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# GRACEFUL CHROMATIC NUMBER OF SOME CARTESIAN PRODUCT GRAPHS<sup>1</sup>

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**Abstract:** A graph  $G(V, E)$  is a system consisting of a finite non empty set of vertices  $V(G)$  and a set of edges  $E(G)$ . A (*proper*) *vertex colouring* of  $G$  is a function  $f : V(G) \rightarrow \{1, 2, \dots, k\}$ , for some positive integer  $k$  such that  $f(u) \neq f(v)$  for every edge  $uv \in E(G)$ . Moreover, if  $|f(u) - f(v)| \neq |f(v) - f(w)|$  for every adjacent edges  $uv, vw \in E(G)$ , then the function  $f$  is called *graceful colouring* for  $G$ . The minimum number  $k$  such that  $f$  is a graceful colouring for  $G$  is called the *graceful chromatic number* of  $G$ . The purpose of this research is to determine graceful chromatic number of Cartesian product graphs  $C_m \times P_n$  for integers  $m \geq 3$  and  $n \geq 2$ , and  $C_m \times C_n$  for integers  $m, n \geq 3$ . Here,  $C_m$  and  $P_m$  are cycle and path with  $m$  vertices, respectively. We found some exact values and bounds for graceful chromatic number of these mentioned Cartesian product graphs.

**Keywords:** Graceful colouring, Graceful chromatic number, Cartesian product.

## 1. Introduction

A graph  $G(V, E)$  is a system consisting of a finite non empty set of vertices  $V(G)$  and a set of edges  $E(G)$ . Let  $G$  and  $H$  be two disjoint graphs. The *Cartesian product* of  $G$  and  $H$ , denoted by  $G \times H$ , is the graph with vertex set  $V(G) \times V(H)$ , and edges  $xy, uv \in V(G) \times V(H)$  are adjacent in  $G \times H$ , if  $x = u$  and  $yv \in E(H)$  or  $y = v$  and  $xu \in E(G)$ . A (*proper*) *vertex colouring* of  $G$  is a way of colouring vertices in  $G$  such that each adjacent vertices are assigned to different colours.

If for a vertex colouring of  $G$  we have that every adjacent edges in  $G$  have different induced colours, then the vertex colouring is called *graceful*. We may think a graceful colouring of  $G$  as a function  $f : V(G) \rightarrow \{1, 2, \dots, k\}$ , for some positive integer  $k$ , such that for every edge  $uv \in E(G)$  we have  $f(u) \neq f(v)$ , and for any vertex  $u \in V(G)$  we have  $|f(u) - f(v)| \neq |f(u) - f(w)|$  for every vertices  $v, w \in V(G)$  which are adjacent to  $u$ . The absolute value  $|f(u) - f(v)|$  for every  $uv \in E(G)$ , is the *induced label* of the edge  $uv \in E(G)$ . In this sense, the terms colour and label are interchangeable. The smallest value of  $k$  for which the function  $f$  is a graceful vertex colouring of  $G$  is called the *graceful chromatic number* of  $G$ . The graceful colouring is a variation of graceful labeling which was introduced by Alexander Rosa in 1967 (see Gallian in [5]). Whereas, the notion of graceful colouring was introduced by Gary Chartrand in 2015, as a variant of the proper vertex  $k$ -colouring problem (see [3]). Since then, researches on graceful colouring numbers started to be celebrated.

Byers in [3] derived exact values for the graceful chromatic number of some graphs: path, cycle, wheel, and caterpillar; and introduced some bounds for certain connected regular graphs.

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Moreover, English, et al. in [4] invented graceful chromatic number of some classes of trees, and gave a lower bound for the graceful chromatic number of connected graphs with certain minimum degree. Mincu et al. in [6] derived graceful chromatic number of some well-known graph classes, such as diamond graph, Petersen graph, Moser spindle graph, Goldner-Harary graph, friendship graphs, and fan graphs. Graceful chromatic number of some particular unicyclic class graphs were presented by Alfarisi et al. (2019) in [1].

Furthermore, in 2022, Asy'ari et al. in [2] presented graceful chromatic numbers of several types of graphs, including star graphs, diamond graphs, book graphs. In addition, Asy'ari, et al. also stated some open problems. One of the problems is to determine the graceful chromatic number of some Cartesian product of certain graphs. Here we derive graceful chromatic number of Cartesian product graph  $C_m \times P_n$ ,  $m \geq 3, n \geq 2$ , where  $C_m$  is the cycle with  $m$  vertices and  $P_n$  is the path with  $n$  vertices. The Cartesian product graph  $C_m \times P_n$  is known as *prism* for  $n = 2$  and as *generalized prism* for  $n \geq 3$ . We also introduce bounds for Cartesian product graph  $C_m \times C_n$ ,  $m, n \geq 3$ .

To proceed with the main results, we need to introduce some introductory facts which will be beneficial for our further discussion.

Let  $G$  be a graph and  $x$  be a vertex of  $G$ . All vertex which are adjacent to  $x$  are called *the neighbors of  $x$* , and denoted by  $N(x)$ . The degree of the vertex  $x$ , denoted by  $\deg(x)$ , is equal to the cardinality of  $N(x)$ ,  $\deg(x) = |N(x)|$ . We will start with the following lemma.

**Lemma 1.** *Let  $G$  be a graph and  $u$  be a vertex in  $G$  with degree  $d \geq 1$ . Let  $f$  be a graceful colouring for  $G$ . If  $f(u) = a$ ,  $1 \leq a \leq d$ , then there is a vertex  $v \in N(u)$  with colour  $f(v) \geq d + a$ .*

*P r o o f.* Let  $f(u) = a$  with  $1 \leq a \leq d$ . If  $a = 1$ , the smaller possible colours we can assign for the all  $d$  neighbors  $v \in N(u)$  of  $u$ , are  $2, 3, \dots, d$  and the colour  $d + 1$ . This means that, there is a vertex  $v \in N(u)$  with  $f(v) \geq d + 1 = d + a$ . We are done for the case  $a = 1$ .

Now, assume  $f(u) = a$ ,  $1 < a \leq d$ . Note that the colours  $k$  and  $2a - k$ , for every  $k, 1 \leq k \leq a - 1$ , can not be assigned simultaneously for the vertices in  $N(u)$ , since they give the same difference from the colour  $a$ . Therefore, the maximum number of colours we may assign from the first  $2(a - 1)$  smallest colours  $\{k, 2a - k : 1 \leq k \leq a - 1\}$  is equal to  $a - 1$ . It implies that the remaining vertices in  $N(u)$  which are not coloured yet, is at least  $d - (a - 1)$  vertices. The colours we need for these vertices are started from a colour  $\geq 2a$ . This means that the next  $d - (a - 1)$  smallest colours we should assign are  $2a, 2a + 1, \dots, 2a + (d - (a - 1) - 1)$ . So, there is a vertex  $v \in N(u)$  such that its colour  $f(v) \geq 2a + (d - (a - 1) - 1) = d + a$ . □

In a specific case, the colour of a vertex  $u$  is equal to the degree of  $u$ ,  $f(u) = \deg(u)$ , we have the following corollary.

**Corollary 1.** *In a graph  $G$  with graceful colouring  $f$ , if the vertex  $u$  has degree  $d \geq 1$  and colour  $d$ , then there is a vertex  $v \in N(u)$  with colour  $f(v) \geq 2d$ .*

*P r o o f.* Let  $G$  be a graph and  $u$  be a vertex of  $G$  with  $\deg(u) = d$ . Let  $f$  be a graceful colouring for  $G$  where  $f(u) = d$ . By Lemma 1, we found a neighbor  $v$  of  $u$  such that  $f(v) \geq d + d = 2d$ . □

The following result was introduced by Byers (2018) in [3].

**Lemma 2** (Byers in [3]). *The graceful chromatic number of cycle  $C_n$  on  $n \geq 3$  vertices is*

$$\chi_g(C_n) = \begin{cases} 4, & \text{if } n \neq 5, \\ 5, & \text{if } n = 5. \end{cases} \tag{1.1}$$

Then, we will introduce some terminologies related with certain ladder graphs.

