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# INVERTIBLE INFINITARY CALCULUS WITHOUT LOOP RULES FOR A RESTRICTED FTL

Abstract. In the paper a fragment of first order linear time logic (with operators "next" and "always") is considered. The object under investigation in this fragment is so-called $t$-$D$-sequents. For considered $t$-$D$-sequents invertible infinitary sequent calculus $G_\omega^+$ is constructed. This calculus has no loop rules, i.e. rules with duplications of a main formula in the premises of the rules. The calculus $G_\omega^+$ along with $\omega$-type rule for the temporal operator "always" contains an integrated separation rule $(IS)$ which includes the traditional loop-type rule $(\Box \to)$, a special rule $(\forall \to)$ (without duplication of a main formula) and traditional rule for the temporal operator "next". The rule $(\to \exists)$ is incorporated in an axiom. The soundness and $\omega$-completeness of the constructed calculus $G_\omega^+$ are proved.

## 1. Introduction

In order to examine a gap between the notions of formal provability and validity in some logical systems, a concept of proof rules is generalized by using infinitary rules. An infinitary rule has infinitely many premises. A rule of this kind was first considered by Hilbert [10] in the classical arithmetics of natural numbers. The rule is called infinite induction (the so-called $\omega$-rule or Carnap rule) and has (in the sequent notation) the following form:

$$\frac{\Gamma \to \Delta, A(0);\ \Gamma \to \Delta, A(1); \dots ; \Gamma \to \Delta, A(k); \dots}{\Gamma \to \Delta, \forall x A(x)} (\to \forall_\omega),$$

here $k \in \omega = \{0, 1, \dots\}$. The completeness and consistency of the classical arithmetics of natural numbers with the rule $(\to \forall_\omega)$ was proved by Novikov [20], Schütte [36] and others. In a series of papers (see, e.g., Dragalin [4], Kuznecov [17], Mints [19], Shoenfield [38]), different "constructive" variants of the rule were considered. Always when the "constructive" rule $(\to \forall_\omega^c)$ has been applied it is assumed that for any $k$ there must exist some constructive method to prove the premise of the rule $(\to \forall_\omega^c)$. In Dragalin [4], Kuznecov [17], Shoenfield [38], the completeness of the classical arithmetics with the rule $(\to \forall_\omega^c)$ is proved. A heightened interest in the $\omega$-rule can also be explained by the known Kreisel hypothesis [14] (see also, e.g., Orevkov [22], [23]).

From Gödel theorem it follows that the arithmetical formal system (with an ordinary induction rule or axiom) is incomplete. We can see similar situation in a first-order linear time temporal logic. Namely, it is known (see, e.g., Gabbay [8], Andreka, Nemeti and Sain [1], Szalas [40], Kröger [16]) that the first-order linear time temporal logic ($FTL$, in short) with operators $\bigcirc$ ("next") and $\Box$ ("always") is not recursively enumerable, i.e., it does not possess a complete finitary axiomatization. However, for this logic one can construct a complete infinitary calculus containing an $\omega$-type rule of the kind:

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta, \bigcirc A; \ldots ; \Gamma \to \Delta, \bigcirc^k A; \ldots}{\Gamma \to \Delta, \Box A} (\to \Box_\omega).$$

A more complex $\omega$-type rule can be present for the binary operator $W$ ("unless"), see, e.g., Pliuškevičius [26]:

$$\frac{\Gamma \to \Delta, W_0;\ \ldots\ ; \Gamma \to \Delta, W_k;\ \ldots}{\Gamma \to \Delta, A\, W\, B} (\to W_\omega),$$

where $W_0 = B \vee A$; $W_k = B \vee (A \wedge \bigcirc W_{k-1})$ $(k = 1, 2, \ldots)$.

The completeness of an infinitary calculus for the first-order linear time temporal logic was proved by Kawai [13], Szalas [41], and others.

The similarity of the linear time temporal logic to the arithmetic of natural numbers also manifests itself in the property of noncompactness. For example, if $N = \{A, \bigcirc A, \ldots, \bigcirc^n A, \ldots\}$ is an infinite set then clearly the formula $\Box A$ semantically follows from $N$ (i.e., $N \models \Box A$). On the other hand, there does not exist a finite subset $N^*$ of $N$ such that $N^* \models \Box A$.

The infinitary axiomatization is wide used and has a rich tradition in the so-called logics of programs. For example, in Kröger [15], the infinitary axiomatization of the Hoare logic with an $\omega$-rule for the operator "while" is proposed; the $\omega$-rule is used in the algorithmic logic (Banachowski, Kreczmar, Mirkowska, Rasiowa, and Salwicki [2]) as well as in the uninterpreted dynamic first-order logic (Harel, Kozen, Tiuryn [9]) and in the constructive propositional dynamic logic (Leivant [18]). The infinitary completeness of the propositional linear time temporal logic is proved in Sundholm [39]. A first-order infinitary linear time temporal logic were considered in Pliuškevičius [24]–[35].

An infinitary axiomatization of a linear time temporal logic has some useful properties, namely: 1) it allows one to establish closer links between the theory of models and proof theory; 2) it explicitly reflects the semantics of the temporal operators under consideration; 3) it allows a natural and rather constructive proof of the completeness of a given infinitary calculus; 4) the complexity of premises of an infinitary rule (e.g., the rule $(\to \Box_\omega)$) is lower than the complexity of a conclusion.

The aim of this paper is to construct invertible infinitary sequent calculus without loop rules for a restricted FTL without non-repeating condition (see, e.g. Pliuškevičius [32]).

An inference rule $(i)$ of some sequent calculus is called a loop rule if the premise (premises) of the rule $(i)$ contains the main formula of the rule. Typical examples of loop rules are, for instance, the rules $(\supset\to)$, $(\neg \to)$ in the intuitionistic propositional calculus, the rule $(\Box \to)$ in the sequent calculus for the modal logic $S4$, and the rules $(\forall \to)$ and $(\to \exists)$ in the sequent calculus for a classical predicate calculus, i.e., the following rules:

$$\frac{A \supset B, \Gamma \to A;\ B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\,(\supset\to); \quad \frac{\neg A, \Gamma \to A}{\neg A, \Gamma \to \Delta}\,(\neg \to);$$

$$\frac{A, \Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\,(\Box \to); \quad \frac{A(t), \forall x A(x), \Gamma \to \Delta}{\forall x A(x), \Gamma \to \Delta}\,(\forall \to);$$

$$\frac{\Gamma \to \Delta, A(t), \exists x A(x)}{\Gamma \to \Delta, \exists x A(x)}\,(\to \exists).$$

In linear time temporal logic with operators $\bigcirc$ ("next") and $\Box$ ("always"), there is an analog of the loop rule as well, namely, the rule of the form:

$$\frac{A, \bigcirc\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\,(\Box \to),$$

where $\bigcirc\Box A$ corresponds to a temporal shift of the main formula $\Box A$ of the rule $(\Box \to)$.

A characteristic feature of the loop rule $(i)$ is the fact that the complexity of a premise (premises) of rule $(i)$ is greater than (or equal to) the complexity of the conclusion of the rule $(i)$. Loop rules may be a cause of severe problems in the proof search.

First investigations in elimination of loop rules in intuitionistic calculus were presented by Vorobjov [42] and Orevkov [21]. Loop-free calculi for the intuitionistic propositional logic were constructed by Dyckhoff [5], Hudelmaier [11], Degtyarev and Voronkov [3]. For the propositional modal logic $S4$ loop-free calculus was constructed by Hudelmaier [12].

