-2}{2}\, \pi\, \kappa^{-4/(q-2)},$$

we obtain

$$\|u\|_q^2 \leqslant (8\pi)^{-(q-2)/q}\, (q-2)^{(q-2)/q}\, \kappa^{-4/q} \left(\|\nabla u\|^2 + \kappa^2 \|u\|^2\right).$$

We optimize the right side by choosing

$$\kappa^2 = (2/(q-2))\|\nabla u\|^2 / \|u\|^2$$

and arrive at (3.1).

If $u \in H_0^1(\Omega)$, we can use the inequality

$$\int_\Omega |u|^2\, dx \leqslant |\Omega|^{1-2/q} \left(\int_\Omega |u|^q\, dx\right)^{2/q}$$

and (3.1) to get

$$\left(\int_\Omega |\nabla u|^2\,dx\right)^{q/2} |\Omega| \geqslant \left(\frac{2}{q}\right)^{q^2/(2(q-2))} (4\pi)^{q/2} \int_\Omega |u|^q\,dx.$$

Thus,

$$\int_\Omega \mathcal{A}\left(\frac{\beta|u|^2}{\|\nabla u\|^2}\right) dx = \sum_{n=2}^\infty \frac{1}{n!} \int_\Omega \left(\frac{\beta|u|^2}{\|\nabla u\|^2}\right)^n dx \leqslant C_\beta\,|\Omega|$$

with

$$C_\beta := \sum_{n=2}^\infty \frac{n^{n^2/(n-1)}}{n!} \left(\frac{\beta}{4\pi}\right)^n.$$

Using Stirling's asymptotics and the root test, we see that C_β converges if $0 \leqslant \beta < 4\pi/e$ and defines a continuous function with $C_0 = 0$. □

Proof of Proposition 2.1. Let C_β be the constant from Lemma 3.1. For the proof of the first part, we simply choose $S_2 > 0$ such that $C_{S_2} \leqslant 1$.

For the proof of the second part, let $\widetilde{Q}$ be the square with the same center as Q but with three times its side length and let $\mathcal{E} : H^1(Q) \to H^1(\widetilde{Q})$ be the extension operator by repeated reflection. Thus, for all $u \in H^1(Q)$,

$$\int_{\widetilde{Q}} |\mathcal{E}u|^2\,dx = 9\int_Q |u|^2\,dx \qquad \text{and} \qquad \int_{\widetilde{Q}} |\nabla\mathcal{E}u|^2\,dx = 9\int_Q |\nabla u|^2\,dx.$$

Let $\eta \in C_0^\infty(\widetilde{Q})$ be a real function with $\eta \equiv 1$ on Q. We choose this function to be of the form $\eta(x) = \eta_0((x-a)/L)$, where a and L are the center and the side length of Q, respectively, and where η_0 is a universal function. Then the operator $\widetilde{\mathcal{E}}$ defined by $\widetilde{\mathcal{E}}u := \eta\mathcal{E}u$ maps $H^1(Q)$ into $H_0^1(\widetilde{Q})$ and satisfies

$$\begin{aligned}\int_{\widetilde{Q}} |\nabla\widetilde{\mathcal{E}}u|^2\,dx &= \int_{\widetilde{Q}} \left(\eta^2|\nabla\mathcal{E}u|^2 - \eta\Delta\eta|u|^2\right) dx \\ &\leqslant 9\|\eta_0\|_\infty^2 \int_Q |\nabla u|^2\,dx + 9\|\eta_0\Delta\eta_0\|_\infty |Q|^{-1} \int_Q |u|^2\,dx.\end{aligned}$$

This inequality is, in particular, valid for the functions $u \in H^1(\Omega)$ that vanish near $Q \cap \partial\Omega$, because such functions can be extended by zero to functions in $H^1(Q)$. If, in addition, $\int_\Omega u\,dx = \int_Q u\,dx = 0$, then we can bound the last

term on the right side by the Poincaré inequality on Q, obtaining finally

$$\int_{\widetilde{Q}} |\nabla \widetilde{\mathcal{E}} u|^2 \, dx \leqslant c \int_Q |\nabla u|^2 \, dx$$

with $c := 9 \left(\|\eta_0\|_\infty^2 + \pi^{-2} \|\eta_0 \Delta \eta_0\|_\infty \right)$.