A *ladder* of  $2m$  vertices,  $m \geq 2$ , denoted by  $L_m$ , is the Cartesian product graph of the path on  $m$  vertices and the path on two vertices. The ladder  $L_2$  is the cycle graph of four vertices. Assume that the vertices of  $L_m$  are  $v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m$  such that its edges are  $v_i v_{i+1}, w_i w_{i+1} : 1 \leq i \leq m-1, v_i w_i : 1 \leq i \leq m$ . For  $m \geq 4$ , if the vertices  $v_1$  and  $v_m$ , and the vertices  $w_1$  and  $w_m$  are identified, then we obtain a prism  $C_{m-1} \times P_2$ . In this resulting  $C_{m-1} \times P_2$ ,  $v_1 = v_m, w_1 = w_m$ , and edge  $v_1 w_1 = v_m w_m$ . Due to this, we may call the ladder  $L_m$  as the *open* graph of  $C_{m-1} \times P_2$  about the edge  $v_1 w_1$ .

On the other side, let  $C_m \times P_2, m \geq 3$ , be a prism. This prism has vertex set  $\{v_1, v_2, \dots, v_m, w_1, w_2, \dots, w_m\}$  and edge set

$$\{v_i v_{i+1}, w_i w_{i+1} : 1 \leq i \leq m-1\} \cup \{v_1 v_m, w_1 w_m\} \cup \{v_i w_i : 1 \leq i \leq m\}.$$

After opening  $C_m \times P_2$  about the edge  $v_1 w_1$  into the ladder  $L_{m+1}$ , the vertices  $v_1$  and  $w_1$  copy themselves into two copies each; the first copy of  $v_1$  (resp.  $w_1$ ) is adjacent with  $v_2$  (resp.  $w_2$ ), and the second copy of  $v_1$  (resp.  $w_1$ ) is adjacent with  $v_m$  (resp.  $w_m$ ). These last vertex copies in the ladder  $L_{m+1}$  are named as  $v_{m+1}$  and  $w_{m+1}$ , respectively. Therefore, if  $f$  a colouring for the prism  $C_m \times P_2$ , then in the ladder  $L_{m+1}$  we have  $f(v_1) = f(v_{m+1})$  as well as  $f(w_1) = f(w_{m+1})$ . In this case, we may also call  $C_m \times P_2$  as the *closed* graph of  $L_{m+1}$  about the edges  $v_1 w_1$  and  $v_m w_m$ .

In the following lemma we will show that a ladder of  $2m$  vertices, with  $m \not\equiv 0 \pmod{4}$ , can not be gracefully coloured using 4 colours.

**Lemma 3.** *Using four different colours, the graph  $C_m \times P_2$ , with  $m \geq 3, m \not\equiv 0 \pmod{4}$ , can not be gracefully coloured.*

**P r o o f.** Let  $a, b, c$  and  $d$  be four different colours, and let  $m = 4k + r, 1 \leq r \leq 3$ . Consider the ladder  $L_{m+1}$  as the opened graph of  $C_m \times P_2$ . Let the vertex and edge sets of the ladder  $L_{m+1}$  be  $\{v_i, w_i : 1 \leq i \leq m+1\}$  and  $\{v_i v_{i+1}, w_i w_{i+1} : 1 \leq i \leq m, v_i w_i : 1 \leq i \leq m+1\}$ , respectively. Observe that the colour of  $v_j$  (resp.  $w_j$ ) must be the same with the colour of  $w_{j+2}$  (resp.  $v_{j+2}$ ) or of  $w_{j-2}$  (resp.  $v_{j-2}$ ) for realizable integer  $j$  (realizable means in the range of discussion). Without loss of generality, let the colour of  $v_1$  is  $a$ . Therefore, the colour of  $w_{4s+3}$  and of  $v_{4t+1}$  is  $a$ , for some realizable non-negative integers  $s, t$ . Now let us see cases:  $r = 1, r = 2$ , and  $r = 3$ . Suppose that  $f$  is a graceful colouring for  $C_m \times P_2$ .

*Case  $r = 1$ .* If we take  $t = k$ , then we have  $f(v_1) = a = f(v_{4k+1}) = f(v_m)$ . Note that  $v_{m+1} = v_{4k+2}$  is adjacent with  $v_m$ . Thus,  $f(v_{m+1})$  can not be  $a$  to maintain proper colouring property. But, in  $C_m \times P_2$ , vertices  $v_1$  and  $v_{m+1}$  are identical which insist  $f(v_{m+1}) = f(v_1) = a$ . This implies a contradiction. So, for  $r = 1$  the graph  $C_m \times P_2$  can not be gracefully coloured.

*Case  $r = 2$ .* Applying a similar argument, by assuming the colour of  $v_1$  is  $a$ , we have that  $f(w_{m+1}) = f(w_{4k+3}) = f(v_1) = a$ . In graph  $C_m \times P_2$ , vertices  $w_1$  and  $w_{m+1}$  are identical. On the other side,  $w_1$  is adjacent with  $v_1$ , so that they can not get the same colour. Thus, a contradiction occurs.

*Case  $r = 3$ .* Again by using a similar reason, we have that  $f(w_m) = f(w_{4k+3}) = f(v_1) = a$ . We know that  $w_{m+1}$  in  $C_m \times P_2$  is identified with  $w_1$ , and therefore is adjacent with both  $w_m$  and  $v_1$ . This implies that the induced edge colours of  $v_1 w_1 (= v_1 w_{m+1})$  and  $w_1 w_m$  are the same which then contradicts the gracefulness property.

In any case we have proven that  $C_m \times P_2, m \not\equiv 0 \pmod{4}$ , can not be gracefully coloured using only 4 colours.  $\square$

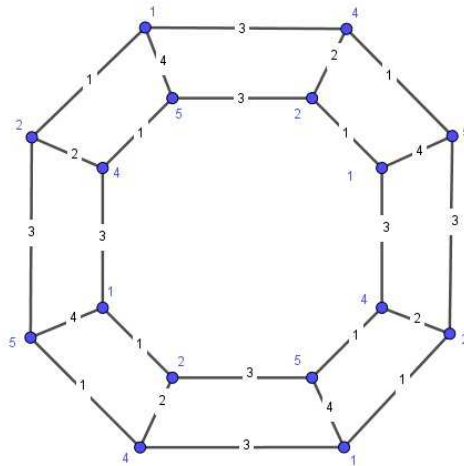


Figure 1. A graceful colouring of  $C_8 \times P_2$ .

## 2. Results on prism and generalized prism graphs

In this section, we will be dealing with the graceful chromatic number of prism  $C_m \times P_2$  first,  $m \geq 3$ , and then with the graceful chromatic number of generalized prism graphs  $C_m \times P_n$ ,  $m, n \geq 3$ . As for some consequences, we also derive some bounds for graceful chromatic number of graph  $C_m \times C_n$ ,  $m, n \geq 3$ , for some specific values of  $m$  and  $n$ .

Our main discussion will be separated into two subsections: For  $C_m \times P_2, m \geq 3$  and for  $C_m \times P_n$ , with  $m, n \geq 3$ .

### 2.1. Prism graph $C_m \times P_2$ for $m \geq 3$ .

**Theorem 1.** *If  $m \equiv 0 \pmod{4}$ , then the graceful chromatic number of graph  $C_m \times P_2$  is equal to 5.*

*P r o o f.* Note that the graph  $C_m \times P_2$  contains subgraph  $C_4$ . Based on Lemma 2, we may conclude that  $\chi_g(C_m \times P_2) \geq 4$ . Since all vertices of  $C_m \times P_2$  has degree 3, if the colour 3 is used, then by Corollary 1, the colour greater than 6 should occur. Therefore, the four colours we will use are 1, 2, 4, and 5. Now we will prove that using these four colours, we are able to colour  $C_m \times P_2$  gracefully. To confirm this, we will do by introducing the following graceful colouring technique for  $C_m \times P_2$  using only labels 1, 2, 4, and 5.

