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A Nonlocal Boundary Value Problem with Constant Coefficients for the Multidimensional Second Order Equation of Mixed Type of the Second Kind

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Multidimensional second order equation of the mixed type of the second kind is considered in the paper. Unique solvability and smoothness of the solution of a nonlocal boundary value problem with constant coefficients in Sobolev spaces are proved under some conditions on coefficients.

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1. Introduction and formulation of the problem

Let $\Omega = \prod_{i=1}^{n} (\alpha_i, \beta_i)$, be n-dimensional parallelepiped in the Euclidean space $\mathbb{R}^n$ of points $(x_1, \ldots, x_n)$, $0 < \alpha_i < \beta_i < +\infty$, $\forall i = \overline{1, n}.$

In domain $Q = \Omega \times (0, T)$ we consider a second order differential equation

$$Lu = K(x,t)\, u_{tt} - \left(a_{ij}(x)\, u_{x_i}\right)_{x_j} + a\,(x,t)\, u_t + c\,(x,t)\, u = f\,(x,t). \tag{1}$$

Here and below repeating indexes mean summation from 1 to n. We assume that all functions below are real-valued and smooth enough.

Let $K\,(x, 0) \leqslant 0 \leqslant K\,(x,\, T)$ at $x \in \overline{\Omega}$. Then equation (1) is an equation of the mixed type of the second kind since function $K(x,t)$ can change sign in the domain $\overline{Q}$ [1–4].

1.1. The nonlocal boundary value problem

We are to find a generalized solution of equation (1) from Sobolev space $W_2^{\,\ell}(Q)$, ($2 \leqslant \ell$ is a natural number) that satisfies nonlocal boundary conditions

$$\gamma \cdot u\,(x, 0) = u\,(x, T), \tag{2}$$

$$\eta_i D^p_{x_i} u|_{x_i=\alpha_i} = D^p_{x_i} u|_{x_i=\beta_i} \tag{3}$$

when $p = 0, 1$, where $D^p_{x_i} u = \dfrac{\partial^p u}{\partial x_i^p}$, $D^0_{x_i} u = u$, γ and $\eta_i, \forall i = \overline{1, n}$ are some constants which are not equal to zero. They will be defined below.

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Nonlocal boundary value problems for the mixed type second order equation both first and second kinds were considered [2, 4–8, 12, 14, 15]. Nonlocal boundary value problems (2), (3) for the mixed type equation of the first kind were studied for the first time by one of the authors of the paper [9].

Here equation (1) is considered in the case $K(x,0) \leqslant 0 \leqslant K(x,T)$. Unique solvability and smoothness of the generalized solution of one nonlocal boundary value problem with constant coefficients (2), (3) in Sobolev spaces $W_2^\ell(Q)$ $(2 \leqslant \ell \in \mathbb{N})$ are studied for the first time.

Let us assume that $a_{ij}(x) = a_{ji}(x)$; $a_{ij}(\alpha_k) = a_{ji}(\beta_k)$, $\forall k = \overline{1,n}$ end $\forall \xi \in \mathbb{R}^n$, $|\xi|^2 = \sum\limits_{i=1}^{n} \xi_i^2$.

Let us also assume that one of the following conditions holds:

(a) $a_{ij}\xi_i\xi_j \geqslant a_0|\xi|^2$, where a_0 is const > 0,

(b) $a_{ij}\xi_i\xi_j \leqslant a_1|\xi|^2$, where a_1 is const < 0.

Further we assume that $|\eta_i| \geqslant 1$, $|\gamma| > 1$ in the case of condition (a), $|\gamma| < 1$ in the case of condition (b).

$W\,_2^l(Q)$ ($2 \leqslant l$-natural number) is the Sobolev space with the scalar product $(,)_l$ and the norm $\|\cdot\|_l$, $W\,_2^0(Q) = L_2(Q)$ is the space of square integrable functions.

Let $\nu = (\nu_t, \nu_{x_1}, \ldots, \nu_{x_n})$ be a unit vector of an exterior normal to the boundary ∂Q, where $\nu_t = \cos(\nu, t)$, $\nu_{x_i} = \cos(\nu, x_i)$, $\forall i = \overline{1,n}$.