Moreover, let $\widetilde{\Omega} \subset \widetilde{Q}$ be the set obtained from Ω by repeated reflection at the boundaries of the square. Then $|\widetilde{\Omega}| = 9|\Omega|$. If $u \in H^1(\Omega)$ vanishes near $Q \cap \partial\Omega$, then $\widetilde{\mathcal{E}} u \in H_0^1(\widetilde{\Omega})$, and therefore by the inequality from Lemma 3.1,

$$\int_\Omega \mathcal{A}\left(\frac{\beta |u|^2}{\|\nabla u\|_{L^2(\Omega)}^2} \right) dx \leqslant \int_{\widetilde{\Omega}} \mathcal{A}\left(\frac{c\beta |\widetilde{\mathcal{E}} u|^2}{\|\nabla \widetilde{\mathcal{E}} u\|_{L^2(\Omega)}} \right) dx \leqslant C_{c\beta} |\widetilde{\Omega}| = 9 C_{c\beta} |\Omega|.$$

Choosing $S_2' > 0$ such that $9 C_{cS_2'} \leqslant 1$, we obtain the claimed inequality. □

Remark 3.2. The inequality stated in Remark 2.2 follows by the same argument as in the previous proof, except that we replace the Poincaré inequality by the inequality

$$|Q|^{-1} \int_\Omega |u|^2 \, dx \leqslant (1 + \theta^{-1}) \pi^{-2} \int_\Omega |\nabla u|^2 \, dx \qquad \text{if} \quad |Q \setminus \Omega| \geqslant \theta |Q|.$$

To see the last inequality, we extend u by zero to Q, put $u_Q = |Q|^{-1} \int_Q u \, dx$, and note that, since u vanishes on $Q \setminus \Omega$, we have

$$|Q \setminus \Omega| |u_Q|^2 \leqslant \int_Q |u - u_Q|^2 \, dx.$$

Thus,

$$\begin{aligned} \int_\Omega |u|^2 \, dx = \int_Q |u|^2 \, dx &= \int_Q |u - u_Q|^2 \, dx + |Q| |u_Q|^2 \\ &\leqslant \left(1 + \frac{|Q|}{|Q \setminus \Omega|} \right) \int_Q |u - u_Q|^2 \, dx. \end{aligned}$$

The claimed inequality now follows from the Poincaré inequality.

§4. The covering lemma

Our goal in the section is to prove Proposition 2.3. As a first step, we will see that around each point we can center a square for which the norm of a given function has a prescribed Orlicz norm. This requires some basic facts about Orlicz spaces, which we recall in Appendix A. At the second step, we

apply the Besicovitch lemma to obtain a suitable finite collection of squares. The relevant version of the Besicovitch lemma will be recalled in Appendix B.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set of finite measure, let*

$$0 \leqslant W \in L^{\mathcal{B}}(\Omega),$$

and let

$$0 < A < \|W\|_{\mathcal{B},\Omega}.$$

Then for any $x \in \overline{\Omega}$ there is an open square Q_x centered at x with

$$\|W\|_{\mathcal{B},Q_x\cap\Omega} = A.$$

If Q_x is chosen maximal with this property, then $\overline{\Omega} \ni x \mapsto |Q_x|$ is upper semicontinuous.

Proof. First, we fix $x \in \overline{\Omega}$ and consider the function

$$j(l) := \|V\|_{\mathcal{B},(x+lQ)\cap\Omega},$$

where $Q := (-1/2, 1/2)^d$, so $x + lQ$ is the open square centered at x with side length l. By a simple property of the norm (see (A.1)), j is a monotone nondecreasing function of l.

We claim that j is continuous on $[0, \infty)$ with $j(0) = 0$. To prove this, we let $(l_n) \subset (0, \infty)$ with $l_n \to l \in [0, \infty)$. Setting $E_n := (x + l_nQ) \cap \Omega$ and $E := (x + lQ) \cap \Omega$, we clearly have $|E_n \Delta E| \to 0$ as $n \to \infty$, and therefore, by Lemma A.5,

$$j(l_n) = \|V\|_{\mathcal{B},E_n} \to \|V\|_{\mathcal{B},E} = j(l),$$

proving the claimed continuity and the fact that $j(0) = 0$.

Also, it is easily seen that

$$\lim_{l\to\infty} j(l) = \|V\|_{\mathcal{B},\Omega}.$$

Thus, for any $0 < A < \|V\|_{\mathcal{B},\Omega}$ there is l such that $j(l) = A$. We denote $l_x := \max\{l : \ j(l) = A\}$, making the dependence on x explicit.