Let the vertices of  $C_m \times P_2$  is the set

$$\{v_{1+i}, v_{2+i}, v_{3+i}, v_{4+i}, w_{1+i}, w_{2+i}, w_{3+i}, w_{4+i} : i = 4k, k = 0, 1, 2, \dots, m/4 - 1\}$$

and its edge set is

$$\{v_1v_m, w_1w_m, v_mv_m, v_iv_{i+1}, w_iw_{i+1}, v_iw_i : i = 1, 2, \dots, m - 1\}.$$

Define a colouring  $f$  for  $C_m \times P_2$  as follows.

$$f(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 \pmod{4}, \\ 4, & \text{if } i \equiv 2 \pmod{4}, \\ 5, & \text{if } i \equiv 3 \pmod{4}, \\ 2, & \text{if } i \equiv 0 \pmod{4}, \end{cases} \quad f(w_i) = \begin{cases} 5, & \text{if } i \equiv 1 \pmod{4}, \\ 2, & \text{if } i \equiv 2 \pmod{4}, \\ 1, & \text{if } i \equiv 3 \pmod{4}, \\ 4, & \text{if } i \equiv 0 \pmod{4}. \end{cases} \quad (2.1)$$

Based on the above function  $f$ , it is clear that for every adjacent vertices  $u$  and  $v$  we have  $f(u) \neq f(v)$ . We can immediately observe that for any adjacent edges  $uw$  and  $wv$  in  $C_m$  we have

$$\{|f(u) - f(w)|, |f(w) - f(v)|\} = \{1, 3\}.$$

Furthermore, we also have

$$\{|f(v_i) - f(w_i)| : 1 \leq i \leq m\} = \{2, 4\}.$$

Remember that each vertex  $u$  in  $C_m \times P_2$  has degree 3; say  $x_1, x_2$ , and  $x_3$  are the vertices adjacent to  $u$ . From the function  $f$  we can immediately conclude that the set

$$\{|f(u) - f(x_1)|, |f(u) - f(x_2)|, |f(u) - f(x_3)|\}$$

is equal to  $\{1, 2, 3\}$  or to  $\{1, 3, 4\}$ . Thus, the function  $f$  satisfies the property to become graceful colouring for  $C_m \times P_2$ . Therefore,  $\chi_g(C_m \times P_2) = 5$ .  $\square$

**Theorem 2.** *If  $m \not\equiv 0 \pmod{4}$ , then the graceful chromatic number of graph  $C_m \times P_2$  is equal to 6.*

*P r o o f.* The proof of Theorem 2 will make use of the result described in the proof of Theorem 1.

For some positive integer  $k \geq 1$ , consider  $C_{4k} \times P_2$  which is coloured as in (2.1). Let the ladder  $L_{4k+1}$  be the open graph of  $C_{4k} \times P_2$  about  $v_1 w_1$ . Since  $C_m \times P_2$  contains subgraph  $C_4$ , to colour it gracefully, one needs at least 4 colours. But, when  $m \equiv 1, 2$  or  $3 \pmod{4}$ , based on Lemma 3, we can not colour the graph  $C_{4k} \times P_2$  gracefully using only 4 colours. Therefore, we have to use at least 5 colours. The smallest five colours are 1, 2, 3, 4, and 5. But, based on Corollary 1, whenever we apply 3 for a vertex colour, the colour 6 or greater colour must occur. Thus, the graceful chromatic number of  $C_m \times P_2$  is at least 6. To conclude that  $\chi_g(C_m \times P_2) = 6$ , we will proceed by showing that a graceful colouring exist with maximum colour 6, as follows.

*Case 1:  $m \equiv 1 \pmod{4}$ .* First, consider  $C_5 \times P_2$  with vertex set  $\{a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5\}$  and with edge set  $\{a_1 a_5, b_1 b_5, a_i a_{i+1}, b_i b_{i+1} : i = 1 \leq i \leq 4\} \cup \{a_i b_i : 1 \leq i \leq 5\}$ . Now, we colour vertices using the following function  $f$ :

$$f(a_i) = \begin{cases} 1, & \text{if } i = 1, \\ 4, & \text{if } i = 2, \\ 3, & \text{if } i = 3, \\ 5, & \text{if } i = 4, \\ 2, & \text{if } i = 5, \end{cases} \quad f(b_i) = \begin{cases} 5, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \\ 6, & \text{if } i = 3, \\ 1, & \text{if } i = 4, \\ 4, & \text{if } i = 5. \end{cases}$$

The coloured  $C_5 \times P_2$  will be used as the seed of our general construction for Case 1, and its diagram is depicted in Fig. 2.

Consider the opened ladder  $L_6$  from the coloured  $C_5 \times P_2$  above about  $a_1 b_1$ . In  $L_6$ , the colours of  $a_1, a_2, a_3, a_4, a_5$ , and  $a_6$  are 1, 2, 5, 3, 4, and 1, while the colours of  $b_1, b_2, b_3, b_4, b_5$ , and  $b_6$  are 5, 4, 1, 6, 2, and 5.

Then, consider the open ladder  $L_{4k+1}$ , for some positive integer  $k \geq 1$ , from the coloured  $C_{4k} \times P_2$  in Theorem 1 about  $v_1 w_1$ . Here, the colours of  $v_1$  and  $w_1$  are also 1 and 5, respectively. The same colours are also for  $v_{4k+1}$  which is 1, and for  $w_{4k+1}$  which is 5. Based on (2.1), we have  $f(v_{4k}) = 2$ , and  $f(w_{4k}) = 4$ . By identifying  $v_{4k+1}$  with  $a_6$  and  $w_{4k+1}$  with  $b_6$ , and maintaining the

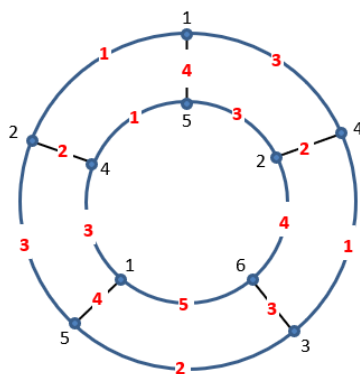


Figure 2. A graceful colouring of  $C_5 \times P_2$ .

other vertex colours, then we get a new ladder on  $4(k + 1) + 2$  vertices,  $L_{4(k+1)+2}$ , with graceful colouring.

Furthermore, we know that  $f(v_2) = 4$ ,  $f(w_2) = 2$ ,  $f(a_2) = 2$ , and  $f(b_2) = 4$ . Thus by identifying  $v_1$  with  $a_1$  and  $w_1$  with  $b_1$  in the ladder  $L_{4(k+1)+2}$ , we obtain  $C_{4(k+1)+1} \times P_2$  with a graceful colouring.

From here, we may infer that the graceful chromatic number of the graph  $C_m \times P_2$ , for  $m \equiv 1 \pmod{4}$  is equal to 6.

Case 2:  $m \equiv 2 \pmod{4}$ . First, consider  $C_6 \times P_2$  with vertex set

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\},$$

and with edge set

$$\{a_1a_6, b_1b_6, a_ia_{i+1}, b_ib_{i+1} : i = 1 \leq i \leq 5, \quad a_ib_i : 1 \leq i \leq 6\}.$$

As a seed graph, we define the following colouring for  $C_6 \times P_2$  as follows.

