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CONTINUOUS DEPENDENCE OF ATTRACTORS ON THE SHAPE OF DOMAIN

Dedicated to N. N. Uraltseva on her birthday

Abstract. *Let Ω_0 be a bounded domain in $\mathbb{R}^n$, let $\mathcal{G}$ be a family of diffeomorphisms, and let $\Omega_G = G(\Omega_0)$ for $G \in \mathcal{G}$. Denote by $\Sigma_t(G)$ the semigroup generated by a fixed parabolic PDE with Dirichlet boundary conditions on the boundary of Ω_G. Let A_G be the global attractor of $\Sigma_t(G)$. Conditions are given under which a generic diffeomorphism $G \in \mathcal{G}$ is a continuity point of the map $G \mapsto A_G$.*

General approach

Let E be a Banach space with norm $\| \cdot \|_E$, let Λ be a topological space. We consider a family of C^r-semigroups ($r \geq 0$) on E

$$S_t(\lambda, x), \tag{1}$$

where $\lambda \in \Lambda, x \in E, t \geq 0$.

For $x \in E, A, B \subset E$ let

$$d_E(x, B) = \inf_{y \in B} \| x - y \|_E, \delta_E(A, B) = \sup_{x \in A} d_E(x, B),$$

$$R_E(A, B) = \max(\delta_E(A, B), \delta_E(B, A)).$$

R_E is an analogue of the Hausdorff metric.

We give some standard definitions. We say that $B \subset E$ *attracts* $C \subset E$ under (1) if

$$\delta_E(S_t(\lambda, C), B) \to 0, t \to \infty.$$

We say that $A \subset E$ is *invariant* under (1) if

$$S_t(\lambda, A) = A, t \geq 0.$$

We say that, for $\lambda \in \Lambda$, $A(\lambda)$ is a *global attractor* for (1) (below we say sometimes simply attractor for (1)) if $A(\lambda)$ is compact, invariant, and attracts any bounded $B \subset E$ under (1).

Many conditions have been given which imply that a semigroup has a global attractor (see [1–3], for example).

We are interested here in the problem of continuous dependence of the attractor $A(\lambda)$ on $\lambda \in \Lambda$ for generic λ (for finite-dimensional dynamical systems results of this sort are described in Chap. 3 of [4]). Let us recall some definitions.

A subset Q of a topological space Λ is called *residual* if Q contains a countable intersection of open dense sets in Λ. If P is a property of elements of Λ, we say that this proprety is *generic* if the set

$$\{\lambda \in \Lambda : \lambda \text{ satisfies } P\}$$

is residual. Sometimes in this case we say that a generic element of Λ satisfies P.

Let X be a compact metric space with metric d. For $A, B \subset X$, define numbers $\delta_X(A,B)$, $R_X(A,B)$ in the same way as δ_E, R_E were defined. Consider a mapping $f : \Lambda \to X^c$, where X^c is the set of compact subsets of X.

We say that f is *upper semicontinuous at* $\lambda_0 \in \Lambda$ if given $\epsilon > 0$ there exists a neighborhood W of λ_0 in Λ such that for $\lambda \in W$

$$\delta_X(f(\lambda), f(\lambda_0)) < \epsilon.$$

We say that f is *upper semicontinuous on* Λ if f is upper semicontinuous at any $\lambda \in \Lambda$. The following statement is well-known [5].

Theorem 1. *If Λ is a topological space, X is a compact metric space, and $f : \Lambda \to X^c$ is upper semicontinuous, then there exists a residual subset $\Lambda_0 \subset \Lambda$ such that every $\lambda_0 \in \Lambda_0$ is a continuity point of f (with respect to R_X).*

Recently conditions for the upper semicontinuity of attractors were obtained by different authors ([1], [6], [7]). Lower semicontinuity (and continuity) of attractors was proved imposing additional conditions on the flow defined by the semigroup restricted to the attractor ([1], [8], [9]).

Combining conditions of upper semicontinuity of attractors with Theorem 1, it is easy to prove the genericity of parameter values that are points of continuity of attractors. Let us formulate one result of this sort.

Theorem 2. *Assume that the family of semigroups* (1) *satisfies the following conditions:*

(1) *S_t is continuous in $\Lambda \times E$ for any fixed t;*

(2) *there exists a compact set $B_0 \subset E$ such that for any bounded set $B \subset E$ and for any $\lambda \in \Lambda$ there is $T(\lambda, B) > 0$ with the property:*

$$S_t(\lambda, B) \subset B_0 \text{ for } t \geq T(\lambda, B).$$

Then:

(1) *for every $\lambda \in \Lambda$* (1) *has a global attractor $A(\lambda)$ in E and $A(\lambda) \subset B_0$,*

(2) *there exists a residual subset* Λ_0 *of* Λ *such that any* $\lambda_0 \in \Lambda_0$ *is a continuity point of the mapping*

$$A : \Lambda \to E^c$$

with respect to R_E.