The present paper is a generalization of the papers [30], [32] where loop-free infinitary calculus for a fragment of $FTL$, namely, so-called $t$-

$D$-sequents (see Definition 21) with non-repeating condition, is constructed. Here the restriction $t$-$D$-sequents to meet non-repeating condition is removed. Analogously as in Pliuškevičius [31], [33], [34] using the constructed infinitary calculus, saturation-based decision procedure for considered fragment of FTL can be obtained. It means that for considered fragment of FTL $\omega$-type rule is *not essential*. However, an infinitary calculus is used to justify a deductive decision procedure for the considered fragment of FTL. The infinitary calculus, which allows a simple decision procedure for induction-free t-D-sequents, is a *first step* constructing decision procedure for arbitrary t-D-sequents. There are theorem proving methods for propositional linear time temporal logic (e.g., Fisher's resolution method [7], Wolper's tableaux method [43]) which are justified not using infinitary calculus.

Our approach uses infinitary calculi. For $FTL$ $\omega$-type rules naturally reflect semantics of induction-like temporal operators and are essential, in general. Infinitary calculi allow to present a rather constructive proof of completeness of considered fragments of $FTL$. When completeness is proved, $\omega$-type rules are replaced by decision or semi-decision procedure, if it is possible. On the other hand, it is expected that the methods presented in the paper may help to construct simpler infinitary calculi even if an $\omega$-type rule is *essential*.

The paper is organized as follows. In section 2 a traditional $\omega$-complete infinitary calculus $G_\omega$ and a calculus $G^*_\omega$ without logical loop rules, i.e., without rules $(\forall \to)$ and $(\to \exists)$, are described. In the calculus $G^*_\omega$ instead of the rule $(\forall \to)$ there is some specialization of the rule without duplication of the main formula in the premise of the rule. The rule $(\to \exists)$ is incorporated in an axiom. In section 3 an equivalence between calculi $G_\omega$ and $G^*_\omega$ with respect to so-called $D$-sequents is proved. In section 4 some auxiliary tools for constructing a loop-free invertible calculus $G^+_\omega$ (i.e., a calculus without the rule $(\Box \to)$) are presented. In section 5 the calculus $G^+_\omega$ is constructed, the soundness and $\omega$-completeness of the calculus $G^+_\omega$ with respect to $t$-$D$-sequents is proved.

## 2. Infinitary calculus without logical loop rules

In this section a traditional infinitary calculus $G_\omega$ and some specification of this calculus without logical loop rules, i.e. without rules $(\forall \to)$, $(\to \exists)$, are described.

**Definition 1** (language, term, formula, atomic formula)**.** *The language is a countable collection of predicate symbols* $P, Q, R, P_1, Q_1, R_1, \dots,$

*functional symbols* $f, g, h, f_1, g_1, h_1, \ldots,$ *constants* $a, b, c, a_1, b_1, c_1, \ldots,$ *variables* $x, y, z, x_1, y_1, z_1, \ldots,$ *logical symbols* $\supset, \wedge, \vee, \neg$, *quantifiers* $\forall, \exists$, *and temporal operators* $\bigcirc$ *("next")*, $\Box$ *("always"). The language does not contain the equality symbol. We assume that all the predicate symbols are flexible (i.e., their value change in time), and the function symbols and constants are rigid (i.e., with time-independent meanings). A term and a formula are defined in the usual way. An atomic formula is an expression of the form* $P(t_1, \ldots, t_n)$, *where* $P$ *is a predicate symbol,* $t_i$ $(1 \leq i \leq n)$ *is a term.*

In first-order linear time temporal logic we have $\bigcirc(A \odot B) \equiv \bigcirc A \odot \bigcirc B$ $(\odot \in \{\supset, \wedge, \vee\})$ and $\bigcirc \sigma A \equiv \sigma \bigcirc A$ $(\sigma \in \{\neg, \Box, \forall x, \exists x\})$. Relying on these equivalences, we can consider occurrences of the "next" operator $\bigcirc$ only entering the formula $\bigcirc^k E$ ($k$-time "next" atomic formula $E$). For the sake of simplicity, we "eliminate" the "next" operator and the formula $\bigcirc^k E$ is abbreviated as $E^k$ (i.e., as an atomic formula with the index $k$). We also use the notation $A^k$ for an arbitrary formula $A$ defined as follows.

**Definition 2** (index, elementary formula, quasi-predicate symbol)**.**

1) *If* $E$ *is an atomic formula,* $i, k \in \omega$, $k \neq 0$ *then* $(E^i)^k := E^{i+k}$ $(E^0 := E)$; $E^l (l \geq 0)$ *is called an elementary formula, and* $E^l$ *becomes atomic if* $l = 0$;

2) $(A \odot B)^k := A^k \odot B^k$ *if* $\odot \in \{\supset, \wedge, \vee\}$; $(\sigma A)^k := \sigma A^k$ *if* $\sigma \in \{\neg, \Box, \forall x, \exists x\}$. *Let* $P^k(t_1, \ldots, t_n)$ *be an elementary formula then* $P^k$ *is a quasi-predicate symbol.*

**Definition 3** (sequent)**.** *A sequent is an expression of the form* $\Gamma \to \Delta$, *where* $\Gamma, \Delta$ *are arbitrary finite multisets (i.e., not sequences or sets) of formulas.*

**Definition 4** (calculi $G_\omega$, $G$)**.** *A calculus* $G_\omega$ *is defined by the following postulates.*

*Axiom:* $\Gamma, A \to \Delta, A$.

*Rules:*

1) *temporal rules:*

$$\frac{A, \Box A^1, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\,(\Box \to) \qquad \frac{\Gamma \to \Delta, A; \ldots; \Gamma \to \Delta, A^k; \ldots}{\Gamma \to \Delta, \Box A}\,(\to \Box_\omega) \quad (k \in \omega);$$

2) *logical rules:*

$$\frac{\Gamma, A \to \Delta, B}{\Gamma \to \Delta, (A \supset B)} (\to\supset) \qquad \frac{\Gamma \to \Delta, A;\ \Gamma, B \to \Delta}{\Gamma, (A \supset B) \to \Delta} (\supset\to)$$

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta, B}{\Gamma \to \Delta, (A \wedge B)} (\to \wedge) \qquad \frac{\Gamma, A, B \to \Delta}{\Gamma, (A \wedge B) \to \Delta} (\wedge \to)$$

$$\frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, (A \vee B)} (\to \vee) \qquad \frac{\Gamma, A \to \Delta;\ \Gamma, B \to \Delta}{\Gamma, (A \vee B) \to \Delta} (\vee \to)$$

$$\frac{\Gamma, A \to \Delta}{\Gamma \to \Delta, \neg A} (\to \neg) \qquad \frac{\Gamma \to \Delta, A}{\Gamma, \neg A \to \Delta} (\neg \to)$$

$$\frac{\Gamma \to \Delta, A[b/x]}{\Gamma \to \Delta, \forall x A} (\to \forall) \qquad \frac{A[t/x], \forall x A, \Gamma \to \Delta}{\forall x A, \Gamma \to \Delta} (\forall \to)$$

$$\frac{\Gamma \to \Delta, A[t/x], \exists x A}{\Gamma \to \Delta, \exists x A} (\to \exists) \qquad \frac{A[b/x], \Gamma \to \Delta}{\exists x A, \Gamma \to \Delta} (\exists \to).$$

*In the rules* $(\to \forall)$, $(\exists \to)$ *the eigen-variable* $b$ *does not enter the conclusion of the rules.*

3) *structural rules: it follows from the definition of a sequent that* $G_\omega$ *implicitly contains the structural rule "exchange".*

*A calculus* $G$ *is obtained from* $G_\omega$ *by dropping the rule* $(\to \Box_\omega)$.

**Definition 5** (height of derivation). *Let* $D$ *be a given derivation of a sequent* $S$ *in* $G_\omega$. *Then the height of the derivation* $D$ *of the sequent* $S$ *(denoted by* $O(D)$*) is defined as follows (*Schwichtenberg [37]*). If* $S$ *is an axiom then* $O(D) = 1$. *Let* $S$ *be a conclusion of rule with* $k$ *premises then* $O(D) = \sup_{i<k}(O(D_i) + 1)$, *where* $D_i$ *are derivations of the premises of the given rule. If a derivation* $D$ *does not contain* $\omega$*-type rule* $(\to \Box_\omega)$ *then the height of the derivation* $D$ *is denoted by* $h(D)$.