Further, the Young inequality is often used

$$\forall u, v > 0,\ \forall \sigma > 0,\ p > 1, \quad u \cdot v \leqslant \frac{\sigma^p u^p}{p} + \frac{v^q}{q\sigma^q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

If $p = q = 2$ then we come to the Cauchy inequality with σ [10].

First, we consider the case $l = 2$, that is, $u \in W_2^2(Q)$ and assume that coefficients of equation (1) are smooth enough functions.

2. Uniqueness of the solution of the problem

Theorem 2.1. *Let us assume that above mentioned conditions on coefficients of equation (1) are fulfilled and* $2a - K_t + \lambda K \geqslant \delta_1 > 0$, $\lambda c - c_t \geqslant \delta_2 > 0$, *where* $\lambda = \frac{2}{T}\ln|\gamma| > 0$ *if* $|\gamma| > 1$ *in the case of condition (a) and* $\lambda = \frac{2}{T}\ln|\gamma| < 0$ *if* $|\gamma| < 1$ *in the case of condition (b),* $|\eta_i| \geqslant 1$, $\forall i = \overline{1,n}$, $c(x,0) \leqslant c(x,T)$. *If a generalized solution of problem (1)–(3) from the space* $W_2^2(Q)$ *exists for any function* $f \in L_2(Q)$ *then the solution is unique and the following inequality holds:*

$$\|u\|_1 \leqslant m\|f\|_0.$$

From this point on m is positive constant.

Proof. Let us assume that a generalized solution of problem (1)–(3) exists in the space $W\,_2^2(Q)$. Taking into account conditions of Theorem 1 and the Cauchy inequality with σ from problem (1)–(3), it is easy to obtain the following inequality

$$2\int_Q Lu \cdot \exp\Big(-\lambda t - \sum_{i=1}^{n} \mu_i x_i\Big) \cdot u_t\,dx\,dt \geqslant \int_Q \exp\Big(-\lambda t - \sum_{i=1}^{n} \mu_i x_i\Big) \{(2a - K_t + \lambda K) \cdot u_t^2 +$$
$$+ \lambda\, a_{ij}\, u_{x_i} u_{x_j} + (\lambda\, c - c_t) \cdot u^2\}\,dx\,dt - \sigma \cdot \|u_x\|_0^2 - \mu^2\sigma^{-1} \cdot \|u_t\|_0^2 +$$
$$+ \int_{\partial Q} \exp\Big(-\lambda\, t - \sum_{i=1}^{n} \mu_i\, x_i\Big) \{K u_t^2 \nu_t - 2a_{ij} u_{x_i} u_t \nu_{x_i} + a_{ij} u_{x_i} u_{x_j}\, \nu_t + c \cdot u^2 \nu_t\}\,ds, \quad (4)$$

where $0 \leqslant \mu_i = \dfrac{2}{\theta_i}\ln|\eta_i|$, $0 < \theta_i = (\beta_i - \alpha_i)$, σ and σ^{-1} are coefficients of the Cauchy inequality with σ. Conditions of Theorem 1 provide non-negativity of the integral over the domain Q and on the boundary ∂Q. Because $u \in W_2^2(Q)$ satisfies boundary conditions (2), (3) and $\gamma^2 = e^{-\lambda\cdot T}$, $\eta_i^2 = e^{\mu_i\cdot\theta_i}$ then

$$\int_{\partial Q}\exp\Big(-\lambda t - \sum_{i=1}^{n}\mu_i x_i\Big)\,\big\{K\,u_t^2\,\nu_t - 2\,a_{ij}u_{x_i}\,u_t\,\nu_{x_i} + a_{ij}u_{x_i}u_{x_j}\,\nu_t + c\,u^2\,\nu_t\big\}\,ds =$$

$$= \int_{\alpha_i}^{\beta_i}\exp\Big(-\sum_{i=1}^{n}\mu_i x_i\Big)\big\{[K(x,T)\,e^{-\lambda T}\gamma^2 - K(x,0)]\;u_t^2(x,0) +$$

$$+ [e^{-\lambda t}\gamma^2 - 1]u_{x_i}^2(x,0) + [c(x,T)e^{-\lambda T}\gamma^2 - c(x,0)]u^2(x,0)\big\}dx -$$

$$-2\big[\exp(-\mu_i\beta_i)\eta_i^2 - \exp(-\mu_i\alpha_i)\big]\int_0^T\exp(-\lambda t)\,u_{x_i}(-\alpha_i,t)u_t(\alpha_i,t)dt \geqslant$$

$$\geqslant \int_{\alpha_i}^{\beta_i}\exp\Big(-\sum_{i=1}^{n}\mu_i x_i\Big)\big\{\big[K(x,T)\,e^{-\lambda T}\gamma^2 - K(x,0)\big]u_t^2(x,0) +$$

$$+ [c(x,T)e^{-\lambda t}\gamma^2 - c(x,0)]u^2(x,0)\big\}dx \geqslant 0. \tag{5}$$