Now we prove the upper semicontinuity statement. This will follow if we can show that for $(x_n) \subset \overline{\Omega}$ and $(l_n) \subset (0, \infty)$ with $x_n \to x \in \overline{\Omega}$ and $l_n \to l \in (0, \infty)$ one has

$$\|V\|_{\mathcal{B},(x_n+l_nQ)\cap\Omega} \to \|V\|_{\mathcal{B},(x+lQ)\cap\Omega}.$$

Indeed, if we apply this statement to $l_n = l_{x_n}$, then we obtain

$$\|V\|_{\mathcal{B},(x+lQ)\cap\Omega} = A,$$

and by the maximality of l_x we conclude that $l_x \geqslant l$, which proves upper semicontinuity.

To prove the statement above, we apply Lemma A.5 again, this time with $E_n := (x_n + l_nQ) \cap \Omega$ and $E := (x + lQ) \cap \Omega$. Again we have $|E_n \Delta E| \to 0$ as

$n \to \infty$, so the assumption of that lemma is satisfied. This finishes the proof of the lemma. □

We are now in a position to prove Proposition 2.3.

Proof of Proposition 2.3. We may assume that $0 < A < \|W\|_{\mathcal{B},\Omega}$. Then for any point $x \in \overline{\Omega}$ Lemma 4.1 yields an open square Q_x centered at x with $\|W\|_{\mathcal{B},Q_x\cap\Omega} = A$. Moreover, the side length $|Q_x|^{1/2}$ depends on x in an upper semicontinuous way. Thus, the Besicovitch lemma (Proposition B.1) yields a countable covering of multiplicity 4 and with the property that the squares can be divided into families Ξ^k, $k = 1, \dots, 17$, each of which consists of disjoint squares.

It remains to show that the covering is finite and with the claimed upper bound on the number M of squares. For any $k = 1, \dots, 17$, by the superadditivity property of the Orlicz norm (Lemma A.1),

$$(\#\Xi^k)A = \sum_{Q\in\Xi^k} \|W\|_{\mathcal{B},Q\cap\Omega} \leqslant \|W\|_{\mathcal{B},\Omega}.$$

Summing over k we obtain

$$MA \leqslant 17\,\|W\|_{\mathcal{B},\Omega},$$

which is the claimed bound. □

Appendix §A. Orlicz spaces

In order to make this paper selfcontained, in this Appendix we provide proofs of the results from the Orlicz space theory that we need. For a deeper treatment we refer, for instance, to [5].

Throughout this section we consider a convex function $\mathcal{A}$ on $[0,\infty)$ satisfying $\mathcal{A}(t) = 0$ if and only if $t = 0$, as well as

$$\lim_{t\to\infty} \frac{\mathcal{A}(t)}{t} = \infty \qquad \text{and} \qquad \lim_{t\to 0} \frac{\mathcal{A}(t)}{t} = 0.$$

(Such functions are called *Young functions.*) The example relevant for the rest of this paper is the function

$$\mathcal{A}(t) = e^t - 1 - t,$$

but our arguments are valid for general $\mathcal{A}$.

Let $\mathcal{B}$ be the Legendre transform of $\mathcal{A}$, that is,

$$\mathcal{B}(s) = \sup_{t\geqslant 0} \big(st - \mathcal{A}(t)\big) \quad \text{for} \quad s \geqslant 0.$$

It can be shown that $\mathcal{B}$ is again a Young function. In the example above, we have

$$\mathcal{B}(s) = (1+s)\ln(1+s) - s.$$

For a finite measure space (X, dx) we denote by $L^{\mathcal{B}}(X)$ the set of measurable functions $g : X \to \mathbb{C}$ for which

$$\|g\|_{\mathcal{B},X} = \sup\left\{ \Big| \int_X fg\,dx \Big| :\ f : X \to \mathbb{C} \text{ measurable such that } \int_X \mathcal{A}(|f|)\,dx \leqslant |X| \right\}$$

is finite (functions equal almost everywhere are identified). Clearly, $\|\cdot\|_{\mathcal{B},X}$ defines a norm. The next result says that this norm is superadditive.