Let $I, I_1, I_2$ be arbitrary calculi then $I \vdash S$ means that the sequent $S$ is derivable in $I$; the notation $I_1 \vdash S_1 \Rightarrow I_2 \vdash S_2$ means that the derivability of $S_1$ in $I_1$ implies the derivability of $S_2$ in $I_2$. In this case, the derivation of $S_1$ in $I_1$ is called a given derivation and the derivation of $S_2$ in $I_2$ is called a resulting derivation.

The concept "sequent $S$ is universally valid in model $M$" (in symbols: $\forall M \models S$) is defined in a traditional way (see, e.g., Kawai [13]). The rule $(j)$ is called admissible in calculus $I$ if, by adding $(j)$, we do not extend the set of derivable sequents in $I$. The rule $(j)$ is called invertible in calculus $I$ if the derivation in $I$ of the conclusion of $(j)$ implies the derivability in $I$ of each premise of $(j)$.

**Lemma 1.** *The structural rule weakening*

$$\frac{\Gamma \to \Delta}{\Pi, \Gamma \to \Delta, \Theta}\ (W)$$

*is admissible in* $G_\omega$.

**Proof.** By induction on the height of the derivation of the premise of the rule $(W)$ and renaming eigen-variables of $(\to \forall), (\exists \to)$.

**Definition 6** (elementary axiom, elementary derivation)**.** *An axiom of the form* $\Gamma, E^l(\bar{t}) \to \Delta, E^l(\bar{t})$ *, where* $E^l(t)$ *is an elementary formula and* $\bar{t} = t_1, \ldots, t_n$ $(n \geq 1)$ *is called an elementary axiom. A derivation* $D$ *in* $G_\omega$ *is called an elementary one if axioms occurring in* $D$ *are elementary axioms.*

**Lemma 2.** *An arbitrary derivation in* $G_\omega$ *may be transformed into an elementary one.*

**Proof.** Let us denote by $g(A)$ the complexity of $A$ defined by the number of occurrences of logical and temporal symbol $\Box$ in $A$. The lemma is proved by induction on $g(A)$.

**Lemma 3** (invertibility of the rules of $G_\omega$)**.** *If* $S_1$ *is a premise and* $S$ *is a conclusion of any rule of* $G_\omega$ *then* $G_\omega \vdash S \Rightarrow G_\omega \vdash S_1$.

**Proof.** By induction on $O(D)$, where $D$ is the given elementary derivation.

**Theorem 1** (soundness and $\omega$-completeness of $G_\omega$; admissibility of $(cut)$ in $G_\omega$)**.** (a) $\forall M \models S$ *if and only if* $G_\omega \vdash S$; (b) *the* $(cut)$ *rule is admissible in* $G_\omega$.

**Proof.** Analogously as in Kawai [13].

Now we introduce a rather specific sequents, so-called $D$-sequents. The $D$-sequents allow us to avoid loop rules on logical level.

**Definition 7** ($D$-sequent, induction-free $D$-sequent, index condition)**.** *For simplicity we consider only one-place predicate symbols and one-place function symbols. A sequent* $S$ *is called a* $D$*-sequent if* $S$ *has a shape* $\Sigma, \Pi^1, \forall\nabla_1, \Box\forall\nabla_2 \to \Box^0 A$, *where* $\Sigma$ *consists of atomic formulas,* $\Pi^1$ *either is empty or consists of elementary formulas,* $\forall\nabla_l$ $(l \in \{1, 2\})$ *consists of formulas of the form* $\forall x_i(Q_i^{n_i}(x_i) \supset P_i^{m_i}(\bar{f}_i(x_i)))$ *(called* $\forall$*-formulas, if* $n_i = 0$ *then* $\forall$*-formula is called an* $a\forall$*-formula), where* $Q_i^{n_i}(x_i)$ *is an elementary formula without functional symbols;* $P_i^{m_i}(\bar{f}_i(x_i))$ *is* *an elementary formula containing some one-place functional symbols;* $\bar{f}_i(x_i) = f_{i1}(\ldots(f_{il_i}(x_i))\ldots)$; $n_i \geq 0$, $m_i > 0$, *and also indexes enjoy a index condition, namely for every* $i$ $n_i < m_i$. *In separate case* $\forall\nabla_1$ *may be empty.* $\Box^0 \in \{\emptyset, \Box\}$; $A = \vee_{i=1}^{k} \exists y_i E_i^{k_i}(y_i)$; $E_i^{k_i}(y_i)$ *is an elementary formula; moreover, for some* $i$ *the variable* $y_i$ *may be a constant (in this case* $\exists y_i$ *is a fictitious quantifier which is omitted). If* $\Box^0$ *is empty then the D-sequent is called induction-free.*

**Remark 1.** The form of the $D$-sequent was proposed in 1998 by V. P. Orevkov in a private conversation with the author. The shape of $D$-sequent is rather similar to normal form proposed by Fisher [6].

Instead of $\forall$-formulas $\forall x(Q^n(x) \supset P^m(\bar{f}(x)))$ (and succedent formulas $E_i^{k_i}(x_i)$, we sometimes write $\forall x(Q(x) \supset P(x))$ ($E_i(x_i)$, respectively), if function symbols and indices are not essential.

For $D$-sequents one can construct a calculus in which the rule $(\rightarrow \exists)$ is omitted altogether. The operation with positive occurrences of the quantifier $\exists$ can be transferred to the axiomatic level. Instead of the rule $(\forall \rightarrow)$ it is possible to use the rule $(\forall^* \rightarrow)$ not containing duplicate of the principal formula in the premise of this rule. Moreover, in the case of induction-free $D$-sequents only linear derivations, i.e., the ones with a single branch, are constructed.

**Definition 8** (calculi $G_\omega^*$, $G^*$). *A calculus* $G_\omega^*$ *is obtained from the calculus* $G_\omega$

(a) *replacing the axiom by the following axiom* $(\exists^*)$

$$\Gamma, E_i^{k_i}(\bar{f}(a)) \rightarrow \overset{n}{\underset{j=1}{\vee}} \exists z_j E_j^{k_j}(\bar{g}(z_j)),$$

*provided that there exists* $j$ *such that* $E_j^{k_j} = E_i^{k_i}$, *and the term* $\bar{f}(a)$ *is an instance of the term* $\bar{g}(z_j)$ *(for example,* $f(g(a))$ *is the instance of the term* $f(z)$*);*

(b) *replacing the logical rules by the following one:*

$$\frac{P^i(t),\ Q^j(\bar{f}(t)),\ \Gamma \rightarrow \Delta}{P^i(t),\ \forall x(P^i(x) \supset Q^j(\bar{f}(x))),\ \Gamma \rightarrow \Delta}\ (\forall^* \rightarrow).$$

*The formula* $P^i(t)$ *is called an elementary main (in short e-main) formula of the rule* $(\forall^* \rightarrow)$; *the formula* $\forall x(P^i(x) \supset Q^j(\bar{f}(x)))$ *is a main formula of the rule* $(\forall^* \rightarrow)$; *the formula* $Q^j(\bar{f}(t))$ *is called a side formula of the rule* $(\forall^* \rightarrow)$.

*The calculus $G^*$ is obtained from the calculus $G^*_\omega$ by dropping the rule* $(\to \Box_\omega)$.

**Remark 2.** (a) The index condition on $D$-sequents, which means that the part $\forall\nabla_i$ ($i \in \{1, 2\}$) of a $D$-sequent consists of formulas of the form $\forall x(P^n(x) \supset Q^m(f(x)))$ and $n < m$, is essential for the correctness of the rule $(\forall^* \to)$. Indeed, consider, for example, the sequent $S = P(c)$, $\Box\forall x(P(x) \supset Q^1(f(x)))$, $\Box\forall y(Q^1(y) \supset P(g(y))) \to Q^1(f(g(f(c))))$. For the sequent $S$ the index condition is violated. It is easy to see that $G \vdash S$, but $G^* \nvdash S$.