Omitting positive boundary integrals, we obtain from (5) the following inequality

$$2\int_Q Lu\cdot\exp(-\lambda t - \sum_{i=1}^{n}\mu_i x_i)\cdot u_t dx\,dt \geqslant \int_Q\exp(-\lambda t - \sum_{i=1}^{n}\mu_i x_i)\;\{(2a - K_t + \lambda K)\cdot u_t^2 +$$

$$+\lambda\,a_\tau\,u_{x_i}^2 + (\lambda\,c - c_t)\cdot u^2\}\,dx\,dt - \sigma\,\|u_{x_i}\|_0^2 - \mu^2\cdot\sigma^{-1}\cdot\|u_t\|_0^2, \tag{6}$$

where $a_\tau = a_0$ in the case of condition (a), $a_\tau = a_1$ in the case of condition (b). Setting coefficients $\lambda a_\tau - \sigma \geqslant \lambda_0 > 0$, $\delta_1 - \mu^2\sigma^{-1} > \delta_0 > 0$, we obtain from inequality (6) the first a priori estimate

$$\|u\|_1 \leqslant m\|f\|_0.$$

Uniqueness of the generalized solution of problem (1)–(3) in $W_2^2(Q)$ follows from this estimate. □

3. The equations of composite type

To prove the existence of the solution of problem (1)–(3) in $W_2^2(Q)$ we use the method of "ε-regularisation" together with Galerkin method [1,3,8,13].

Let us consider a nonlocal problem for composite type equation

$$L_\varepsilon u_\varepsilon = -\varepsilon\frac{\partial}{\partial t}\Delta u_\varepsilon + Lu_\varepsilon = f(x,t), \tag{7}$$

$$\gamma D_t^q\,u_\varepsilon|_{t=0} = D_t^q\,u_\varepsilon|_{t=T}, \quad q = 0,1,2, \tag{8}$$

$$\eta_i D_{x_i}^p\,u_\varepsilon|_{x_i=\alpha_i} = D_{x_i}^p\,u_\varepsilon|_{x_i=\beta_i}, \quad p = 0,1, \tag{9}$$

where $\Delta u = \dfrac{\partial^2 u}{\partial t^2} + \sum\limits_{i=1}^{n}\dfrac{\partial^2 u}{\partial x_i^2}$ is the Laplace operator, $D_{x_i}^{\,p}u = \dfrac{\partial^{\,p}u}{\partial\,x_i^p}$, $D_{x_i}^{\,0}u = u$, $p = 0,1$, $D_t^{\,q}u = \dfrac{\partial^{\,q}u}{\partial\,t^{\,q}}$, $q = 1,2$; $D_t^{\,0}u = u$, ε is a small enough positive number, $\eta_i, \gamma = \text{const} \neq 0$, such that $|\,\gamma\,| > 1$ in the case of condition (a), $|\,\gamma\,| < 1$ in the case of condition (b), $|\eta_i| \geqslant 1$, $\forall i = \overline{1,n}$.

In what follows we use composite type equation (7) as the ε-regularization equation for equation (1) [1,8].

Let us denote a class of functions such that $u_\varepsilon(x,t) \in W_2^2(Q)$ and $\dfrac{\partial \Delta u_\varepsilon}{\partial t} \in L_2(Q)$ satisfying conditions (8),(9) by W.