Remark.. If the second assumption holds, we say that B_0 is an *absorbing set* for (1).

Proof. Standard considerations (see, for example, [1] or [10]) show that the assumptions of Theorem 2 imply the first statement.

To prove the second one note that for any $\lambda_0 \in \Lambda$, $\epsilon > 0$ there exists $T(\epsilon)$ such that

$$\delta_E(S_{T(\epsilon)}(\lambda_0, B_0), A(\lambda_0)) < \epsilon.$$

For any $x_0 \in B_0$ there are neighborhoods $N(x_0)$ and $W(\lambda_0)$ in E, Λ such that for $(x, \lambda) \in N(x_0) \times W(\lambda_0)$

$$\delta_E(S_{T(\epsilon)}(\lambda, x), A(\lambda_0)) < \epsilon.$$

It follows from the compactness of B_0 that B_0 can be covered by a finite collection of neighborhoods $N(x)$, hence there exists a neighborhood W of λ_0 such that for $\lambda \in W$

$$\delta_E(S_{T(\epsilon)}(\lambda, B_0), A(\lambda_0)) < \epsilon,$$

hence

$$\delta_E(A(\lambda), A(\lambda_0)) < \epsilon.$$

This proves that

$$A : \Lambda \to B_0^c$$

is upper semicontinuous. It remains to apply Theorem 1.

We give here an example of application of Theorem 2 to the problem of continuous dependence of attractors of PDE with respect to the shape of the domain in which the PDE are considered. This problem was stated by J. Hale in his talk at EQUADIFF8, Bratislava, 1993.

Main result

Let Ω_0 be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega_0$. We consider a family $\mathcal{G}$ of diffeomorphisms G such that:

(a.1) any $G \in \mathcal{G}$ is a diffeomorphism of class C^1 of a neighborhood of $\overline{\Omega}_0$ onto its image.

We denote $\Omega_G = G(\Omega_0)$. Let

$$\| G \|_{C^0(\overline{\Omega}_0)} = \max_{x \in \overline{\Omega}_0} |G(x)|,$$

$$\| G \|_{C^1(\overline{\Omega}_0)} = \| G \|_{C^0(\overline{\Omega}_0)} + \max_{x \in \overline{\Omega}_0} \| \frac{\partial G}{\partial x}(x) \|,$$

and define similarly

$$\| G^{-1} \|_{C^0(\overline{\Omega}_G)}, \| G^{-1} \|_{C^1(\overline{\Omega}_G)} .$$

We assume that

$$\text{(a.2)} \sup_{G \in \mathcal{G}} \| G \|_{C^1(\overline{\Omega}_0)} < +\infty, \sup_{G \in \mathcal{G}} \| G^{-1} \|_{C^1(\overline{\Omega}_G)} < +\infty.$$

The family $\mathcal{G}$ is considered with a topology $\mathcal{T}$ such that the convergence $G_j \to G$ with respect to $\mathcal{T}$ implies

$$\| G_j - G \|_{C^0(\overline{\Omega}_0)} \to 0.$$

Let

$$\Sigma_t(G, u_0) \tag{2}$$

be the semigroup generated by a parabolic PDE

$$\frac{du}{dt} - \Delta u + f(u) = 0 \tag{3}$$

on $\Omega_G \times [0, +\infty)$ with boundary conditions

$$u = 0 \text{ on } \partial\Omega_G$$

and initial conditions

$$u(0, x) = u_0.$$

We suppose that there exists c such that

(a.3) $f(u)u \geq -c$;

(a.4) $|f'(u)| \leq c(1 + |u|^p)$, where $p \leq \min(\frac{2}{n-2}, \frac{4}{n})$ for $n > 2$.

Let $E = L_2(\Omega_0), E_G = L_2(\Omega_G)$. We define

$$G^* : E_G \to E :$$

for $u(y) \in E_G$ let $G^*u(y) = u(G^{-1}(y))$.

Theorem 3. *If conditions* (a.1)–(a.4) *hold, then:*

(1) *for every* $G \in \mathcal{G}$ *there exists a global attractor* A_G *of* (2) *in* E_G*;*

(2) *there exists a residual (in topology $\mathcal{T}$) subset $\mathcal{G}_0$ of $\mathcal{G}$ such that every $G \in \mathcal{G}_0$ is a continuity point of the mapping*

$$A : \mathcal{G} \to E^c,$$

here $A(G) = G^*(A_G)$.