(b) The rule $(\forall^* \to)$ is noninvertible. Indeed, let $S_1 = E(c), E(d)$, $\forall x(E(x) \supset P^1(f(x))) \to P^1(f(d))$. Then it is easy to construct the derivation $S_1$ in $G^*$:

$$\frac{E(c), E(d), P^1(f(d)) \to P^1(f(d))}{E(c), E(d), \forall x(E(x) \supset P^1(f(x))) \to P^1(f(d))}\ (\forall^* \to),$$

however $G^* \nvdash E(c), E(d), P^1(f(c)) \to P^1(f(d))$.

**Definition 9** (regular application of the rule $(\forall^* \to)$, $\forall^*$-regular derivation in the calculus $G^*$)**.** *Let $V$ be a derivation in the calculus $G^*$. The application of the rule $(\forall^* \to)$ in $G^*$ is called regular if there are no applications of the rule $(\Box \to)$ above this application of the rule $(\forall^* \to)$ in the derivation $V$. A derivation $V$ in the calculus $G^*$ is called $\forall^*$-regular if in $V$ all the applications of the rule $(\forall^* \to)$ are regular.*

**Lemma 4.** *Let $S$ be an induction-free $D$-sequent. Then any derivation $V$ in the calculus $G^*$ of sequents $S$ can be transformed into a $\forall^*$-regular derivation with the same end-sequent $S$.*

**Proof.** We shall use induction on the number $n$ of irregular applications of the rule $(\forall^* \to)$ in $V$.

The basis of induction: $n = 0$; then $V$ is a $\forall^*$-regular derivation. The induction step: $n > 0$. Let us examine the topmost irregular application of the rule $(\forall^* \to)$ in $V$:

$$V_1 \left\{ \frac{V_{11}\left\{ \overline{S_1}\ (\Box \to) \right.}{\Sigma, \forall x B, \forall\nabla_1, \Box\forall\nabla_2 \to A}\ (\forall^* \to). \right.$$

Let $m(V_{11})$ be the number of applications of $(\Box \to)$ in a derivation $V_{11}$. The induction step is founded using the induction on $m(V_{11})$.

The case when $m(V_{11}) = 0$ is trivial. Let $m(V_{11}) > 0$ and $\Sigma = E(t), \Sigma_1$; $\forall x B = \forall x(E(x) \supset R(x))$; $\forall\nabla_2 = \Box C, \Box\forall\nabla_{21}$. Then the end of the derivation $V_1$ has the following shape:

$$V_1 \left\{ \dfrac{V_{11}\left\{ \dfrac{E(t), \Sigma_1, R(t), \forall\nabla_1, C, \Box C^1, \Box\forall\nabla_{21} \to A}{E(t), \Sigma_1, R(t), \forall\nabla_1, \Box C, \Box\forall\nabla_{21} \to A} (\Box \to) \right.}{E(t), \Sigma_1, \forall x(E(x) \supset R(x)), \forall\nabla_1, \Box C, \Box\forall\nabla_{21} \to A} (\forall^* \to). \right.$$

Let us interchange the applications of the rules $(\Box \to)$ and $(\forall^* \to)$, i.e. let us construct the following derivation:

$$V_1' \left\{ \dfrac{V_{11}'\left\{ \dfrac{E(t), \Sigma_1, R(t), \forall\nabla_1, C, \Box C^1, \Box\forall\nabla_{21} \to A}{E(t), \Sigma_1, \forall x(E(x) \supset R(x)), \forall\nabla_1, C, \Box C^1, \Box\forall\nabla_{21} \to A} (\forall^* \to) \right.}{E(t), \Sigma_1, \forall x(E(x) \supset R(x)), \forall\nabla_1, \Box C, \Box\forall\nabla_{21} \to A} (\Box \to). \right.$$

Since $m(V_{11}') < m(V_{11})$ derivation $V_{11}'$ can be transformed into a $\forall^*$-regular derivation in $G^*$.

## 3. Equivalence of the calculi $G_\omega^*$ and $G_\omega$ for $D$-sequents

In this section we prove that the calculi $G_\omega^*$ and $G_\omega$ are equivalent with respect to $D$-sequents. To prove this equivalence we follow techniques described in [30]. First, the following calculus is introduced.

**Definition 10** (calculus $G_1$). *The axioms of a calculus $G_1$ are elementary ones only. The rules of the calculus $G_1$ are given by the following list:*

$$\frac{\forall x A, \Gamma \to \Delta, E(t);\ \forall x A, Q(t), \Gamma \to \Delta}{\forall x(E(x) \supset Q(x)), \Gamma \to \Delta} (\forall^+ \to),$$

$$\text{where } A = (E(x) \supset Q(x));$$

$$\frac{\forall x A, \Box\forall x A^1 \to \Delta}{\Box\forall x A, \Gamma \to \Delta} (\Box \to);$$

$$\frac{\Gamma \to \Delta, E_1(t_1), \dots, E_n(t_n), \exists x_1 E_1(x_1), \dots, \exists x_n E_n(x_n)}{\Gamma \to \Delta, \bigvee_{i=1}^{n} \exists x_i E_i(x_i)} (\to \exists^+).$$

The rule $(\forall^+ \to)$ covers the rules $(\forall \to), (\supset\to)$. The rule $(\to \exists^+)$ includes the rules $(\to \vee), (\to \exists)$.

**Lemma 5.** *Let $S$ be an induction-free $D$-sequent. Then the condition $G \vdash S$ implies $G_1 \vdash S$.*

**Proof.** Relying on Lemma 2 we can assume that the given derivation is an elementary one. From the definition of $D$-sequent it follows that each application of the rule $(\rightarrow \exists)$ is a special case of the rule $(\rightarrow \exists^+)$ when $n = 1$. Therefore to prove the lemma it is sufficient to show that each application of the rules $(\forall \rightarrow)$, $(\rightarrow \vee)$ can be replaced by the applications of the rules $(\forall^+ \rightarrow)$, $(\rightarrow \exists^+)$. Let us consider the lowest application of the rule $(\forall \rightarrow)$. From the definition of $D$-sequent it follows that the application of the rule $(\forall \rightarrow)$ has the following form:

$$\frac{V\,\{(E(t) \supset Q(t)), \forall x A, \Gamma \rightarrow \Delta}{\forall x(E(x) \supset Q(x)), \Gamma \rightarrow \Delta}\,(\forall \rightarrow),$$

where $A = E(x) \supset Q(x)$.

Using the invertibility of the rule $(\supset\rightarrow)$ (see Lemma 3), from the derivation $V$ we can get two derivations of the sequents $S_1 = \forall x A, \Gamma \rightarrow \Delta, E(t)$ and $S_2 = \forall x A, Q(t), \Gamma \rightarrow \Delta$. Applying the rule $(\forall^+ \rightarrow)$ to the sequents $S_1, S_2$, we get a derivation of the sequent $\forall x(E(x) \supset Q(x)), \Gamma \rightarrow \Delta$. In the same way we can replace all applications of the rule $(\forall \rightarrow)$ by the rule $(\forall^+ \rightarrow)$.

Let us consider the lowest application of the rule $(\rightarrow \vee)$. From the definition of $D$-sequent it follows that the application of the rule has the following form:

$$\frac{V\,\{\Gamma \rightarrow \Delta, \exists x_1 E_1(x_1), \ldots, \exists x_n E_n(x_n)}{\Gamma \rightarrow \Delta, \bigvee\limits_{i=1}^{n} \exists x_i(x_i)}\,(\rightarrow \vee).$$

Using the admissibility of the structural rule $(W)$ from the derivation $V$ we can get a derivation of the sequent $S_1 = \Gamma \rightarrow \Delta$, $E_1(t_1), \ldots, E_n(t_n)$, $\exists x_1 E_1(x_1), \ldots, \exists x_n E_n(x_n)$. Applying the rule $(\rightarrow \exists^+)$ to the sequent $S_1$, we get a derivation of the sequent $\Gamma \rightarrow \Delta, \bigvee\limits_{i=1}^{n} \exists x_i(x_i)$. In such a way we can replace other applications of the rule $(\rightarrow \vee)$.