Lemma A.1. *Let $g \in L^{\mathcal{B}}(X)$, and let $E_1, E_2, \dots$ be pairwise disjoint measurable subsets of X. Then*

$$\sum_j \|g\|_{\mathcal{B},E_j} \leqslant \|g\|_{\mathcal{B},X}.$$

Note that this implies, in particular, that

$$\|g\|_{\mathcal{B},E} \leqslant \|g\|_{\mathcal{B},X} \qquad \text{if} \quad E \subset X. \tag{A.1}$$

Proof. Consider any sequence of measurable functions $f_1, f_2, \dots$ on $E_1, E_2, \dots$ with

$$\int_{E_j} \mathcal{A}(|f_j|)\,dx \leqslant |E_j| \qquad \text{for all } j.$$

We define functions s_j on E_j with $|s_j| = 1$ and $s_j f_j g = |f_j g|$ pointwise on E_j. We define a function $\widetilde{f}$ on X by $\widetilde{f}|_{E_j} := s_j f_j$ for each j and $\widetilde{f} := 0$ on $X \setminus \bigcup_j E_j$. Then

$$\int_X \mathcal{A}(|\widetilde{f}|)\,dx = \sum_j \int_{E_j} \mathcal{A}(|s_j f_j|)\,dx = \sum_j \int_{E_j} \mathcal{A}(|f_j|)\,dx \leqslant \sum_j |E_j| \leqslant |E|.$$

Thus,

$$\sum_j \Big| \int_{E_j} f_j g\,dx \Big| = \sum_j \int_{E_j} \widetilde{f}_j g\,dx = \int_X \widetilde{f} g\,dx \leqslant \|g\|_{\mathcal{B},X}.$$

Taking the supremum over all functions f_j with the specified properties, we arrive at the inequality in the lemma. □

Lemma A.2. *Let $g \in L^{\mathcal{B}}(X)$, let $E, F \subset X$ be measurable with $|E| \leqslant |F|$, and assume that g vanishes on $X \setminus (E \cap F)$. Then*

$$\|g\|_{\mathcal{B},E} \leqslant \|g\|_{\mathcal{B},F} \leqslant \frac{|F|}{|E|}\|g\|_{\mathcal{B},E}.$$

Proof. First we observe that, since g vanishes outside $E \cap F$ and since $\mathcal{A}$ is monotone, we have

$$\|g\|_{\mathcal{B},E} = \sup\left\{\left|\int_{E\cap F} fg\,dx\right| : f : E\cap F \to \mathbb{C} \text{ is measurable with } \int_{E\cap F}\mathcal{A}(|f|)\,dx \leqslant |E|\right\}$$

and

$$\|g\|_{\mathcal{B},F} = \sup\left\{\left|\int_{E\cap F} fg\,dx\right| : f : E\cap F \to \mathbb{C} \text{ is measurable with } \int_{E\cap F}\mathcal{A}(|f|)\,dx \leqslant |F|\right\}.$$

Since $|E| \leqslant |F|$, this immediately implies $\|g\|_{\mathcal{B},E} \leqslant \|g\|_{\mathcal{B},F}$.

To prove the second inequality, let f be a measurable function on $E \cap F$ with

$$\int_{E\cap F} \mathcal{A}(|f|)\,dx \leqslant |F|.$$

Since $\mathcal{A}$ is convex with $\mathcal{A}(0) = 0$, we have $\mathcal{A}(\theta t) \leqslant \theta\mathcal{A}(t)$ for any $0 \leqslant \theta \leqslant 1$ and any $t \geqslant 0$. Thus, $\widetilde{f} := (|E|/|F|)f$ satisfies

$$\int_{E\cap F} \mathcal{A}(|\widetilde{f}|)\,dx \leqslant |E|,$$

and therefore

$$\|g\|_{\mathcal{B},E} \geqslant \left|\int_{E\cap F} \widetilde{f}g\,dx\right| = \frac{|E|}{|F|}\left|\int_{E\cap F} fg\,dx\right|.$$

Taking the supremum over all f as before, we deduce that

$$\|g\|_{\mathcal{B},E} \geqslant \big(|E|/|F|\big)\|g\|_{\mathcal{B},F},$$

as claimed. □

Lemma A.3. *For any $g \in L^{\mathcal{B}}(X)$, we have*

$$\int_X \mathcal{B}\left(\frac{|X|\,|g|}{\|g\|_{\mathcal{B},X}}\right)dx \leqslant |X|.$$