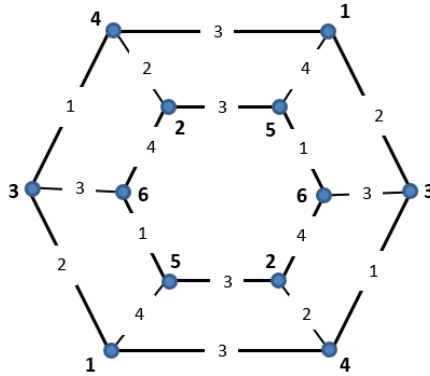
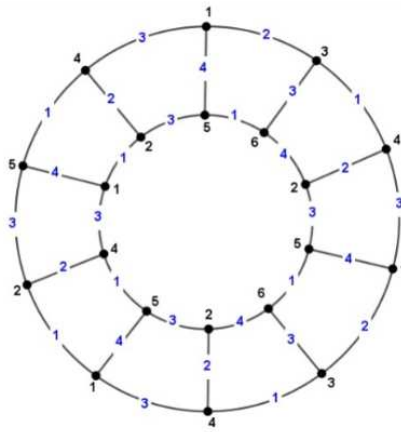
$$f(a_i) = \begin{cases} 1, & \text{if } i = 1, \\ 3, & \text{if } i = 2, \\ 4, & \text{if } i = 3, \\ 1, & \text{if } i = 4, \\ 3, & \text{if } i = 5, \\ 4, & \text{if } i = 6, \end{cases} \quad f(b_i) = \begin{cases} 5, & \text{if } i = 1, \\ 6, & \text{if } i = 2, \\ 2, & \text{if } i = 3, \\ 5, & \text{if } i = 4, \\ 6, & \text{if } i = 5, \\ 2, & \text{if } i = 6. \end{cases}$$

By inspection we can verify that the above colouring for  $C_6 \times P_2$  is graceful. The diagram of the coloured graph is shown in Fig. 3.

Let the ladder of 7 vertices,  $L_7$ , is the open graph from the  $C_6 \times P_2$  above about  $v_1w_1$ . We emphasize here that in this ladder  $L_7$ , vertices  $a_7$  and  $b_7$  have colours 1 and 5, respectively; the same as the colours of  $a_1$  and  $b_1$ , respectively.

We use again the same ladder  $L_{4k+1}$ ,  $k \geq 1$ , as in Case 1. Now we identify  $v_{4k+1}$  with  $a_7$  and  $w_{4k+1}$  with  $b_7$ , and maintaining the other vertex colours. Then we get a new ladder on  $4(k + 1) + 3$  vertices,  $L_{4(k+1)+3}$ , with graceful colouring.

Furthermore, we identify  $v_1$  with  $a_1$  and  $w_1$  with  $b_1$  in the ladder  $L_{4(k+1)+3}$ . Based on the previous colours, we know that the colours of  $v_2, w_2, a_2, b_2, v_1 = a_1, w_1 = b_1$ , are 4, 2, 3, 6, 1, 5, respectively. This means that after the last identification, the graceful colouring of  $C_{4(k+1)+2}$  are maintained. Thus, we may conclude that  $C_{4(k+1)+2} \times P_2$  is with graceful colouring.

Figure 3. A graceful colouring of  $C_6 \times P_2$ .Figure 4. A graceful colouring of  $C_{10} \times P_2$ .

A graceful labeled  $C_{10} \times P_2$  which is constructed using this method is depicted in Fig. 4.

From here, we may infer that the graceful chromatic number of the graph  $C_m \times P_2$ , for  $m \equiv 2 \pmod{4}$  is equal to 6.

*Case 3:  $m \equiv 3 \pmod{4}$ .* Here we will introduce a construction for graceful colouring of  $C_m \times P_2$  with  $m \equiv 3 \pmod{4}$ . We start with  $C_3 \times P_2$  with vertex set  $\{a_1, a_2, a_3, b_1, b_2, b_3\}$  and edge set  $\{a_3a_1, a_1a_2, a_2a_3, b_3b_1, b_1b_2, b_2b_3, a_1b_1, a_2b_2, a_3b_3\}$ . Then we colour  $C_3 \times P_2$  using the following colouring  $f$ .

$$f(a_i) = \begin{cases} 1, & \text{if } i = 1, \\ 3, & \text{if } i = 2, \\ 4, & \text{if } i = 3, \end{cases} \quad f(b_i) = \begin{cases} 5, & \text{if } i = 1, \\ 6, & \text{if } i = 2, \\ 2, & \text{if } i = 3. \end{cases}$$

We can immediately check that this colouring  $f$  is graceful. The diagram of the gracefully coloured graph  $C_3 \times P_2$  is shown in Fig. 5. We can verify that the graceful chromatic number of this graph is 6.

We should mention again that this above colouring of  $C_3 \times P_2$  is graceful. As we did for Case 1 and Case 2, first we will observe the open ladder  $L_4$  from  $C_3 \times P_2$  about  $a_1b_1$ . In this  $L_4$ , the



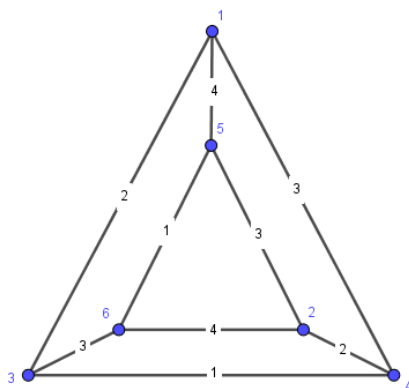


Figure 5. A graceful colouring of  $C_3 \times P_2$ .

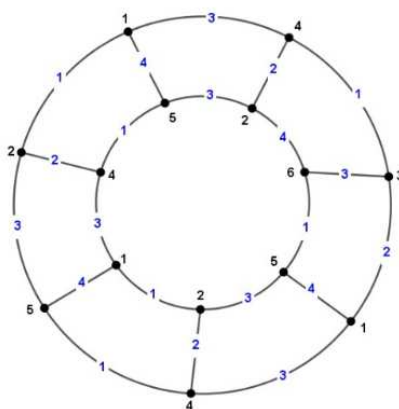


Figure 6. A graceful colouring of  $C_7 \times P_2$ .

colour of vertices  $a_4 = a_1 = 1$  and  $b_4 = b_1 = 5$ . Observe back the open ladder  $L_{4k+1}$  in Case 1 (and Case 2).

Now we identify  $v_{4k+1}$  with  $a_4$  and  $w_{4k+1}$  with  $b_4$  to obtain a graceful colouring ladder  $L_{4k+4}$ . Let us denote the colouring as  $\alpha$ . We can easily see that in this ladder we have  $\alpha(a_1) = \alpha(v_1) = 1$  and  $\alpha(b_1) = \alpha(w_1) = 5$ . Moreover, we have also  $\alpha(a_2) = f(a_2) = 3$ ,  $\alpha(b_2) = f(b_2) = 6$ ,  $\alpha(v_2) = 4$ , and  $\alpha(w_2) = 2$ . Thus, by identifying  $v_1$  with  $a_1$  and  $w_1$  with  $b_1$ , we get a graceful colouring  $C_{4k+3} \times P_2$ , with graceful chromatic number is 6. See the labeled graph  $C_7 \times P_2$  in Fig. 6 as an example of the graph resulted from the construction.

Therefore, we may conclude that the graceful chromatic number of the graph  $C_m \times P_2$ , with  $m \equiv 3 \pmod{4}$  is also 6.

Since in all cases of  $m$  we proved that  $C_m \times P_2$  has graceful chromatic number 6, we may conclude that  $\chi_g(C_m \times P_2) = 6$ . □

### 2.2. Results on generalized prism graphs $C_m \times P_n$ , $m, n \geq 3$ .

For a graph  $G$ , let  $f$  be a graceful colouring for  $G$ . It is obvious that for a vertex  $u \in V(G)$ , if  $v, w \in N(u)$ , then  $f(v) \neq f(w)$ . Therefore, we can immediately observe that the graph  $P_3 \times P_3$  can not be coloured by only four different colours. This observation gives

$$\chi_g(P_3 \times P_3) \geq 5.$$

But, if we use only five colours 1, 2, 3, 4 and 5, the center vertex of  $P_3 \times P_3$  must be 1 or 5. Then, by inspection we can show that using only five colours, we can not colour  $P_3 \times P_3$  gracefully. This gives the following lemma.

**Lemma 4.** *The graceful chromatic number of the graph  $P_3 \times P_3$ ,  $\chi_g(P_3 \times P_3) \geq 6$ .*

The following Lemma 5 will be an important tool for the proofs of our main results encountered in this section.