Proof. Let $H_1(\Omega_G)$ be the completion of $C_0^\infty(\Omega_G)$ in the norm of the Sobolev space

$$\| u \|^2_{H_1(\Omega_G)} = \sum_{i=1}^{n} \int_{\Omega_G} (\frac{\partial u}{\partial x_i})^2 dx + \int_{\Omega_G} |u|^2 dx.$$

It is well-known [1] that under our assumptions for any $u_0 \in E_G$ there exists a unique solution $u(t)$ of (3) such that for every $T > 0$

$$u \in L_2([0,T], H_1(\Omega_G)) \cap L_\infty([0,T], L_2(\Omega_G)),$$

$$\frac{du}{dt} \in L_2([0,T], H_1(\Omega_G)^*).$$

This defines a semigroup $\Sigma_t(G, u_0) : E_G \to E_G$ by

$$\Sigma_t(G, u_0) = u(t).$$

This semigroup has a compact absorbing set

$$B_G = \{u \in E_G : \| u \|_{H_1(\Omega_G)} \leq R\},$$

where R is a sufficiently large constant which depends on constants c in (a.3), (a.4) and on diamΩ_G. Hence, $\Sigma_t(G, .)$ has a global attrator $A_G \in E_G$.

It follows from (a.2) that

$$\sup_{G \in \mathcal{G}} \text{diam}\Omega_G < \infty,$$

so that we can choose R independent of G.

Now we define the semigroup of operators

$$S_t(G, .) : E \to E, G \in \mathcal{G},$$

by

$$S_t(G, v_0) = G^* \Sigma_t(G, (G^*)^{-1} v_0).$$

It follows from (a.2) that

$$\sup_{x \in \Omega, G \in \mathcal{G}} |\det(\frac{\partial G}{\partial x}(x))| < \infty, \quad \sup_{y \in \Omega_G, G \in \mathcal{G}} |\det(\frac{\partial G^{-1}}{\partial y}(y))| < \infty,$$

hence

$$G^* : E_G \to E, G^* : H_1(\Omega_G) \to H_1(\Omega_0)$$

are bounded linear operators, and their inverses are bounded (uniformly on $G \in \mathcal{G}$). So there exists C (depending only on $\mathcal{G}$) such that for every $S_t(G,.), G \in \mathcal{G}$, the set

$$B_0 = \{u \in E : \| u \|_{H_1(\Omega_0)} \leq CR\}$$

is absorbing. As the set B_0 is compact in $E = L_2(\Omega_0)$, it remains to establish the continuity of S_t in $\mathcal{G} \times E$ and to apply Theorem 2.

Take $(G_0, v_0) \in \mathcal{G} \times E$ and consider sequences $G_j \in \mathcal{G}, v_{j0} \in E$ such that

$$G_j \to G_0 \text{ in } \mathcal{G}, v_{j0} \to v_0 \text{ as } j \to \infty.$$

Corresponding solutions of (3)

$$u_j(t) = \Sigma_t(G_j, u_{j0}), u_{j0} = (G_j^*)^{-1} v_{j0},$$

are bounded in

$$L_2([0,T], H_1(\Omega_{G_j})), L_\infty([0,T], L_2(\Omega_{G_j})),$$

and for $0 < \delta < t$ $u_j(t)$ are bounded in $H_1(\Omega_{G_j})$ by the smoothing property uniformly in j.

Hence,

$$v_j(t) = G_j^* u_j(t)$$

are bounded in the corresponding spaces of functions on Ω_0. There exists a subsequence v_j of v_j (we use the same labelling for subsequences) which weakly converges to a function v in these spaces, and $\frac{dv_j}{dt}$ converge in $L_2([0,T], H_1(\Omega)^*)$ to $\frac{dv}{dt}$. Let $u = (G_0^*)^{-1} v$.

Convergence $G_j \to G_0$ in $\mathcal{G}$ implies

$$\| G_j - G_0 \|_{C^0(\overline{\Omega_0})} \to 0,$$

hence for any subdomain O with $\overline{O} \subset \Omega_{G_0}$ we have

$$\overline{O} \subset \Omega_{G_j} = G_j(\Omega_0)$$

for j sufficiently large.

Since $u_j(t)$ are solutions of (3) in $\Omega_{G_j} \times (0,T)$ in the sense of distributions, we easily obtain that $u(t) = (G_0^*)^{-1} v(t)$ is a solution of this equation in $\Omega_{G_0} \times (0,T)$ as well. Dirichlet boundary condition holds since

$$\partial(G_0(\Omega_0)) = G_0(\partial\Omega_0).$$

At the same time we have a solution

$$u^*(t) = \Sigma_t(G_0, u_0), u_0 = (G_0^*)^{-1} v_0,$$

and $u^*(t) = u(t)$ by the uniqueness of solutions of (3).