Proof. As a preliminary remark, we note that

$$s\mathcal{B}'(s) = \mathcal{A}(\mathcal{B}'(s)) + \mathcal{B}(s) \qquad \text{for all } s \geqslant 0; \tag{A.2}$$

here and in what follows we denote by $\mathcal{B}'$ the right-sided derivative of $\mathcal{B}$ (which exists by convexity). In fact, by convexity we have

$$\mathcal{B}(\sigma) \geqslant \mathcal{B}(s) + \mathcal{B}'(s)(\sigma - s) \quad \text{for all} \quad \sigma,$$

and therefore

$$s\mathcal{B}'(s) - \mathcal{B}(s) = \sup_{\sigma} \left(\sigma\mathcal{B}'(s) - \mathcal{B}(\sigma)\right).$$

Since $\mathcal{A}$ is the Legendre transform of $\mathcal{B}$, we obtain (A.2).

Now we turn to the proof of the lemma. Clearly, we may assume that $\|g\|_{\mathcal{B},X} = |X|$. Let $f := \overline{\operatorname{sgn} g}\mathcal{B}'(|g|)$. We shall show momentarily that

$$\int_X \mathcal{A}(|f|)\,dx \leqslant |X|. \tag{A.3}$$

This inequality and the definition of $\|g\|_{\mathcal{B}}$ imply that

$$|X| = \|g\|_{\mathcal{B},X} \geqslant \left|\int_X fg\,dx\right| = \int_X |f||g|\,dx.$$

On the other hand, because of (A.2) we have

$$|f||g| = \mathcal{A}(|f|) + \mathcal{B}(|g|), \tag{A.4}$$

and therefore

$$\int_X |f||g|\,dx = \int_X \mathcal{A}(|f|)\,dx + \int_X \mathcal{B}(|g|)\,dx \geqslant \int_X \mathcal{B}(|g|)\,dx,$$

which yields the inequality in the lemma.

It remains to prove (A.3). For $M > 0$, let $f_M := f\mathbb{1}_{\{|f|\leqslant M\}}$ and note that (A.4) yields

$$|f_M||g| = \mathcal{A}(|f_M|) + \mathbb{1}_{\{|f|\leqslant M\}}\mathcal{B}(|g|) \geqslant \mathcal{A}(|f_M|).$$

We choose M so large that $\mathbb{1}_{\{|f|\leqslant M\}}\mathcal{B}(|g|)$ does not vanish almost everywhere and obtain

$$\int_X |f_M||g|\,dx > \int_X \mathcal{A}(|f_M|)\,dx. \tag{A.5}$$

Now we show that

$$\gamma := \int_X \mathcal{A}(|f_M|)\,dx \leqslant |X|,$$

which implies (A.3) by monotone convergence. We argue by contradiction and assume that $\gamma > |X|$. Note that $\gamma < \infty$ because f_M is bounded. As in the previous proof, by convexity we have

$$\mathcal{A}(|X|\,|f_M|/\gamma) \leqslant (|X|/\gamma)\mathcal{A}(|f_M|),$$

and, therefore,

$$\int_X \mathcal{A}(|X|\,|f_M|/\gamma)\,dx \leqslant |X|.$$

By the definition of $\|g\|_{\mathcal{B},X}$,

$$\int_X |f_M||g|\,dx = \frac{\gamma}{|X|}\int_X \frac{|X|\,f_M}{\gamma} g\,dx \leqslant \frac{\gamma}{|X|}\|g\|_{\mathcal{B}} = \gamma.$$

This contradicts (A.5), and we see that $\gamma \leqslant |X|$. □

Lemma A.4. *Assume that $\mathcal{B}$ satisfies*

$$\limsup_{t\to\infty} \frac{\mathcal{B}(2t)}{\mathcal{B}(t)} < \infty. \tag{A.6}$$

Then $L^\infty(X)$ is dense in $L^{\mathcal{B}}(X)$.

In the theory of Orlicz spaces, (A.6) is called the Δ_2 *condition*. Note that the function $\mathcal{B}(t) = (1+|t|)\ln(1+|t|) - |t|$ satisfies this condition (while $\mathcal{A}(t) = e^{|t|} - 1 - |t|$ does not).

Proof. Let $g \in L^{\mathcal{B}}(X)$. We show that $g_M := g\mathbb{1}_{\{|g|\leqslant M\}} \to g$ in $L^{\mathcal{B}}(X)$ as $M \to \infty$.