**Definition 11** (regular application of rule $(\rightarrow \exists^+)$, $\exists$-regular derivation)**.** *Let $V$ be a derivation in the calculus $G_1$. An application of the rule $(\rightarrow \exists^+)$ is called regular if a premise of this application is an axiom. A derivation $V$ in the calculus $G_1$ is called $\exists$-regular if, either in $V$ there is no application of the rule $(\rightarrow \exists^+)$, or all applications of the rule $(\rightarrow \exists^+)$ in $V$ are regular.*

**Lemma 6.** *The rule* $(\forall^+ \to)$ *is invertible in the calculus* $G_1$.

**Proof.** Follows from the admissibility of the structural rule weakening.

**Lemma 7.** *Let* $S$ *be an induction-free D-sequent. Then any derivation* $V$ *in the calculus* $G_1$ *of the sequent* $S$ *can be transformed into an* $\exists$*-regular derivation with the same end-sequent* $S$.

**Proof.** We use induction on the number $m$ of irregular applications of the rule $(\to \exists^+)$ in $V$. The basis of induction: $m = 0$; then $V$ is an $\exists$-regular derivation. The induction step: $m > 0$. Let us consider a topmost irregular application of the rule $(\to \exists^+)$ in $V$. The derivation $V$ is as follows:

$$V \left\{ \begin{array}{c} V_1' \left\{ \dfrac{\ldots}{\Gamma \to \Sigma_2, \Theta, \exists x_1 E_1(x_1), \ldots, \exists x_n E_n(x_n)} (j) \right. \\ \hline \Gamma \to \Sigma_2, \bigvee_{i=1}^{n} \exists x_i E_i(x_i) \end{array} (\to \exists^+) \right. \\ \vdots \\ S$$

where $\Sigma_2$ consists of elementary formulas, $\Theta = E_1(t_1), \ldots, E_n(t_n)$ ($t_i$, $1 \leq i \leq n$, is some term). The case when $j$ is an regular application of $(\to \exists^+)$ is obvious. Therefore we consider only the cases when $(j) \in \{(\forall^+ \to), (\Box \to)\}$. To justify the induction on $m$, we make use of induction on the number $\delta$ of the application of the rules $(j) \in \{(\forall^+ \to), (\Box \to)\}$ in $V_1'$. The induction basis on $\delta$ (i.e., when $\delta = 0$) is trivial. Let $\delta > 0$. Then it is possible to interchange the application of the rules $(j)$ and $(\to \exists^+)$, which reduces $\delta$.

**Definition 12** (calculus $G_2$). *A calculus* $G_2$ *is obtained from the calculus* $G_1$ *by replacing the rule* $(\to \exists^+)$ *and the axiom by the axiom* $(\exists^*)$ *(see Definition 8).*

**Lemma 8.** *Let* $S$ *be an induction-free D-sequent. Then the condition* $G_1 \vdash S$ *implies* $G_2 \vdash S$.

**Proof.** Let $V$ be a derivation of the sequent $S$ in $G_1$. By Lemma 7 we can assume that all applications of the rule $(\to \exists^+)$ in $V$ are regular, i.e., the premise of any application of the rule $(\to \exists^+)$ is an axiom. Then any regular application of the rule $(\to \exists^+)$ can be replaced by the axiom $(\exists^*)$.

**Definition 13** (regular application of the rule $(\forall^+ \to)$, $\forall$-regular derivation in the calculus $G_2$). *Let $V$ be a derivation in the calculus $G_2$. The application of the rule $(\forall^+ \to)$ in $G_2$ is called regular if there are no applications of the rule $(\Box \to)$ above this application of the rule $(\forall^+ \to)$ in the derivation $V$. The derivation $V$ in the calculus $G_2$ is called $\forall$-regular if in $V$ all the applications of the rule $(\forall^+ \to)$ are regular.*

**Lemma 9.** *Let $S$ be an induction-free D-sequent. Then any derivation $V$ in the calculus $G_2$ of sequents $S$ can be transformed into a $\forall$-regular derivation with the same end-sequent $S$.*

**Proof.** We use induction on the number $n$ of irregular applications of the rule $(\forall^+ \to)$ in $V$.

The basis of induction: $n = 0$; then $V$ is a $\forall$-regular derivation. The induction step: $n > 0$. Let us examine the topmost irregular application of the rule $(\forall^+ \to)$ in $V$:

$$V_1 \left\{ \frac{V_{11} \left\{ \overline{S_1}\,(i) \right. ; \quad V_{12} \left\{ \overline{S_2}\,(j) \right.}{\Sigma_1, \forall x B, \forall \nabla_1, \Box \forall y \nabla_2 \to \Sigma^*} \, (\forall^+ \to), \right.$$

Let $m(\Box)$ be the number of applications of $(\Box \to)$ in $V_{11}$, $V_{12}$. In the induction step we use the induction on $m(\Box)$.

The following three cases are possible:

(1) $(i) = (\Box \to)$, $(j) = (\forall^+ \to)$, i.e., the derivation $V_{12}$ is $\forall$-regular;

(2) $(i) = (\forall^+ \to)$ (i.e., the derivation $V_{11}$ is $\forall$-regular), $(j) = (\Box \to)$;

(3) $(i) = (j) = (\Box \to)$.

We consider the case (1) only (the cases (2) and (3) are investigated in a similar manner). The end of the derivation $V_1$ is of the form:

$$V_1 \left\{ \frac{V'_{11} \left\{ \frac{S_1^*}{S_1} (\Box \to) \right. \qquad V_{12} \left\{ S_2 \right.}{\Sigma_1, \forall x B, \forall \nabla_1, \Box \forall y A, \Box \forall \nabla_2 \to \Sigma_2^*} \, (\forall^+ \to), \right.$$

where $V'_{11}$ and $V_{12}$ are $\forall$-regular derivations, $B = E(x) \supset Q(x)$; $S_1 = \Sigma_1, \forall x B, \forall \nabla_1, \Box \forall y A, \Box \forall \nabla_2 \to \Sigma_2^*, E(t)$; $S_2 = \Sigma_1, \forall x B, \forall \nabla_1, \Box \forall y A, \Box \forall \nabla_2, Q(t) \to \Sigma_2^*$; $S_1^* = \Sigma_1, \forall x B, \forall \nabla_1, \forall y A, \Box \forall y A^1, \Box \forall \nabla_2 \to \Sigma_2^*, E(t)$.

We transform the derivation $V_1$ as follows:

$$V_1^* \left\{ \dfrac{\dfrac{V'_{11}\,\{S_1^* \; ; \quad V'_{12}\,\{S'_2}{\Sigma_1, \forall xB, \forall\nabla_1, \Box\forall yA, \Box\forall yA^1, \Box\forall\nabla_2 \to \Sigma_2^*}\,(\forall^+ \to)}{\Sigma_1, \forall xB, \forall\nabla_1, \Box\forall yA, \Box\forall\nabla_2 \to \Sigma_2^*}\,(\Box \to), \right.$$

where $S'_2 = \Sigma_1, \forall xB,\ \forall\nabla_1, \forall yA,\ \Box\forall yA^1,\ \Box\forall\nabla_2,\ Q(t) \to \Sigma_2^*$. By relying on the admissibility of the structural rule weakening in $G_2$ (which can be proved by the evident induction on height), the $\forall$-regular derivation $V_{12}$ can be transformed into a $\forall$-regular derivation $V'_{12}$ of the sequent $S'_2$. Using induction on $m(\Box)$, we transform the derivation $V_1^*$ into an $\forall$-regular derivation $V_1^{**}$. Having replaced the derivation $V_1$, in the given derivation of the sequent $S$ by the $\forall$-regular derivation $V_1^{**}$, we obtain the derivation $V^*$ of the sequent $S$, which contains $n-1$ irregular applications of the rule $(\forall^+ \to)$. According to the induction hypothesis, the derivation $V^*$ can be transformed into a $\forall$-regular derivation.