Definition. *Function $u_\varepsilon(x,t) \in W$ satisfying equation (7) is denoted the regular solution of problem (7)–(9).*

Theorem 3.1. *Let us assume that above mentioned coefficient conditions for equation (1) are fulfilled and $2a - |K_t| + \lambda \geqslant \delta_1 > 0,\ \lambda\, c - c_t \geqslant \delta_2 > 0,$ where $\lambda = \dfrac{2}{T} \ln|\gamma| > 0$ if $|\gamma| > 1$ in the case of condition (a) and $\lambda = \dfrac{2}{T}\ln|\gamma| < 0$ if $|\gamma| < 1$ in the case of condition (b), $|\eta_i| \geqslant 1$, $c(x,0) = c(x,T)$, $a(x,0) = a(x,T)$, $a(\alpha_i,t) = a(\beta_i,t)$, $K(\alpha_i,t) = K(\beta_i,t)$, $\forall i = \overline{1,n}$. Then for any function $f, f_t \in L_2(Q)$, such that $\gamma \cdot f(x,0) = f(x,T)$ there is a unique regular solution of problem (7)–(9), and the following inequalities are true:*

$$I) \qquad \varepsilon(\,\|u_{\varepsilon tt}\|_0^2 + \|u_{\varepsilon tx}\|_0^2) + \|u_\varepsilon\|_1^2 \leqslant m\,\|f\|_0^2\,,$$

$$II) \qquad \varepsilon \left\|\frac{\partial \Delta u_\varepsilon}{\partial\, t}\right\|_0^2 + \|u_\varepsilon\|_2^2 \leqslant m\left[\|f\|_0^2 + \|f_t\|_0^2\right].$$

Proof. The proof of Theorem 2 is carried out using Galerkin method with special basis functions. [8,10].

3.1. Proof of the first a priori estimate I)

Consider the following spectral problems. Let $\phi_j(x,t)$ be eigenfunction of the following problem

$$\Delta\phi_j = \frac{\partial^2\phi_j}{\partial^2 t} + \frac{\partial^2\phi_j}{\partial^2 x} = -\nu_j^2\phi_j, \tag{10}$$

$$D_t^p\,\phi_j|_{t=0} = D_t^p\,\phi_j|_{t=T}\,, \qquad p = 0,1, \tag{11}$$

$$D_x^p\,\phi_j|_{x=0} = D_x^p\,\phi_j|_{x=\ell}. \tag{12}$$

It follows from the general theory of linear self-adjoint elliptic operators that all $\{\,\phi_j(x,t)\,\}$ are eigenfunctions of problem (10)–(12). They form fundamental system in $W_2^2(Q)$, and they are orthonormal in $L_2(Q)$ [10, 11]. Then we construct the solution of an auxiliary problem using these functions:

$$\exp\left[\frac{-1}{2}\left(\lambda t + \sum_{i=1}^n \mu_i x_i\right)\right]\omega_{jt} = \phi_j, \tag{13}$$

$$\gamma\cdot\omega_j(x,0) = \omega_j(x,T), \tag{14}$$

where, $\gamma = \text{const} \neq 0$, such that $|\,\gamma\,| > 1$ in the case of condition (a), $|\,\gamma\,| < 1$ in the case of condition (b), $0 \leqslant \mu_i = \dfrac{2}{\theta_i}\ln|\eta_i|$, $|\eta_i| \geqslant 1, \forall i = \overline{1,n}$. Obviously, problem (13), (14) is uniquely solvable and its solution has the from

$$\ell^{-1}\phi_j = \omega_j = \exp\left(\frac{\sum\limits_{i=1}^n \mu_i\cdot x_i}{2}\right)\cdot\left[\int_0^t \exp\left(\frac{\lambda\tau}{2}\right)\phi_j d\tau + \frac{1}{\gamma-1}\int_0^T \exp\left(\frac{\lambda t}{2}\right)\phi_j dt\right]. \tag{15}$$

It is clear that functions $\omega_j(x,t)$ are linearly independent. Indeed, if $\sum\limits_{j=1}^{N} c_j\omega_j = 0$ for some set of functions $\omega_1, \omega_2, \ldots, \omega_N$ then acting on this sum by the operator ℓ, we have $\sum\limits_{j=1}^{N} c_j\ell\omega_j = = \sum\limits_{j=1}^{N} c_j\phi_j = 0$. Then we obtain that $c_j = 0$ for any $j = \overline{1,N}$. It follows from the construction of function $\phi_j(x,t)$ that functions $\omega_j(x,t)$ satisfy the following conditions

$$\gamma D_t^q\,\omega_i|_{t=0} = D_t^q\,\omega_i|_{t=T}, \qquad q = 0,1,2 \tag{16}$$

$$\eta_i D_{x_i}^p\,\omega_i|_{x_i=\alpha_i} = D_{x_i}^p\,\omega_i|_{x_i=\beta_i}, \quad p = 0,1. \tag{17}$$