**Lemma 5.** *The graceful chromatic number of the graph  $P_5 \times P_5$ ,  $\chi_g(P_5 \times P_5) \geq 7$ .*

**P r o o f.** Let the vertices of  $P_5 \times P_5$  be  $V(P_5 \times P_5) = \{v_{ij} : i, j = 0, 1, 2, 3, 4\}$  and  $E(P_5 \times P_5) = \{v_{ij}v_{i(j+1)}, v_{ij}v_{(i+1)j} : i, j = 0, 1, 2, 3\}$ . Now, observe the subgraph  $P_3 \times P_3$  with  $V(P_3 \times P_3) = \{v_{ij} : i, j = 1, 2, 3\}$  and

$$E(P_3 \times P_3) = \{v_{ij}v_{(i+1)j}, v_{ij}v_{(i)(j+1)} : i, j = 1, 2\}.$$

In  $P_5 \times P_5$ , every vertex of the subgraph  $P_3 \times P_3$  has degree 4. Based on Lemma 4, for gracefully colouring  $P_3 \times P_3$ , we need at least five colours. If the colour 3 or 4 is assigned for a vertex of  $P_3 \times P_3$ , then based on Lemma 1 the colour greater than or equal to  $4 + 3 = 7$  must appear in  $P_5 \times P_5$ . If the colors 3 and 4 both are not assigned for any vertex of  $P_3 \times P_3$ , then, since we need at least five colours, we need some color greater than or equal to 7 for gracefully colouring  $P_5 \times P_5$ .  $\square$

Now, observe the graph  $P_4 \times P_3$ . We will make use of this observation for facilitating the result which will be formulated in Lemma 6. Let  $V(P_4 \times P_3) = \{v_{ij} : i = 0, 1, 2, 3; j = 0, 1, 2\}$ , and  $E(P_4 \times P_3) = \{v_{ij}v_{i(j+1)} : i = 0, 1, 2, 3; j = 0, 1\} \cup \{v_{ij}v_{(i+1)j} : i = 0, 1, 2; j = 0, 1, 2\}$ . The picture in Fig. 7 is the diagram of graph  $P_4 \times P_3$  with vertex names.

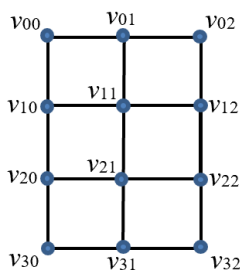


Figure 7. The graph  $P_4 \times P_3$  with vertex names.

In here, we will restrict a vertex colouring  $\alpha$  for  $P_4 \times P_3$  as  $\alpha(v_{0j}) = \alpha(v_{3j}), \forall j = 0, 1, 2$ . We will show that under this restriction, using only six colours, the vertex colouring  $\alpha$  can not be graceful.

Let the six colours be 1, 2, 3, 4, 5 and 6. Based on Lemma 1, since the degree of vertices  $v_{11}$  and  $v_{21}$  each is four, the colours 3 and 4 both can not be used for these two vertices. So, there are four colours: 1, 2, 5, and 6 that can be assigned for the vertices  $v_{11}$  and  $v_{21}$ . In total, there are six different combinations for colouring these two vertices:  $\{\alpha(v_{11}), \alpha(v_{21})\} = \{a, b\}$ ,  $a, b \in \{1, 2, 5, 6\}$ , with  $a \neq b$ . We can check by inspection that any one of these combinations results in the colouring  $\alpha$  is not graceful. But, for the space consideration, we will only describe the detail process for combination  $\{\alpha(v_{11}), \alpha(v_{21})\} = \{1, 2\}$  as in Fig. 8. Note that the case  $\alpha(v_{11}) = a$  and  $\alpha(v_{21}) = b$  is similar to the case  $\alpha(v_{11}) = b$  and  $\alpha(v_{21}) = a$ .

The explanation of the colouring process in Fig. 8 is the following:

- 1) The colours  $\alpha(v_{11}) = 1$  and  $\alpha(v_{21}) = 2$  are fixed as the initial combination.
- 2) The next vertex colouring follows the following vertices order:  $v_{20}, v_{10}, v_{00}, v_{01}, v_{02}, v_{12}, v_{22}$ . Note that  $\alpha(v_{3j}) := \alpha(v_{0j}), \forall j = 0, 1, 2$ , based on the restriction imposed for  $\alpha$ .
- 3) For some colours  $x, y$  and  $z$ , a notation  $x/y/z$  means that we assign the colour  $y$  (indicated with bold face) for the related vertex among the possible colours  $x, y$  and  $z$ .
- 4) The colour which stands alone (written in red bold face), indicates that the colour is the only possible colour for the related vertex.
- 5) The red cross sign **X** informs that the colouring process is discontinued at the related vertex, since there is no possible choice of colours to colour the vertex. The appearance of **X** indicates that the colouring fails to be graceful.

From Fig. 8 we can see that each colouring process ends to be not graceful which is indicated by the appearance of the sign **X**. Thus, we may conclude that under the restriction  $\alpha(v_{1j}) = \alpha(v_{4j}), j = 0, 1, 2$ , using exactly six different colours, we can not colour the graph  $P_4 \times P_3$  gracefully.

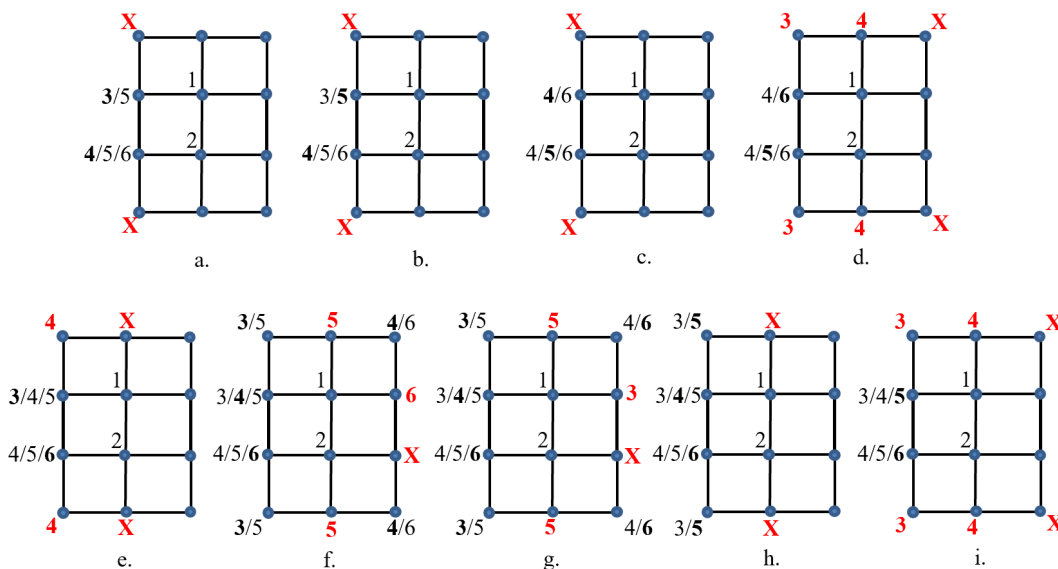


Figure 8. The colouring process for  $P_4 \times P_3$  with  $\alpha(v_{11}) = 1$  and  $\alpha(v_{21}) = 2$ .

If we extend this last observation to graph  $P_4 \times P_n, n \geq 3$ , with

$$V(P_4 \times P_n) = \{v_{ij} : i=0, 1, 2, 3; j=0, 1, \dots, n-1\},$$

and

$$E(P_m \times P_n) = \{v_{ij}v_{i(j+1)} : i=0, 1, 2, 3; j=0, 1, \dots, n-2\} \cup \{v_{ij}v_{(i+1)j} : i=0, 1, 2; j=0, 1, \dots, n-1\},$$

under restriction that  $\alpha(v_{0j}) = \alpha(v_{3j}), j = 0, 1, \dots, n-1$ , we may also conclude that we need at least seven colours to maintain the colouring  $\alpha$  becomes graceful for  $P_4 \times P_n$ .