**Definition 14** (calculi $G_3$, $G_3^+$)**.** *A calculus $G_3$ is obtained from the calculus $G_2$ by replacing the rule $(\forall^+ \to)$ by the following one-premise rule:*

$$\frac{E(t), Q(t), \forall x(E(x) \supset Q(x)), \Gamma \to \Delta}{E(t), \forall x(E(x) \supset Q(x)), \Gamma \to \Delta}\,(\forall^{++} \to).$$

*A calculus $G_3^+$ is obtained from the calculus $G_2$ by adding the rule $(\forall^{++} \to)$.*

A regular application of the rule $(\forall^{++} \to)$ and $\forall$-regular derivation in the calculi $G_3$, $G_3^+$ are defined analogously as in Definition 13.

**Lemma 10.** *Let $S$ be an induction-free D-sequent. Then any derivation $V$ in the calculus $G_3^+$ of the sequent $S$ can be transformed into the derivation of the sequent $S$ in the calculus $G_3$.*

**Proof.** Clearly, if in Lemma 9 the calculus $G_2$ is replaced by the calculus $G_3^+$, Lemma 9 still holds. Therefore, we can assume that the given derivation $V$ is $\forall$-regular, i.e., there are no applications of the rule $(\Box \to)$ above the application of the rules $(\forall^+ \to)$ and $(\forall^{++} \to)$. In order to prove Lemma 10 we use induction on $n(\forall^+ \to)$ (the number of applications of the rule $(\forall^+ \to)$ in the given derivation).

The basis of induction: $n(\forall^+ \to) = 0$; then the given derivation is the one we wish to have.

The induction step: $n(\forall^+ \to) > 0$. Consider the topmost application of the rule $(\forall^+ \to)$ in the derivation $V$:

$$V_1 \left\{ \frac{V_{11}\,\{\forall xA, \Gamma \to E(t), \Delta;\ \forall xA, Q(t), \Gamma \to \Delta}{\forall xA, \Gamma \to \Delta} (\forall^+ \to), \right.$$

where $\forall xA = \forall x(E(x) \supset Q(x))$. To justify the step of induction on $n(\forall^+ \to)$, we apply induction on $h(V_{11})$, i.e., on the height of derivation of the left premise of the considered application of the rule $(\forall^+ \to)$.

Below is the list of all possible cases.

1. $h(V_{11}) = 1$, i.e., the sequent $\forall xA, \Gamma \to E(t), \Delta$ is an axiom.

1.1. The chosen occurrence of the formula $E(t)$ is not a main formula of an axiom. Then the conclusion of the rule $(\forall^+ \to)$, i.e., the sequent $\forall xA, \Gamma \to \Delta$ is also an axiom. Thus, in this case, it is possible to reduce the induction parameter $n(\forall^+ \to)$.

1.2. The formula $E(t)$ is a main formula of an axiom. In this case, the considered application of the rule $(\forall^+ \to)$ can be replaced by an application of the rule $(\forall^{++} \to)$.

2. $h(V_{11}) > 1$. Since the considered application of the rule $(\forall^+ \to)$ is the topmost in the $\forall$-regular derivation $V$ and $h(V_{11}) > 1$, the sequent $\forall xA, \Gamma \to E(t), \Delta$ is the conclusion of the rule $(\forall^{++} \to)$.

2.1. The main formula of the considered application of the rule $(\forall^{++} \to)$ is the formula $\forall xA$. Then the end of the derivation $V_1$ is of the following form:

$$V_1 \left\{ \frac{V_{11} \left\{ \dfrac{E(p), \forall xA, Q(p), \Gamma \to E(t), \Delta}{E(p), \forall xA, \Gamma \to E(t), \Delta} (\forall^{++} \to) \right. \quad S_2}{E(p), \forall xA, \Gamma \to \Delta} (\forall^+ \to), \right.$$

where $S_2 = E(p), \forall xA, Q(t), \Gamma \to \Delta$; $p$ is some term. Now we construct a derivation:

$$V_1{'} \left\{ \frac{\dfrac{V_{11}{'}\,\{E(p), \forall xA, Q(p), \Gamma \to E(t), \Delta; \quad S_2^*}{E(p), \forall xA, Q(p), \Gamma \to \Delta} (\forall^+ \to)}{E(p), \forall xA, \Gamma \to \Delta} (\forall^{++} \to), \right.$$

where $S_2^* = E(p), \forall xA, Q(t), Q(p), \Gamma \to \Delta$. The derivation of the sequent $S_2^*$ can be obtained with the aim of the structural rule weakening (which is evidently admissible in the calculus $G_3^+$) from the derivation of the sequent $S_2 = E(p), \forall x A, Q(t), \Gamma \to \Delta$. Since $h(V'_{11}) < h(V_{11})$, the derivation $V'_1$ can be transformed into a derivation in the calculus $G_3$.

2.2. The main formula of the considered application of the rule $(\forall^{++} \to)$ differs from the chosen occurrence of the formula $\forall x A$. This case is treated quite similarly to the case 2.1.

**Lemma 11.** *Let $S$ be an induction-free $D$-sequent. Then any derivation in the calculus $G_2$ can be transformed into a derivation in the calculus $G_3$ with the same end-sequent.*

**Proof.** Follows from Lemma 10.

**Remark 3.** As it follows from the shape of the rule $(\forall^{++} \to)$ any derivation in the calculus $G_3$ has a linear form.

**Definition 15** (descendant, ancestor, $\Sigma$ and $\Pi$ parts of the $D$-sequent, trace)**.** *Let us consider the rule*

$$\frac{E(t), \forall x(E(x) \supset P(\bar{f}(x))), P(\bar{f}(t)), \Gamma \to \Delta}{E(t), \forall x(E(x) \supset P(\bar{f}(x))), \Gamma \to \Delta} \, (\forall^{++} \to).$$

*The formula $E(t)$ is called an elementary main (in short: $e$-main) formula of the rule $(\forall^{++} \to)$; $\forall x(E(x) \supset P(\bar{f}(x)))$ is called a main formula of the rule $(\forall^{++} \to)$; $P(\bar{f}(t))$ is called a side formula of the rule $(\forall^{++} \to)$. The side formula is called a descendant (or an immediate descendant) of the $e$-main formula, while the $e$-main formula is called an ancestor (or an immediate ancestor) of the side formula. Let us consider the part of a $\forall$-regular derivation in the calculus $G_3$ of the $D$-sequent $S$ consisting of applications of the rule $(\forall^{++} \to)$ and denoted by $V^*$:*

$$V^* \left\{ \begin{array}{c} \dfrac{\Sigma, \Pi, E_i(t_i), \dots, E_{i+p}(t_{i+p}), E_{i+p+1}(t_{i+p+1}), \Gamma \to \Theta}{\Sigma, \Pi, E_i(t_i), \dots, E_{i+p}(t_{i+p}), \Gamma \to \Theta} \, (\forall^{++} \to) \\ \vdots \\ \dfrac{\Sigma, \Pi, E_i(t_i), E_{i+1}(t_{i+1}), \Gamma \to \Theta}{\Sigma, \Pi, E_i(t_i), \Gamma \to \Theta} \, (\forall^{++} \to), \end{array} \right.$$

*where $\Sigma$ consists of elementary formulas of an initial $D$-sequent $S$ and is called a $\Sigma$ part of the $D$-sequent; part $\Pi$ of $D$-sequents in the derivation $V^*$ is either empty or consists of elementary formulas obtained in the process of construction of the derivation; $\Gamma = \forall \nabla_1, \Box \forall \nabla_2$; $E_i(t_i)$ $(E_{i+1}(t_{i+1}))$ is an $e$-main (side, respectively) formula of the lower application of the rule $(\forall^{++} \to)$ in $V^*$ and $E_{i+p}(t_{i+p})$, $(E_{i+p+1}(t_{i+p+1}))$ is an $e$-main (side, respectively) formula of the upper application of the rule*