We take the approximate solution of (7)–(9) in the from $w = u_\varepsilon^N = \sum\limits_{j=1}^{N} c_j\omega_j$ where coefficients c_j are defined for any $j = \overline{1,N}$ as solutions of the linear algebraic system

$$\int_Q L_\varepsilon u_\varepsilon^N \cdot e^{\frac{-(\lambda\cdot t+\sum\limits_{i=1}^{n}\mu_i\cdot x_i)}{2}}\phi_j\,dxdt \;=\; \int_Q f\cdot e^{\frac{-(\lambda\cdot t+\sum\limits_{i=1}^{n}\mu_i\cdot x_i)}{2}}\phi_j\,dxdt. \tag{18}$$

We prove the unique solvability of algebraic system (18). Multiplying every equation of (18) by $2c_j$ and summing up with respect to j from 1 to N and taking into account (12), (13), (18), we obtain

$$\int_Q L_\varepsilon w\cdot e^{-(\lambda t+\sum\limits_{i=1}^{n}\mu_i x_i)}\cdot w_t dxdt = \int_Q f\cdot e^{-(\lambda t+\sum\limits_{i=1}^{n}\mu_i x_i)}\cdot w_t dxdt. \tag{19}$$

Upon integrating identity (19), by virtue of theorem 2 we obtain for the approximate solution of problem (7)–(9) the estimates I), i.e.

$$\varepsilon\big(\left\|u_{\varepsilon tt}^N\right\|_0^2 + \left\|u_{\varepsilon tx}^N\right\|_0^2\big) + \left\|u_\varepsilon^N\right\|_1^2 \leqslant m\,\|f\|_0^2\,. \tag{20}$$

This implies the solvability of algebraic system (18). In particular, from estimate (20) we obtain a weak solution of problem (7)–(9) [3,10].

3.2. Proof of the second a priori estimate II.)

Taking into account problem (10)–(14), from identity (18) we obtain

$$-\frac{1}{\nu_j^2}\int_Q L_\varepsilon w\; e^{\frac{-(\lambda\cdot t+\sum\limits_{i=1}^{n}\mu_i\cdot x_i)}{2}}\;\Delta\ell\omega_j\,dxdt = -\frac{1}{\nu_j^2}\int_Q f\; e^{\frac{-(\lambda\cdot t+\sum\limits_{i=1}^{n}\mu_i\cdot x_i)}{2}}\;\Delta\ell\omega_j\;\;dxdt, \tag{21}$$

where,

$$\Delta\,\ell\,\omega_j = \exp\left[\frac{-(\lambda\,t+\sum\limits_{i=1}^{n}\mu_i x_i)}{2}\right]\left(\Delta\,\omega_{j_t} - \lambda\,\omega_{j_{tt}} - \mu_j\;\omega_{j_{xx}} + \frac{\lambda^2+\mu_j^2}{4}\omega_{j_t}\right),\; \Delta\omega_j = \omega_{j_{tt}} + \omega_{j_{xx}}.$$

Multiplying each equation of (21) by $2\nu_j^2 c_j$ and summing up with respect to j from 1 to N and considering (15), (16), (21), we have the following identity

$$-2\int_Q L_\varepsilon w\;\cdot e^{\frac{-(\lambda\cdot t+\sum\limits_{i=1}^{n}\mu_i x_i)}{2}}\cdot\Delta\,\ell\,w\,dxdt \;=\; -2\int_Q f\cdot\,e^{\frac{-(\lambda\cdot t+\sum\limits_{i=1}^{n}\mu_i x_i)}{2}}\cdot\Delta\,\ell\,w\;dxdt\,. \tag{22}$$