From this last observation we can formulate the following result.

**Lemma 6.** For  $n \geq 3$ , the graceful chromatic number of the graph  $C_3 \times P_n, \chi_g(C_3 \times P_n) \geq 7$ .

**P r o o f.** The generalized prism graph  $C_3 \times P_n, n \geq 3$ , can be obtained by identifying vertices  $v_{0j}$  and  $v_{3j}$  for every  $j = 0, 1, 2, \dots, n - 1$  as it is in the last observation. By considering a graceful colouring  $\alpha$  for the graph  $P_4 \times P_n$  under the above mentioned restriction, we are done.  $\square$

For facilitating the discussion of our main results in this section, we need the following definition, as we defined a ladder as an open graph of  $C_m \times P_2$  in the previous section. Here we will define a similar one as an open graph from the graph  $C_m \times P_n, m, n \geq 3$ . Let the vertex set of graph  $C_m \times P_n, m, n \geq 3$ , be

$$\{v_{ij}, 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\},$$

and its edge set be

$$\{v_{ij}v_{kl}, \text{ if } i = k \text{ and } |j - l| = 1 \text{ or } j = l \text{ and } |i - k| \equiv 1 \pmod{m}\}.$$

Consider the open graph of  $C_m \times P_n, m, n \geq 3$ , about the path  $P$  which has end vertices  $v_{00}$  and  $v_{0n}$ , and has vertex set and edge set  $\{v_{0j}, j = 0, 1, \dots, n - 1\}$  and  $\{v_{0j}v_{0(j+1)}, j = 0, 1, \dots, n - 2\}$ , respectively. Denote this open graph by  $\mathcal{L}_{m+1,n}$ . This graph is a grid graph having  $(m + 1) \times n$  vertices which involves two copies of path  $P$ . These two copies of path  $P$ , each has vertices  $v_{0j}, j = 0, 1, \dots, n - 1$  and edges  $v_{0j}v_{0(j+1)}, j = 0, 1, \dots, n - 2$ . In the open graph  $\mathcal{L}_{m+1,n}$ , the vertices and edges of the second copy of  $P$  will be denoted by  $v_{mj}, j = 0, 1, \dots, n - 1$ , and  $v_{mj}v_{(m)(j+1)}, j = 0, 1, \dots, n - 1$ , respectively. It is clear that the vertex  $v_{mj}$  is adjacent with  $v_{(m-1)j}$  for every  $j = 0, 1, \dots, n - 2$ . In this case,  $C_m \times P_n$  can be reconstructed from  $\mathcal{L}_{m+1,n}$  by identifying vertex  $v_{0j}$  and  $v_{mj}$  for every  $j = 0, 1, \dots, n - 1$ .

**Theorem 3.** For any positive integers  $m, n \geq 3$ , with  $m \equiv 0 \pmod{3}$ ,  $\chi_g(C_m \times P_n) = 7$ .

**P r o o f.** From Lemma 4 we know that the graceful chromatic number of  $C_m \times P_n$  is at least seven. Now we will show that a graceful colouring exists for  $C_m \times P_n$  such that it uses only seven different colours, and therefore  $\chi_g(C_m \times P_n) = 7$ .

Let the vertex set of  $C_m \times P_n$  is  $\{v_{ij} | 0 \leq i \leq m - 1; 0 \leq j \leq n - 1\}$ , and edge set

$$\{v_{ij}v_{rs} | i = r \text{ and } |s - j| \equiv 1 \pmod{n} \text{ or } j = s \text{ and } |i - r| \equiv 1 \pmod{m}\}.$$

To this end, here we define a colouring function  $f$  for  $C_m \times P_n$  as follows.

$$f(v_{ij}) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod{3}, j \equiv 0 \pmod{6}, \\ 5, & \text{if } i \equiv 0 \pmod{3}, j \equiv 1 \pmod{6}, \\ 6, & \text{if } i \equiv 0 \pmod{3}, j \equiv 2 \pmod{6}, \\ 2, & \text{if } i \equiv 0 \pmod{3}, j \equiv 3 \pmod{6}, \\ 3, & \text{if } i \equiv 0 \pmod{3}, j \equiv 4 \pmod{6}, \\ 7, & \text{if } i \equiv 0 \pmod{3}, j \equiv 5 \pmod{6}, \\ 3, & \text{if } i \equiv 1 \pmod{3}, j \equiv 0 \pmod{6}, \\ 7, & \text{if } i \equiv 1 \pmod{3}, j \equiv 1 \pmod{6}, \\ 1, & \text{if } i \equiv 1 \pmod{3}, j \equiv 2 \pmod{6}, \\ 5, & \text{if } i \equiv 1 \pmod{3}, j \equiv 3 \pmod{6}, \\ 6, & \text{if } i \equiv 1 \pmod{3}, j \equiv 4 \pmod{6}, \\ 2, & \text{if } i \equiv 1 \pmod{3}, j \equiv 5 \pmod{6}, \\ 6, & \text{if } i \equiv 2 \pmod{3}, j \equiv 0 \pmod{6}, \\ 2, & \text{if } i \equiv 2 \pmod{3}, j \equiv 1 \pmod{6}, \\ 3, & \text{if } i \equiv 2 \pmod{3}, j \equiv 2 \pmod{6}, \\ 7, & \text{if } i \equiv 2 \pmod{3}, j \equiv 3 \pmod{6}, \\ 1, & \text{if } i \equiv 2 \pmod{3}, j \equiv 4 \pmod{6}, \\ 5, & \text{if } i \equiv 2 \pmod{3}, j \equiv 5 \pmod{6}. \end{cases} \tag{2.2}$$

An example of a graceful coloured graph  $C_6 \times P_n$  using (2.2) is shown in Fig. 9. In this figure we may also see the related open graph  $\mathcal{L}_{7,n}$  of  $C_6 \times P_n$ .

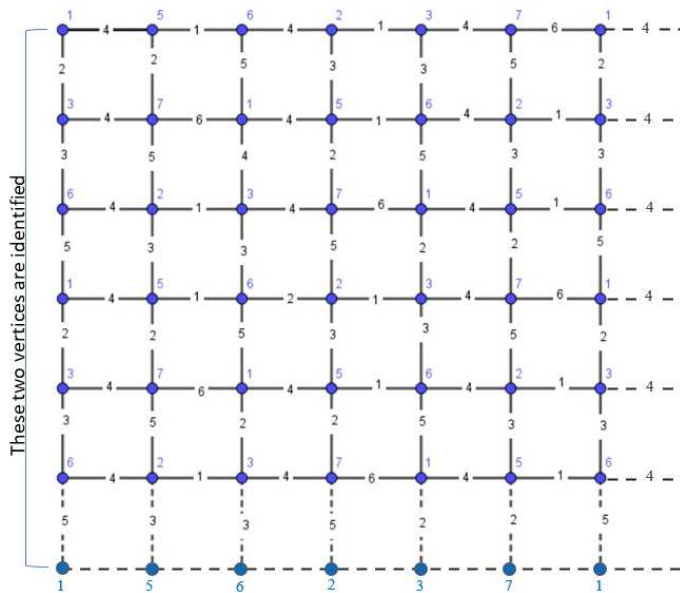


Figure 9. A graceful colouring of  $C_6 \times P_n$ ,  $n \geq 3$ .

Fig. 9 also helps us to be able to check by inspection that  $f$  is a graceful colouring for the graph  $C_m \times P_n$ , with  $m \equiv 0 \pmod{3}$ . Therefore, we may conclude that this graph has chromatic number 7.  $\square$

Furthermore, based on (2.2) we see that for every  $i$ ,  $0 \leq i \leq m - 1$ , we have  $f(v_{ij}) = f(v_{ik})$  provided  $|j - k| \equiv 0 \pmod{6}$ .