$(\forall^{++} \to)$ *in* $V^*$. *We say that* $E_{i+p+1}(t_{i+p+1})$ *is a descendant of the formula* $E_i(t_i)$, *and* $E_i(t_i)$ *is an ancestor of the formula* $E_{i+p+1}(t_{i+p+1})$ *if the list* $E_i(t_i)$, $E_{i+1}(t_{i+1}), \ldots, E_{i+p}(t_{i+p})$, $E_{i+p+1}(t_{i+p+1})$ *(called a trace of the formula* $E_i(t_i)$*) satisfies the following condition:* $\forall \rho$ $(0 \leq \rho \leq p)$ $E_{i+\rho}(t_{i+\rho})$ *is an e-main, and* $E_{i+\rho+1}(t_{i+\rho+1})$ *is a side formula of the same application of the rule* $(\forall^{++} \to)$. *If the last member of this list is the main formula of an axiom then such a trace is called axiomatic.*

**Definition 16** (essential application of the rule $(\forall^{++} \to)$). *The application of the rule* $(\forall^{++} \to)$ *is called essential if the e-main formula (and hence, the side formula) of this application is an ancestor of the main formula of an axiom.*

**Lemma 12.** *Let* $V$ *be a* $\forall$*-regular derivation of the induction-free D-sequent* $S$ *in the calculus* $G_3$. *Then it is possible to construct a* $\forall$*-regular derivation* $V^*$ *of the sequent* $S$, *in which all applications of the rule* $(\forall^{++} \to)$ *are essential.*

**Proof.** Since the deletion of nonessential applications of the rule $(\forall^{++} \to)$ will change neither the axiom nor all essential applications of the rules $(\forall^{++} \to)$, we get the desired derivation $V^*$ of the same sequent $S$ in the calculus $G_3$.

**Definition 17** ($\delta$-application of the rule $(\forall^{++} \to)$, $\delta \in \{\Sigma, \Pi\}$). *Let an e-main formula of application of the rule* $(\forall^{++} \to)$ *belong to the* $\delta$*-part* $(\delta \in \{\Sigma, \Pi\})$ *of a D-sequent. Then the application of the rule* $(\forall^{++} \to)$ *is called a* $\delta$*-application. Any* $\Pi$*-application of the rule* $(\forall^{++} \to)$ *is called regular if above this* $\Pi$*-application of the rule* $(\forall^{++} \to)$ *one can find only* $\Pi$*-applications of the rule* $(\forall^{++} \to)$. *Any* $\forall$*-regular derivation in the calculus* $G_3$ *is called* $\Pi\forall$*-regular if all* $\Pi$*-applications of the rule* $(\forall^{++} \to)$ *are regular.*

**Lemma 13.** *Let* $S$ *be an induction-free D-sequent. Then any* $\forall$*-regular derivation in the calculus* $G_3$ *can be transformed into a* $\Pi\forall$*-regular derivation with the same end-sequent.*

**Proof.** We use induction on the number $n(\Pi)$ of irregular $\Pi$-applications of the rule $(\forall^{++} \to)$.

If $n(\Pi) = 0$ then the given derivation is the desired one. Let $n(\Pi) > 0$. Then we use induction on the number $m(\Sigma)$ of the $\Sigma$-applications of the rule $(\forall^{++} \to)$, which are above the $\Pi$-applications of the rule $(\forall^{++} \to)$. The basis of induction (i.e., the case $m(\Sigma) = 0$) is trivial. Let $m(\Sigma) > 0$.

Consider the topmost irregular Π-application of the rule $(\forall^{++} \to)$:

$$V_1 \left\{ \begin{array}{c} E(c), \Sigma, \Pi, E_i(t_i), E_{i+1}(t_{i+1}), P, \Gamma \to \Theta \\ \hline E(c), \Sigma, \Pi, E_i(t_i), E_{i+1}(t_{i+1}), \Gamma \to \Theta \\ \hline E(c), \Sigma, \Pi, E_i(t_i), \Gamma \to \Theta \end{array} \right. \begin{array}{l} (\forall^{++} \to) \\ \\ (\forall^{++} \to), \end{array}$$

where $E_i(t_i)$ is an $e$-main formula of the Π-application of the rule $(\forall^{++} \to)$, $E(c)$ is an $e$-main formula of the Σ-application of the rule $(\forall^{++} \to)$, and $P$ is the $e$-side formula of the Σ-application of the rule $(\forall^{++} \to)$. Then it is possible to interchange the Σ-application under consideration with a Π-application of the rule $(\forall^{++} \to)$; this will result in reduced $m(\Sigma)$.

**Lemma 14** (trace property)**.** *Let there be given a* Π∀*-regular derivation* $V$ *of the induction-free* $D$*-sequent* $S$ *in the calculus* $G_3$*. Let* $E_i^{k_0}(t_i)$ *be an* $e$*-main formula of some* Π*-application of the rule* $(\forall^{++} \to)$*. Let us consider an arbitrary trace of the formula* $E_i^{k_0}(t_i)$ *(i.e., a list of elementary formulas* $E_i^{k_0}(t_i)$, $E_{i+1}^{k_1}(t_{i+1})$, $\dots$, $E_{i+p}^{k_p}(t_{i+p})$*). Then the sequence of indices of elementary formulas in the trace* $k_0, k_1, \dots, k_p$ *is strictly increasing, i.e.,* $k_0 < k_1 < \dots < k_p$*.*

**Proof.** By induction on the length of the trace of the formula $E_i^{k_0}(t_i)$: $l(E_i^{k_0}(t_i))$. The basis of induction (i.e., when $l(E_i^{k_0}(t_i) = 1)$ is trivial. Let $l(E_i^{k_0}(t_i)) > 1$. According to the inductive assumption, for the trace $E_i^{k_0}(t_i)$, $\dots$, $E_{i+p-1}^{k_{p-1}}(t_{i+p-1})$ we have $k_0 < \dots < k_{p-1}$. Let us consider the application of the rule $(\forall^{++} \to)$ with $e$-main formula $E_{i+p-1}^{k_{p-1}}(t_{i+p-1})$:

$$\frac{\Sigma, \Pi, E_{i+p-1}^{k_{p-1}}(t_{i+p-1}), E_{i+p}^{k_p}(t_{i+p}), \forall x A, \Gamma \to \Delta}{\Sigma, \Pi, E_{i+p-1}^{k_{p-1}}(t_{i+p-1}), \forall x A, \Gamma \to \Delta} (\forall^{++} \to),$$

here $A = (E_{i+p-1}^{k_{p-1}}(x) \supset E_{i+p}^{k_p}(\bar{f}(x)))$ and $t_{i+p} = \bar{f}(t_{i+p-1})$. According to the index condition on $D$-sequents (Definition 7), we have $k_{p-1} < k_p$. Therefore, the given trace of the formula $E_i^{k_0}(t_i)$ is such that the list of indices is strictly increasing, i.e., $k_0 < k_1 < \dots < k_p$.

**Lemma 15.** *Let* $S$ *be an induction-free* $D$*-sequent. Then* $G_3 \vdash S$ *implies* $G^* \vdash S$*.*

**Proof.** Let $V$ be a given derivation of an induction-free $D$-sequent $S$ in $G_3$. A pair of elementary formulas $E_{i,2}^{m_i}(t)$, $E_{i,2}^{m_i}(q)$, where $t$ and $q$ are terms that are values of the variable $x_i$ from the formula $\forall x_i(E^{n_i}_{i,1}(x_i) \supset E^{m_i}_{i,2}(x_i)))$, is called a singular pair, while the elementary formulas themselves are called repeating formulas. Let $S^*$ be the axiom of the derivation $V$. To prove the lemma we employ induction on $|S^*|$, the number of repeating formulas in $S^*$.

Basis of induction: $|S^*| = 0$; this means that in the derivation $V$ the rule $(\forall^{++} \to)$ with the same main formula was not applied twice. In this case, every application of the rule $(\forall^{++} \to)$ can be replaced by the application of the rule $(\forall^{+} \to)$. From this fact it follows that $G^* \vdash S$.