Integrating (22) and taking into account conditions of Theorem 2.1 and boundary conditions (15), (16), we obtain the following inequality

$$m\cdot\left[\|f_t\|_0^2+\|f\|_0^2\right]\geqslant\varepsilon\left\|\frac{\partial\,\Delta w}{\partial\,t}\right\|_0^2+\int_Q e^{-(\lambda\cdot t+\sum\limits_{i=1}^{n}\mu_i x_i)}\{(2\alpha-|K_t|+\lambda K)\,w_{tt}^2+$$

$$+(2\alpha-|K_t|+\lambda K)w_{tx_i}^2+\lambda\,w_{x_ix_i}^2+\lambda\,w_{tx_i}^2\}\,dxdt+\int_{\partial Q} e^{-(\lambda\cdot t+\sum\limits_{i=1}^{n}\mu_i x_i)}[(K\,w_{tt}^2-2\alpha\,w_t\,w_{tt}+$$

$$+w_{x_ix_i}^2+2w_{x_ix_i}\,w_{tt}-w_{x_it}^2+Kw_{x_it}^2+2c\,w\,(w_{tt}+w_{x_ix_i})\nu_t+$$

$$+(2K\,w_{tt}\,w_{x_it}-2w_{tt}\,w_{x_it}+2\alpha\,w_t\,w_{x_it})\,\nu_{x_i}]ds-\sigma\,(\,\|w_{xx}\|_0^2+\|w_{xt}\|_0^2\,)-$$

$$-\,\mu^2\sigma^{-1}\,\|u_{tt}\|_0^2-m\,(\,\|f\|_0^2)=\sum_{i=1}^{2}J_i,\quad(23)$$

where, J_1 is the integral over the domain, J_2 is the integral over the boundary.

Taking into account conditions of Theorem 2.1 and boundary conditions (14), (15), we obtain for coefficients $\lambda-\sigma\geqslant\lambda_0>0$, $\delta_1-\mu^2\sigma^{-1}>\delta_0>0$ that $J_1>0$ and $J_2\geqslant 0$. Now we have from inequality (23) the second estimate

$$\varepsilon\cdot\left\|\frac{\partial}{\partial t}\Delta u_\varepsilon^N\right\|_0^2+\left\|u_\varepsilon^N\right\|_2^2\leqslant m\cdot\left[\|f\|_0^2+\|f_t\|_0^2\right].\quad(24)$$

Hence, from the well-known theorem on weak compactness [10] the obtained estimations (20), (24) allow one to take the limit $N\to\infty$ and to conclude that a subsequence $\left\{u_\varepsilon^{N_k}\right\}$ converges in $L_2(Q)$ together with the first and the second order derivatives to the unique regular solution $u_\varepsilon(x,t)$ of problem (7)–(9) with the properties specified in Theorem 2.1 [3,6,8,10].

By virtue of (24) the following inequality holds for $u_\varepsilon(x,t)$

$$\varepsilon\left\|\frac{\partial}{\partial t}\Delta u_\varepsilon\right\|_0^2+\|u_\varepsilon\|_2^2\leqslant m\;\left[\;\|f\|_0^2+\|f_t\|_0^2\right].\quad(25)$$

Theorem 2.1 is proved. □

4. Existence of solution for the problem

4.1. The method of "ε-regularization"

Now by means of the method of "ε-regularization" we prove solvability of problem (1)–(3).

Theorem 4.1. *Let us assume that all conditions of theorem 2.1 are satisfied. Then the generalized solution of problem (1)(3) in space $W_2^2(Q)$ exists and it is unique*

Proof. The uniqueness of the solution of problem (1)–(3) in $W_2^2(Q)$ is proved in Theorem 1.1. Now we prove existence of the generalized solution of problem (1)–(3) in $W_2^2(Q)$. For this purpose, we consider equation (7) in the domain Q with nonlocal boundary conditions (8), (9) at $\varepsilon>0$. Because all conditions of Theorem 2.1 are fulfilled then there exists unique regular solution of problem (7)–(9) at $\varepsilon>0$, and estimates I),II) are true for it.

It follows from the well-known theorem on weak compactness [10] that it is possible to take from the set of functions $\{u_\varepsilon\}$, $\varepsilon>0$ weakly converging sub sequence of functions in W such that $\{u_{\varepsilon_i}\}\to u$ at $\varepsilon_i\to 0$. Let us show that limit function $u(x,t)$ satisfies the equation $Lu=f$ (1).