Let $\lambda := \|g\|_{\mathcal{B},X}/|X|$, so that $\int_X \mathcal{B}(|g|/\lambda)\,dx < \infty$ by Lemma A.3. Moreover, $\mathcal{B}(|g_M - g|/\lambda) \leqslant \mathcal{B}(|g|/\lambda)$ and therefore, by dominated convergence, $\int_X \mathcal{B}(|g_M - g|/\lambda)\,dx \to 0$ as $M \to \infty$. It is easily seen that the assumption (A.6) implies that for any $\varepsilon > 0$ and $k \in \mathbb{N}$ there is $C_{k,\varepsilon} < \infty$ such that

$$\mathcal{B}(2^k t) \leqslant C_{k,\varepsilon}\mathcal{B}(t) + \varepsilon \qquad \text{for all } t \geqslant 0.$$

Let f be a measurable function with $\int_X \mathcal{A}(|f|)\,dx \leqslant |X|$. Then by the definition of $\mathcal{B}$ as a Legendre transform,

$$|f|2^k|g_M - g|/\lambda \leqslant \mathcal{A}(|f|) + \mathcal{B}(2^k|g_M - g|/\lambda) \leqslant \mathcal{A}(|f|) + C_{k,\varepsilon}\mathcal{B}(|g_M - g|/\lambda) + \varepsilon.$$

Thus,

$$\left|\int_X f 2^k (g_M - g)/\lambda\,dx\right| \leqslant (1+\varepsilon)|X| + C_{k,\varepsilon}\int_X \mathcal{B}(|g_M - g|/\lambda)\,dx$$

and, taking the supremum over f, we arrive at

$$(2^k/\lambda)\|g_M - g\|_{\mathcal{B},X} = \|2^k(g_M - g)/\lambda\|_{\mathcal{B},X} \leqslant (1+\varepsilon)|X| + C_{k,\varepsilon}\int_X \mathcal{B}(|g_M - g|/\lambda)\,dx.$$

Letting $M \to \infty$ gives

$$\limsup_{M\to\infty} \|g_M - g\|_{\mathcal{B},X} \leqslant \lambda 2^{-k}(1+\varepsilon)|X|,$$

and letting $k \to \infty$ gives $g_M \to g$ in $L^{\mathcal{B}}(X)$. □

Lemma A.5. *Assume that $\mathcal{B}$ satisfies* (A.6). *Let $g \in L^{\mathcal{B}}(X)$, and let $E_1, E_2, \dots$ and E be measurable subsets of X with $|E_n \Delta E| \to 0$ as $n \to \infty$. Then, as $n \to \infty$,*

$$\|g\|_{\mathcal{B},E_n} \to \|g\|_{\mathcal{B},E}.$$

Proof. First we claim that we may assume that g is bounded. Indeed, from Lemma A.4 we know for any $\varepsilon > 0$ there is $g_\varepsilon \in L^\infty(X)$ such that $\|g_\varepsilon - g\|_{\mathcal{B},X} \leqslant \varepsilon$. Thus, by the triangle inequality and by (A.1), for any measurable $F \subset X$ we have $|\|g_\varepsilon\|_{\mathcal{B},F} - \|g\|_{\mathcal{B},F}| \leqslant \|g_\varepsilon - g\|_{\mathcal{B},F} \leqslant \|g_\varepsilon - g\|_{\mathcal{B},X} \leqslant \varepsilon$. Applying this with $F = E_n$ and with $F = E$, we see that it suffices to prove the lemma for $g \in L^\infty(X)$, as claimed.

Let us define $g_n := g \mathbb{1}_{E_n \cap E}$. Then

$$\begin{aligned} |\|g\|_{\mathcal{B},E_n} - \|g\|_{\mathcal{B},E}| \leqslant |\|g\|_{\mathcal{B},E_n} - \|g_n\|_{\mathcal{B},E_n}| + |\|g_n\|_{\mathcal{B},E_n} - \|g_n\|_{\mathcal{B},E}| \\ + |\|g_n\|_{\mathcal{B},E} - \|g\|_{\mathcal{B},E}|. \end{aligned}$$