**Corollary 2.** For any positive integers  $m, n \geq 3$ , with  $m \equiv 0 \pmod{3}$  and with  $n \equiv 0 \pmod{6}$ ,  $\chi_g(C_m \times C_n) = 7$ .

*P r o o f.* The proof of this corollary may be derived from (2.2). From Theorem 3 we conclude that  $\chi_g(C_m \times P_n) = 7$ , if  $m \equiv 0 \pmod{3}$ , and  $n \geq 3$ . From (2.2) we know that  $f(v_{ij}) = f(v_{ik})$  whenever  $|j - k| \equiv 0 \pmod{6}$ . Thus, if  $n \equiv 0 \pmod{6}$ , then if we identify vertex  $v_{i0}$  and  $v_{in}$  for every  $i, 0 \leq i \leq m - 1$  in  $C_m \times P_n$ , then we get a graceful coloured graph  $C_m \times C_n$ ,  $m \equiv 0 \pmod{3}$  and  $n \equiv 0 \pmod{6}$ . Therefore, we may conclude that  $\chi_g(C_m \times C_n) = 7$  where  $m \equiv 0 \pmod{3}$  and  $n \equiv 0 \pmod{6}$ .  $\square$

In the remaining part of this section we will see the graceful colouring number for  $C_m \times P_n$ , with  $m \not\equiv 0 \pmod{3}$ ,  $n \geq 3$ . We start to observe the case  $m \equiv 1 \pmod{3}$  as we formulate in the following theorem.

**Theorem 4.** If  $m \equiv 1 \pmod{3}$ , then  $7 \leq \chi_g(C_m \times P_n) \leq 8$ .

*P r o o f.* We will make use of prism graph  $C_4 \times P_n$  as the seed of our graceful colouring construction. We first introduce a colouring for the graph  $C_4 \times P_n$ ,  $n \geq 3$ .

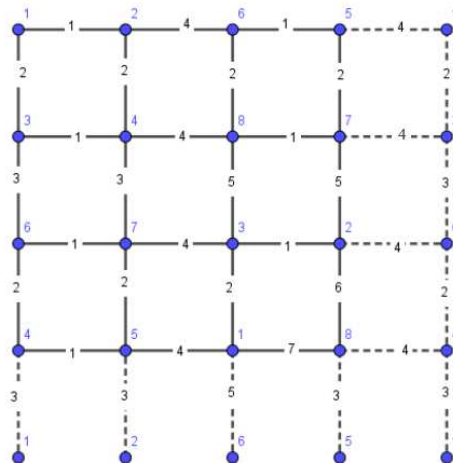


Figure 10. A graceful colouring of  $C_4 \times P_n$ .

Let the vertex set of  $C_4 \times P_n$  is

$$\{v_{ij} | 0 \leq i \leq 3; 0 \leq j \leq n - 1\},$$

and edge set

$$\{v_{ij}v_{rs} | i = r \text{ and } |s - j| \equiv 1 \pmod{n} \text{ or } j = s \text{ and } |i - r| \equiv 1 \pmod{4}\}.$$

To this end, we define a colouring function  $f$  as follows.

$$f(v_{ij}) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod{4}, j \equiv 0 \pmod{4}, \\ 2, & \text{if } i \equiv 0 \pmod{4}, j \equiv 1 \pmod{4}, \\ 6, & \text{if } i \equiv 0 \pmod{4}, j \equiv 2 \pmod{4}, \\ 5, & \text{if } i \equiv 0 \pmod{4}, j \equiv 3 \pmod{4}, \\ 3, & \text{if } i \equiv 1 \pmod{4}, j \equiv 0 \pmod{4}, \\ 4, & \text{if } i \equiv 1 \pmod{4}, j \equiv 1 \pmod{4}, \\ 8, & \text{if } i \equiv 1 \pmod{4}, j \equiv 2 \pmod{4}, \\ 7, & \text{if } i \equiv 1 \pmod{4}, j \equiv 3 \pmod{4}, \\ 6, & \text{if } i \equiv 2 \pmod{4}, j \equiv 0 \pmod{4}, \\ 7, & \text{if } i \equiv 2 \pmod{4}, j \equiv 1 \pmod{4}, \\ 3, & \text{if } i \equiv 2 \pmod{4}, j \equiv 2 \pmod{4}, \\ 2, & \text{if } i \equiv 2 \pmod{4}, j \equiv 3 \pmod{4}, \\ 4, & \text{if } i \equiv 3 \pmod{4}, j \equiv 0 \pmod{4}, \\ 5, & \text{if } i \equiv 3 \pmod{4}, j \equiv 1 \pmod{4}, \\ 1, & \text{if } i \equiv 3 \pmod{4}, j \equiv 2 \pmod{4}, \\ 8, & \text{if } i \equiv 3 \pmod{4}, j \equiv 3 \pmod{4}. \end{cases} \tag{2.3}$$

For an illustration one can see in Fig. 10

Fig. 10 helps us to see that (2.3) gives a graceful colouring for  $C_4 \times P_n$  for every  $n \geq 3$  with  $\chi_g(C_4 \times P_n) \leq 8$ . Therefore, based on Lemma 4, we may conclude that  $7 \leq \chi_g(C_4 \times P_n) \leq 8$ .

Furthermore, the graceful colouring of  $C_m \times P_n$ , with  $m \equiv 1 \pmod{3}$  and  $n \geq 3$  in general, is obtained by extending graceful coloured graph  $C_4 \times P_n$  using the prism graph  $C_3 \times P_n$  which has colouring as we will show below.

Let the vertex set of  $C_3 \times P_n$  is

$$\{v_{ij} | 0 \leq i \leq 2; 0 \leq j \leq n - 1\},$$

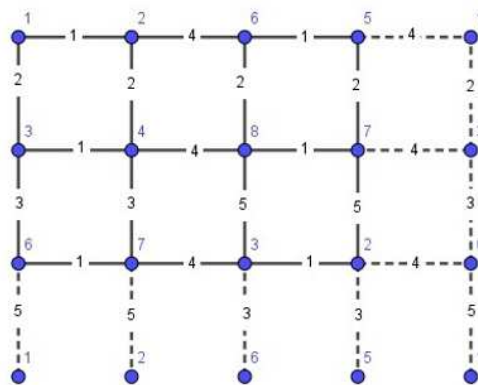


Figure 11. A graceful colouring of  $C_3 \times P_n$ .

and edge set

$$\{v_{ij}v_{rs} \mid i = r \text{ and } |s - j| \equiv 1 \pmod{n} \text{ or } j = s \text{ and } |i - r| \equiv 1 \pmod{3}\}.$$

To this end, we define a colouring function  $f$  as follows.

$$f(v_{ij}) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod{3}, j \equiv 0 \pmod{4}, \\ 2, & \text{if } i \equiv 0 \pmod{3}, j \equiv 1 \pmod{4}, \\ 6, & \text{if } i \equiv 0 \pmod{3}, j \equiv 2 \pmod{4}, \\ 5, & \text{if } i \equiv 0 \pmod{3}, j \equiv 3 \pmod{4}, \\ 3, & \text{if } i \equiv 1 \pmod{3}, j \equiv 0 \pmod{4}, \\ 4, & \text{if } i \equiv 1 \pmod{3}, j \equiv 1 \pmod{4}, \\ 8, & \text{if } i \equiv 1 \pmod{3}, j \equiv 2 \pmod{4}, \\ 7, & \text{if } i \equiv 1 \pmod{3}, j \equiv 3 \pmod{4}, \\ 6, & \text{if } i \equiv 2 \pmod{3}, j \equiv 0 \pmod{4}, \\ 7, & \text{if } i \equiv 2 \pmod{3}, j \equiv 1 \pmod{4}, \\ 3, & \text{if } i \equiv 2 \pmod{3}, j \equiv 2 \pmod{4}, \\ 2, & \text{if } i \equiv 2 \pmod{3}, j \equiv 3 \pmod{4}. \end{cases} \tag{2.4}$$

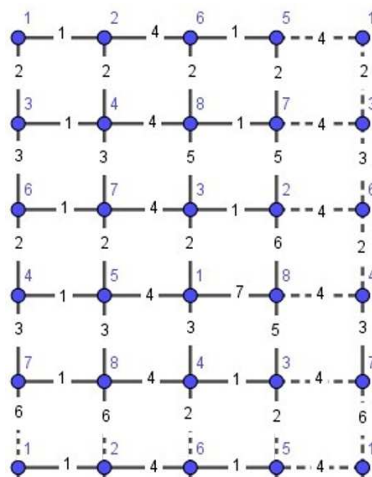
The diagram of coloured graph  $\mathcal{L}_{4,n}$  from  $C_3 \times P_n$  is depicted in Fig. 11. The coloured graph  $C_3 \times P_n$  is obtained by identifying  $v_{0j}$  and  $v_{3j}$  for all  $j, 0 \leq j \leq n - 1$ . We can immediately observe that (2.4) gives a graceful colouring for the prism graph  $C_3 \times P_n$  with  $\chi_g(C_3 \times P_n) \leq 8$ . Again based on Lemma 4, we conclude that  $7 \leq \chi_g(C_3 \times P_n) \leq 8$ .