Indeed, as sequence $\{u_{\varepsilon_i}\}$ converges weakly in $W_2^2(Q)$, sequence $\dfrac{\partial \Delta u_\varepsilon}{\partial t}$, $(\varepsilon > 0)$ is uniformly bounded in $L_2(Q)$, and operator L is linear, then we have

$$Lu - f = Lu - Lu_{\varepsilon_i} + \varepsilon_i \cdot \frac{\partial \Delta u_{\varepsilon_i}}{\partial t} = L(u - u_{\varepsilon_i}) + \varepsilon_i \cdot \frac{\partial \Delta u_{\varepsilon_i}}{\partial t}. \tag{26}$$

Taking the limit $\varepsilon_i \to 0$, we obtain from (26) the unique solution of problem (1)–(3) in $W_2^2(Q)$ [1,6,8].

Theorem 3.1 is proved. □

5. Smoothness of solution for the problem

Now we prove a more general case $l \geqslant 3$. Further we assume that coefficients of equation (1) are infinitely differentiated in the closed domain $\overline{Q}$.

Theorem 5.1. *Let us assume that conditions of Theorem 3.1 are fulfilled and*

$2(\alpha + pK_t) - |K_t| + \lambda K \geqslant \delta > 0,$

$D_t^m K|_{t=0} = D_t^m K|_{t=T}, \ D_t^m a|_{t=0} = D_t^m a|_{t=T}, \ D_t^m c|_{t=0} = D_t^m c|_{t=T}.$

Then for any function $f(x,t)$ such that $f \in W_2^p(Q), D_t^{p+1} f \in L_2(Q)$, $\gamma D_t^m f|_{t=0} = D_t^m f|_{t=T}$ where $m = 0,1,2,3,\dots,p$ there exists unique generalized solution of problem (1)–(3) in the space $W_2^{p+2}(Q)$, where $p = 1,2,3,\dots$.

Proof. It follows from smoothness of the solution of problem (10)–(14) that the approximate solution of problem (7)–(9) satisfies conditions $w = u_\varepsilon^N \in C^\infty(\overline{Q})$;

$$\gamma D_t^q w|_{t=0} = D_t^q w|_{t=T}, \quad q = 0,1,2,\dots,$$

$$\eta_i D_{x_i}^p w|_{x_i=-\alpha_i} = D_{x_i}^p w|_{x_i=\beta_i}, \quad p = 0,1.$$

Taking into account conditions of Theorem 2.1 at $\varepsilon > 0$, nonlocal conditions at $t = 0$, $t = T$ and equality

$$(e^{-\frac{\lambda t}{2}} \cdot L_\varepsilon u_\varepsilon)\,|_{t=0}^{t=T} = (-\varepsilon \cdot e^{\frac{-\lambda t}{2}} \cdot \frac{\partial \Delta u_\varepsilon}{\partial t} + e^{\frac{-\lambda t}{2}} \cdot Lu_\varepsilon)\,|_{t=0}^{t=T} = (e^{\frac{-\lambda t}{2}} \cdot f(x,t))\,|_{t=0}^{t=T},$$

we obtain

$$\|\gamma \cdot u_{\varepsilon\, ttt}(x,0) - u_{\varepsilon\, ttt}(x,T)\|_0 \leqslant \text{const}.$$

Hence, function $v_\varepsilon(x,t) = u_{\varepsilon\ t}(x,t)$ belongs to W and satisfies the following equation

$$P_\varepsilon v_\varepsilon = L_\varepsilon v_\varepsilon = f_t - a_t\ u_{\varepsilon\, t} - c_t\, u_\varepsilon = F_\varepsilon. \tag{27}$$

It follows from theorem 2.1 that the set of functions $\{F_\varepsilon\}$ is uniformly bounded in the space $L_2(Q)$, i.e.

$$\|F_\varepsilon\|_0 \leqslant m\ \left[\ \|f\|_0^2 + \|f_t\|_0^2\right].$$

Further, it can be easily obtained from conditions of Theorem 3.1 that coefficients of the operators P_ε $(\varepsilon > 0)$ satisfy conditions of Theorem 4.1. Then on the basis of estimates I), II) for function $\{v_\varepsilon\}$ we obtain similar estimates

$$\varepsilon\left(\|v_{\varepsilon tt}\|_0^2 + \|v_{\varepsilon tx}\|_1^2\right) + \|v_\varepsilon\|_1^2 \leqslant m\left(\|f\|_0^2 + \|f_t\|_0^2\right), \tag{28}$$

$$\varepsilon \left\| \frac{\partial \Delta v_\varepsilon}{\partial t} \right\|_0^2 + \|v_\varepsilon\|_2^2 \leqslant m \left[\|f\|_1^2 + \|f_{tt}\|_0^2 \right] . \tag{29}$$