Using (A.1), we have

$$\begin{aligned} |\|g_n\|_{\mathcal{B},E_n} - \|g\|_{\mathcal{B},E_n}| \leqslant \|g_n - g\|_{\mathcal{B},E_n} = \|g\chi_{E_n \setminus E}\|_{\mathcal{B},E_n} \leqslant \|g\|_\infty \|\chi_{E_n \setminus E}\|_{\mathcal{B},E_n} \\ \leqslant \|g\|_\infty \|\chi_{E_n \setminus E}\|_{\mathcal{B},X}, \end{aligned}$$

and similarly

$$\begin{aligned} |\|g_n\|_{\mathcal{B},E} - \|g\|_{\mathcal{B},E}| \leqslant \|g_n - g\|_{\mathcal{B},E} = \|g\chi_{E \setminus E_n}\|_{\mathcal{B},E} \leqslant \|g\|_\infty \|\chi_{E \setminus E_n}\|_{\mathcal{B},E} \\ \leqslant \|g\|_\infty \|\chi_{E \setminus E_n}\|_{\mathcal{B},X}. \end{aligned}$$

It is a simple exercise (using Jensen's inequality) to compute that for any measurable $F \subset X$,

$$\|\chi_F\|_{\mathcal{B},X} = |F| \, \mathcal{A}^{-1}(|X|/|F|),$$

where $\mathcal{A}^{-1}$ is the inverse function of $\mathcal{A}$; see [5, Subsection II.9.3]. Since $\mathcal{A}(t)/t \to \infty$ as $t \to \infty$, we deduce that $\|\chi_F\|_{\mathcal{B},X} \to 0$ as $|F| \to 0$. Thus, the assumption $|E_n \Delta E| \to 0$ implies that

$$|\|g\|_{\mathcal{B},E_n} - \|g_n\|_{\mathcal{B},E_n}| + |\|g_n\|_{\mathcal{B},E} - \|g\|_{\mathcal{B},E}| \to 0$$

as $n \to \infty$.

By (A.1) and Lemma A.2, we have

$$|\|g_n\|_{\mathcal{B},E_n} - \|g_n\|_{\mathcal{B},E}| \leqslant \left(\frac{\max\{|E_n|, |E|\}}{\min\{|E_n|, |E|\}} - 1 \right) \|g\|_{\mathcal{B},X}.$$

Again, since $|E_n \Delta E| \to 0$, we deduce that

$$\|g_n\|_{\mathcal{B},E_n} - \|g_n\|_{\mathcal{B},E} \to 0$$

as $n \to \infty$. This completes the proof. □

Appendix §B. The Besicovitch lemma

In this section we state and prove a version of Besicovitch's covering lemma. We follow the exposition in [2], but since we get a better constant under an additional semicontinuity assumption (which is satisfied in our application), we include the details. We prove the result in general dimension d.

By a *cube* we always mean an open cube with edges parallel to coordinate axes, and by a *covering* of a set $K \subset \mathbb{R}^d$ by cubes $Q_1, \dots, Q_M$ we mean that $K \subset \bigcup_m Q_m$. The *multiplicity* of such a covering is $\sup_{x\in\mathbb{R}^d} \#\{m : \ x \in Q_m\}$.

We denote $Q := (-1/2, 1/2)^d$, so that $a + lQ$ is the cube centered at $a \in \mathbb{R}^d$ with side length $l > 0$.

Proposition B.1. *Let $d \geqslant 1$, let $K \subset \mathbb{R}^d$ be a compact set, and let l be a positive, upper semicontinuous function on K. Then there is a (finite or infinite) sequence $(x_m) \subset K$ such that the cubes*

$$Q_m = x_m + l(x_m)Q, \quad m = 1, 2, 3, \dots,$$

form a covering of K with multiplicity at most 2^d. Moreover, the sequence (Q_m) can be divided into $4^d{+}1$ sequences $\Xi^k = (Q_m^k)$ such that for any $1 \leqslant k \leqslant 4^d{+}1$, the cubes in Ξ^k are disjoint (that is, $Q_{m_1}^k \cap Q_{m_2}^k = \varnothing$ if $m_1 \neq m_2$).

In the proof we use the notation $|x|_\infty = \max\{|x^{(j)}| : 1 \leqslant j \leqslant d\}$ for $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$.

Proof. Since the function l is upper semicontinuous on the compact set K, it attains its maximum at some point $x_1 \in K$. Now assume that for some $m \in \mathbb{N}$, the points $x_1, \dots, x_m$ have already been chosen. If

$$K \setminus \bigcup_{n=1}^{m} Q_m = \varnothing,$$

then the selection process is finished. Otherwise, we take

$$x_{m+1} \in K \setminus \bigcup_{n=1}^{m} Q_n$$

such that the maximum of l over the compact set

$$K \setminus \bigcup_{n=1}^{m} Q_n$$

is attained at x_{m+1}. This procedure leads to a finite or infinite sequence of points x_m. Let us show that they have all the required properties.