For producing a graceful colouring for  $C_m \times P_n, m \equiv 1 \pmod{3}$  we use  $\mathcal{L}_{5,n}$  from  $C_4 \times P_n$  and  $\mathcal{L}_{4,n}$  from  $C_3 \times P_n$ , by identifying  $v_{5j}$  of  $\mathcal{L}_{5,n}$  and  $v_{0j}$  of  $\mathcal{L}_{4,n}$  for all  $j, 0 \leq j \leq n - 1$ . This identification results in a graceful coloured grid graph  $\mathcal{L}_{8,n}$ . Then, if we identify  $v_{8j}$  of  $\mathcal{L}_{8,n}$  and  $v_{0j}$  of  $\mathcal{L}_{4,n}$  for all  $j, 0 \leq j \leq n - 1$ , we get a graceful coloured grid graph  $\mathcal{L}_{11,n}$ . Continuing the same procedure, then we get a graceful coloured grid graph  $\mathcal{L}_{(m+1),n}$ . Then by identifying vertex  $v_{0j}$  and  $v_{mj}$  from  $\mathcal{L}_{(m+1),n}$  we obtain  $C_m \times P_n$  with  $m \equiv 1 \pmod{3}$  and  $n \geq 3$ .  $\square$

As one consequence, as we formulated Corollary 2 based on Theorem 3, we also formulate a corollary based on Theorem 4 as the following.

**Corollary 3.** *If  $m \equiv 1 \pmod{3}$  and  $n \equiv 0 \pmod{4}$ , then  $7 \leq \chi_g(C_m \times C_n) \leq 8$ .*

Now we go to the next case  $m \equiv 2 \pmod{3}$ . The result is formulated in the following theorem.

Figure 12. A graceful colouring of grid graph  $\mathcal{L}_{6,n}$  of  $C_5 \times P_n$ .

**Theorem 5.** *If  $m \equiv 2 \pmod{3}$ , then  $7 \leq \chi_g(C_m \times P_n) \leq 8$ .*

*P r o o f.* To proof this theorem, we will start with a graceful colouring for  $C_5 \times P_n$ ,  $m \equiv 2 \pmod{3}$ ,  $n \geq 3$ . We introduce the following colouring for the graph  $C_5 \times P_n$ ,  $n \geq 3$ .

$$f(v_{ij}) = \begin{cases} 1, & \text{if } i \equiv 0 \pmod{5}, & j \equiv 0 \pmod{4}, \\ 2, & \text{if } i \equiv 0 \pmod{5}, & j \equiv 1 \pmod{4}, \\ 6, & \text{if } i \equiv 0 \pmod{5}, & j \equiv 2 \pmod{4}, \\ 5, & \text{if } i \equiv 0 \pmod{5}, & j \equiv 3 \pmod{4}, \\ 3, & \text{if } i \equiv 1 \pmod{5}, & j \equiv 0 \pmod{4}, \\ 4, & \text{if } i \equiv 1 \pmod{5}, & j \equiv 1 \pmod{4}, \\ 8, & \text{if } i \equiv 1 \pmod{5}, & j \equiv 2 \pmod{4}, \\ 7, & \text{if } i \equiv 1 \pmod{5}, & j \equiv 3 \pmod{4}, \\ 6, & \text{if } i \equiv 2 \pmod{5}, & j \equiv 0 \pmod{4}, \\ 7, & \text{if } i \equiv 2 \pmod{5}, & j \equiv 1 \pmod{4}, \\ 3, & \text{if } i \equiv 2 \pmod{5}, & j \equiv 2 \pmod{4}, \\ 2, & \text{if } i \equiv 2 \pmod{5}, & j \equiv 3 \pmod{4}, \\ 4, & \text{if } i \equiv 3 \pmod{5}, & j \equiv 0 \pmod{4}, \\ 5, & \text{if } i \equiv 3 \pmod{5}, & j \equiv 1 \pmod{4}, \\ 1, & \text{if } i \equiv 3 \pmod{5}, & j \equiv 2 \pmod{4}, \\ 8, & \text{if } i \equiv 3 \pmod{5}, & j \equiv 3 \pmod{4}, \\ 7, & \text{if } i \equiv 4 \pmod{5}, & j \equiv 0 \pmod{4}, \\ 8, & \text{if } i \equiv 4 \pmod{5}, & j \equiv 1 \pmod{4}, \\ 4, & \text{if } i \equiv 4 \pmod{5}, & j \equiv 2 \pmod{4}, \\ 3, & \text{if } i \equiv 4 \pmod{5}, & j \equiv 3 \pmod{4}. \end{cases} \quad (2.5)$$

The diagram for coloured grid graph  $\mathcal{L}_{6,n}$  of  $C_5 \times P_n$ , which is derived from (2.5), can be seen in Fig. 12. Using this diagram we may conclude that the colouring is graceful. It is clear that  $\chi_g(C_5 \times P_n) \leq 8$ .

The process of expanding to get coloured graph  $C_m \times P_n$ ,  $m \equiv 2 \pmod{3}$ ,  $n \geq 3$ , is similar to the previous process as was described in the proof of Theorem 4. Here we use graceful coloured grid graph  $\mathcal{L}_{6,n}$  from graceful coloured graph  $C_5 \times P_n$ , and graceful coloured grid graph  $\mathcal{L}_{4,n}$  from graceful coloured graph  $C_3 \times P_n$ . Again by considering Lemma 4, we then conclude that



$$7 \leq \chi_g(C_5 \times P_n) \leq 8. \quad \square$$

Similar to the previous corollaries, here we formulate the following corollary as a consequence of Theorem 5.

**Corollary 4.** *If  $m \equiv 2 \pmod{3}$  and  $n \equiv 0 \pmod{4}$ , then  $7 \leq \chi_g(C_m \times C_n) \leq 8$ .*

### 3. Conclusion

In the discussion above, it has been proven that prism graph  $C_m \times P_2$  has a chromatic number equal to 5 when  $m \equiv 0 \pmod{4}$ , and equal to 6 when  $m \not\equiv 0 \pmod{4}$ . While for generalized prism  $C_m \times P_n$  we found that its chromatic number is equal to 7 while  $m \equiv 0 \pmod{3}$ . Whereas for  $m \not\equiv 0 \pmod{3}$ , we got that  $7 \leq \chi_g(C_m \times P_n) \leq 8$ . Based on these results, we could also derive some exact and bound values of graceful chromatics number of  $C_m \times C_n$  for certain  $m, n \geq 3$ . Regarding this last observation, we propose the following open problem and conjecture.

*Conjecture.* If  $m \not\equiv 0 \pmod{3}$  and  $n \geq 3$ , then  $\chi_g(C_m \times P_n) = 8$ .

*Open problem.* What is  $\chi_g(C_m \times C_n)$ , if  $m \not\equiv 0 \pmod{3}$ ,  $m, n \geq 3$ ?

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