Since any sequence $v_j(t), t > 0 (t \geq \delta > 0)$ has a converging subsequence due to the boundedness of $v_j(t)$ in $H_1(\Omega_0)$ and the compactness of the embedding

$$H_1(\Omega_{G_0}) \subset L_2(\Omega_{G_0}),$$

we obtain that

$$v_j(t) \to v(t)$$

for any such subsequence.

Thus, for any sequenses $G_j \to G_0, v_j \to v_0$ the following holds: there exists a subsequence of $v_j(t)$ which converges in E, the limit is independent on the subsequence, and it equals

$$v(t) = G_0^*(\Sigma_t(G_0, (G_0^*)^{-1} v_0)).$$

Hence, S_t is continuous at (G_0, v_0). This completes the proof.

Remark 1. An analogous statement is true for systems of PDE of the form (3).

Remark 2. Using the same ideas, it is possible to weaken our assumptions on the smoothness of $G \in \mathcal{G}$. An analogue of Theorem 3 is true for families $\mathcal{G}$ of Lipschitz homeomorphisms G such that the Lipschitz constants of G, G^{-1} are uniformly bounded for $G \in \mathcal{G}$.

Remark 3. In [11], D. Henry considered the equation

$$\frac{du}{dt} = \Delta u + f(x, u, \nabla u) \tag{4}$$

in a domain $\Omega \subset \mathbb{R}^n$ with boundary conditions

$$u = 0 \text{ or } \frac{\partial u}{\partial N} = g(x, u) \text{ on } \partial\Omega.$$

Equilibrium solutions of (4) satisfy

$$\Delta u + f(x, u, \nabla u) = 0 \text{ in } \Omega. \tag{5}$$

Infinite-dimensional transversality theory is applied in [11] to show that for a generic (with respect to C^k-topology, $k \geq 1$) bounded C^2 domain $\Omega \subset \mathbb{R}^n$ all solutions of (5) are hyperbolic.

J. Hale and G. Raugel studied in [9] families $T_h(t)$ of gradient C^1-semigroups. They showed that if:

(1) each $T_h(t)$ is asymptotically smooth with the set E_h of equilibrium points bounded (thus, $T_h(t)$ has a global compact attractor A_h);

(2) all the equilibrium points of $T_0(t)$ are hyperbolic,

then (under some additional conditions) attractors A_h are continuous at $h = 0$.

So, if we take the domain Ω in (4) as a parameter (and denote by $A(\Omega)$ the corresponding attractor), it is possible (using results of [9], [11]) to prove the genericity of domains Ω which are continuity points of the mapping A.

Let us explain the difference between our approach and the approach of [11]. In [11], given a bounded domain Ω_0, the collection of all domains C^k-diffeomorphic to Ω_0 is considered. Let $\mathcal{X}$ be the corresponding set of all diffeomorphisms (or a C^k-neighborhood of id_{Ω_0}). The result of [11] implies that there is a residual set $\mathcal{X}_0$ in $\mathcal{X}$ such that for all $h \in \mathcal{X}_0$ solutions of (5) are hyperbolic if $\Omega = h(\Omega_0)$.

In our Theorem 3, the family $\mathcal{G}$ is arbitrary. For example, we can consider a family of diffeomorphisms $G_\lambda : \mathbb{R}^2 \to \mathbb{R}^2, \lambda > 0$, given by:

$$G_\lambda(x_1, x_2) = (x_1, \lambda x_2).$$

Let $A(\lambda)$ be the corresponding attractors. It follows from Theorem 3 that there exists a residual subset $\Lambda \subset (0, +\infty)$ such that the points of Λ are continuity points of A, while the approach based on [11] does not work in this case.

It can be seen from the sketches of proofs in [11] (no complete proofs are published till now) that it is possible to generalize the results. Nevertheless, the considered set $\mathcal{X}$ of diffeomorphisms is to be "large enough" depending on the nonlinearity f in (4) (as usually when transversality theorems are applied).

Besides, smoothness requirements are different:

– in Theorem 3, Ω_0 is an arbitrary bounded domain, $\mathcal{G}$ can be considered with C^0-topology;

– in [11], the domain Ω_0 is C^2, and the topology is C^k with $k \geq 1$.

Remark 4. Since our results hold for systems, stability properties of attractors are not determined by properties of equilibrium points (attractors may include periodic orbits etc). Therefore, our results are applicable in principally more general situation than it was before.

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Preprint [12] contains the results of this paper.

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