We claim that

$$\Big(x_m + \frac{l(x_m)}{2}Q\Big) \cap \Big(x_n + \frac{l(x_n)}{2}Q\Big) = \varnothing \qquad \text{if } m \neq n. \tag{B.1}$$

Indeed, to show this we may assume that $m < n$. Then $x_n \notin Q_m$ by construction, and therefore $|x_m - x_n|_\infty \geqslant l(x_m)/2$. By construction we have $l(x_m) \geqslant l(x_n)$, so that

$$|x_m - x_n|_\infty \geqslant (l(x_m) + l(x_n))/4.$$

This implies (B.1).

We also claim that

$$x_m \notin Q_n \qquad \text{if } m \neq n. \tag{B.2}$$

Indeed, this is clear from the construction if $m > n$. On the other hand, if $m < n$, then, again from the construction, we have $l(x_m) \geqslant l(x_n)$ and $x_n \notin Q_m$, which implies that

$$|x_n - x_m|_\infty \geqslant l(x_m)/2 \geqslant l(x_n)/2.$$

Thus, (B.2) also holds true in this case.

Using the compactness of K, from (B.1) it is easy to deduces that if the sequence (Q_m) is infinite, then

$$l(x_m) \to 0 \qquad \text{as } m \to \infty. \tag{B.3}$$

We now prove that (Q_m) covers K. This is clear from the construction if the sequence (Q_m) is finite. So we may assume that it is infinite. We argue by contradiction and assume that there is $x \in K \setminus \bigcup_m Q_m$. Then, because of (B.3), there is m such that $l(x_m) < l(x)$. However, this contradicts the construction of x_m.

Next, we show that the multiplicity of the covering is at most 2^d, that is, any point $x \in K$ belongs to at most 2^d of the cubes Q_m. For this, we divide $\mathbb{R}^d$ into 2^d hyper-quadrants with boundaries passing through x and parallel to the d coordinate hyper-planes. It suffices to show that in each closed hyper-quadrant there is at most one point x_m such that $x \in Q_m$. We argue by contradiction and assume that there are two distinct points x_m and x_n in the same closed hyper-quadrant with $x \in Q_m \cap Q_n$. We may assume that $|x_n - x|_\infty \geqslant |x_m - x|_\infty$. Since $x \in Q_n$, the set of all points y in the same hyper-quadrant as x_n satisfying $|y - x|_\infty \leqslant |x_n - x|_\infty$ is contained in Q_n. In particular, we have $x_m \in Q_n$. This contradicts (B.2).

Finally, we have to rearrange the sequence into $4^d + 1$ disjoint sequences. First we claim that for any m there are at most 4^d cubes among the cubes $Q_1, \dots, Q_{m-1}$ that have nonempty intersection with Q_m. To see this, note that if $n < m$ and if $Q_n \cap Q_m \neq \varnothing$, then Q_n contains at least one of the 2^d vertices of Q_m. (This follows from the fact that $l(x_n) \geqslant l(x_m)$ for $n < m$.) However,

by the bound on the multiplicity, any fixed vertex of Q_m is contained in at most 2^d cubes. Thus, there are at most $2^d \times 2^d$ cubes Q_n with $n < m$ that have nonempty intersection with Q_m.

Now we use this fact to decompose our sequence. We are going to define inductively $r = 4^d + 1$ sequences $\Xi^1, \dots, \Xi^r$ of cubes. To start, we set $Q_m \in \Xi^m$ for $m = 1, \dots, r$. Now let $m \geqslant r + 1$ and assume that the families $\Xi^1, \dots, \Xi^r$ contain all the cubes $Q_1, \dots, Q_{m-1}$ and that each Ξ^k consists of disjoint cubes. By the above fact, Q_m can intersect at most $r - 1$ cubes among the cubes $Q_1, \dots, Q_{m-1}$. Since there are r families of cubes in total, there must be $k \in \{1, \dots, r\}$ such that Q_m does not intersect any of the cubes in Ξ^k. We put $Q_m \in \Xi^k$. This defines inductively the claimed partitioning of the Q_m. The proof of the proposition is complete. □

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Поступило 8 декабря, 2017