Function $\{u_\varepsilon\}$ satisfies parabolic equation with conditions (2), (3)

$$\Pi\, u_\varepsilon = u_{\varepsilon t} - \sum_{i,j=1}^{n} \left(a_{ij} u_{\varepsilon\, x_i} \right)_{x_j} = f + \varepsilon \frac{\partial \Delta u_\varepsilon}{\partial t} - K(x,t) u_{\varepsilon tt} - (a-1) u_{\varepsilon t} - c\, u_\varepsilon = \Phi_\varepsilon, \tag{30}$$

here $\Phi_\varepsilon \in L_2(Q)$. Set of functions $\{\Phi_\varepsilon\}$ is uniformly bounded in $W_2^2(Q)$, i.e.

$$\|\Phi_\varepsilon\|_0^2 \leqslant m \left[\|f\|_1^2 + \|f_{tt}\|_0^2 \right] \leqslant m\, \|f\|_2^2 . \tag{31}$$

On the basis of a priory estimates for parabolic equations [1], [10] and inequality (31) we obtain

$$\|\, u_\varepsilon\|_3^2 \leqslant m\, \|f\|_2^2 .$$

Further, one can prove in a similar way that $\|u_\varepsilon\|_{p+2}^2 \leqslant m\, \|f\|_{p+1}^2$, where $p = 2, 3, \ldots$. □

Remark. In the formulation of problem (1)–(3) the sign at the quadratic form does not play an essential role. However, in the case

(a) $a_{ij}(x)\xi_i\xi_j \geqslant a_0|\xi|^2$; $a_{ij} = a_{ji}$, where $a_0 = const > 0$, $x \in \Omega$, $\xi \in \mathbb{R}^n$

the class of equations (1) includes parabolic equations and in the case

(b) $a_{ij}(x)\xi_i\xi_j \leqslant a_1|\xi\,|^2$; $a_{ij} = a_{ji}$, where $a_1 = const < 0$, $x \in \Omega$

the class of equations (1) includes inverse parabolic equations. Nevertheless, similar results are obtained only with the change in the value of γ for problem (1)–(3) in the case of conditions (a) and (b).

Therefore, the following question arises: whether or not restrictions on γ are essential? In this connection we consider the following examples.

Examples. In the rectangle $Q = (0, \ell) \times (0, T)$ we consider the following problem

$$\Pi_1 u = u_t - u_{xx} = 0, \tag{32}$$

$$\gamma\, u\,(x, 0) = u\,(x, T), \tag{33}$$

$$u(0, t) = u(\ell, t) = 0. \tag{34}$$

Solving problem (32)–(34) by the Fourier method, we find $\gamma_k = \exp(-\lambda_k T) < 1$, $\lambda_k = \dfrac{2\pi k}{\ell}$, $k = 0, 1, 2, \ldots$. It is easy to verify that all conditions of Theorem 1 are fulfilled but functions $u_k = C_k e^{-\lambda_k t} \sin \lambda_k x$ (where C_k are arbitrary constants) are nontrivial solutions of this boundary value problem.

In the same way, we consider the following problem

$$\Pi_2 u = u_t + u_{xx} = 0 \;, \tag{35}$$

$$\gamma\, u\,(x, 0) = u\,(x, T), \tag{36}$$

$$u(0, t) = u(\ell, t) = 0. \tag{37}$$

Solving problem (35)–(37) by the Fourier method, we find that functions $u_k = C_k e^{\lambda_k t} \sin \lambda_k x$ with any C_k are nontrivial solutions of this boundary value problem. In this case $\gamma_k = \exp(\lambda_k T) > 1$.

Hence, we see that restrictions on γ for both conditions (a) and (b) are essential. If these conditions are not satisfied then we do not have the uniqueness of the problem as shown above.

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Об одной нелокальной краевой задаче с постоянным коэффициентом для многомерного уравнения смешанного типа второго рода, второго порядка

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В данной работе при выполнении некоторых условий на коэффициенты многомерного уравнения смешанного типа второго рода в пространстве доказываются однозначная разрешимость и гладкость решения одной нелокальной краевой задачи с постоянным коэффициентом в пространствах С.Л.Соболева.

Ключевые слова: многомерные уравнения, разрешимость, обобщенное решение.