Induction step: $|S^*| > 0$. First , let us make the following *preliminary remark*: by Lemmas 12 and 13 one can assume that all applications of the rule $(\forall^{++} \to)$ are essential and that the derivation $V$ is $\Pi\forall$-regular. Let us consider the topmost singular pair $E^{m_i}_{i,2}(\bar{f}(t))$, $E^{m_i}_{i,2}(\bar{f}(q))$ arises from. Let given derivation $V$ be as follows:

$$\dfrac{\begin{matrix}\vdots\\ \Sigma, \forall x_i A, E^{m_i}_{i,2}(\bar{f}(t)), E^{m_i}_{i,2}(\bar{f}(q)), E^{n_i}_{i,1}(q),\ \Gamma \to \Delta\end{matrix}}{V\prime \left\{\begin{matrix}\Sigma, \forall x_i A, E^{m_i}_{i,2}(\bar{f}(t)), E^{n_i}_{i,1}(q), \Gamma \to \Delta\\ \vdots \\ S\end{matrix}\right.}\ (\forall^{++} \to)$$

here $\forall x_i A = \forall x_i(E^{n_i}_{i,1}(x_i) \supset E^{m_i}_{i,2}(\bar{f}(x_i)))$, $E^{n_i}_{i,1}(q)$ $(E^{m_i}_{i,2}(\bar{f}(q)))$ is the $e$-main (side, respectively) formula of the considered application of the rule $(\forall^{++} \to)$. In accordance with the index condition for $D$-sequents we have $n_i < m_i$.

We examine the following cases.

1.$E^{m_i}_{i,2}(\bar{f}(q))$ is not an ancestor of the main formula of the axiom.

Then the considered application of the rule $(\forall^{++} \to)$ is nonessential. This contradicts to the preliminary remark.

2. $E^{m_i}_{i,2}(\bar{f}(q))$ is an ancestor of the main formula of the axiom.

2.1. $E^{m_i}_{i,2}(\bar{f}(t))$ is the $e$-main formula of the rule $(\forall^{++} \to)$ in the part $V'$ of the derivation $V$. According to the preliminary remark, this application of the rule $(\forall^{++} \to)$ is essential. Therefore $E^{m_i}_{i,2}(\bar{f}(t))$ is an ancestor of the main formula of the axiom, and hence an ancestor of the formula $E^{n_i}_{i,1}(q)$ (since $E^{n_i}_{i,1}(q)$ is an ancestor of the formula $E^{m_i}_{i,2}(\bar{f}(q))$, which, according to the assumption of case 2, is the ancestor of the main formula of axiom). This means that there exists an axiomatic trace of the formula $E^{m_i}_{i,2}(\bar{f}(t))$ containing the formula $E^{n_i}_{i,1}(q)$, where (as follows from the index condition on $D$-sequents) $n_i < m_i$. This is impossible by Lemma 14. Therefore, $E^{m_i}_{i,2}(\bar{f}(t))$ cannot be the $e$-main formula of the essential application of the rule $(\forall^{++} \to)$. This contradicts to the preliminary remark.

2.2. $E^{m_i}_{i,2}(\bar{f}(t))$ is not the $e$-main formula of the rule $(\forall^{++} \to)$ in the part $V'$ of the derivation $V$. Then $E^{m_i}_{i,2}(\bar{f}(t))$ is the side formula of an application of the rule $(\forall^{++} \to)$. Moreover, this application is essential, i.e., $E^{m_i}_{i,2}(\bar{f}(t))$ is an ancestor of the main formula of axiom, and thus an ancestor of the formula $E^{n_i}_{i,1}(q)$. Now, using the argument described in case 2.1 we get that $E^{m_i}_{i,2}(\bar{f}(t))$ cannot be the side formula of the essential application of the rule $(\forall^{++} \to)$. This contradicts to the preliminary remark.

**Lemma 16.** *If $G \vdash S$ then $G^* \vdash S$, where $S$ is an induction-free $D$-sequent.*

**Proof.** Follows from Lemmas 5, 8, 11, and 15.

**Lemma 17.** *If $G^* \vdash S$ then $G \vdash S$, where $S$ is an induction-free $D$-sequent.*

**Proof.** Follows from admissibility of structural rule weakening in $G$, and applying the rules $(\supset\to)$ and $(\forall \to)$.

**Theorem 2.** *Let $S$ be a $D$-sequent then $G_\omega \vdash S$ if and only if $G^*_\omega \vdash S$.*

**Proof.** Follows from the invertibility of the rule $(\to \Box_\omega)$ and Lemmas 16, 17.

**Theorem 3** (soundness and $\omega$-completeness of the calculus $G^*_\omega$ for $D$-sequents)**.** *Let $S$ be a $D$-sequent. Then $\forall M \models S$ if and only if $G^*_\omega \vdash S$.*

**Proof.** Follows from Theorems 1, and 2.

## 4. Auxiliary tools for constructing an infinitary calculus without loop rules

On the logical level the calculus $G^*_\omega$ (constructed and founded in the previous sections) does not contain loop rules. However, the premise of the rule $(\Box \to)$ still contains a duplicate of the main formula corresponding to a time shift.

**Example 1.** Let us consider the following $D$-sequent $S = E(t), E^1(t), \Box A \to \exists y Q^2(y)$, where $A = \forall x(E(x) \supset Q^1(f(x)))$. Let us construct a derivation of the $D$-sequent $S$ in $G^*$:

$$\dfrac{\dfrac{\dfrac{E(t), E^1(t), Q^2(f(t)), A, \Box A^2 \to \exists y Q^2(y)}{E(t), E^1(t), A, A^1, \Box A^2 \to \exists y Q^2(y)}(\forall^* \to)}{E(t), E^1(t), A, \Box A^1 \to \exists y Q^2(y)}(\Box \to)}{E(t), E^1(t), \Box A \to \exists y Q^2(y)}(\Box \to)$$

It is easy to verify that without the duplication of the main formula in the rule $(\Box \to)$ the sequent $S$ is not derivable in $G^*$.

First we introduce some generalization of the rule $(\forall^* \to)$ which allows to achieve the existential invertibility for $a\forall$-formulas of this generalized rule not introducing the non-repeating condition (see Pliuškevičius [32]).

**Definition 18** (operation (*)). *Let $\Sigma, \Pi^1, \forall\nabla_1, \Box\forall\nabla_2 \to \Box^0 A$ be a $D$-sequent and $E^k(t) \in \Sigma, \Pi^1$, and $\forall x(P^n(x) \supset R^l(x)) \in \forall\nabla_1$. Then $(E^k(t), \forall x(P^n(x) \supset R^l(x)))^* = E^k(t), \forall x(P^n(x) \supset R^l(x))$, if $E^k \neq P^n$. Let $E^k = P^n$ then $(E^k(t), \forall x(E^k(x) \supset R^l(x)))^* = E^k(t), R^l(t)$. Let $\forall\nabla_1 = \forall x_1(E_1^{k_1}(x_1) \supset R_1^{l_1}(x_1)), \dots, \forall x_n(E_n^{k_n}(x_n) \supset R_n^{l_n}(x_n))$ then $(E^k(t), \forall\nabla_1)^* = (E^k(t), \forall x_1(E_1^{k_1}(x_1) \supset R_1^{l_1}(x_1)))^*, \dots, (E^k(t), \forall x_n(E_n^{k_n}(x_n) \supset R_n^{l_n}(x_n)))^*$. Let $\Sigma, \Pi^1 = E^k(t_1), \dots, E^k(t_n),\ \Sigma_1,\ \Pi_1^1$ (where $k \geq 0$ and for any $i$ $E^k(t_i) \notin \Sigma_1,\ \Pi_1^1$). Then $(\Sigma, \Pi^1,\ \forall\nabla_1)^* = (E^k(t_1), \forall\nabla_1)^*, \dots, (E^k(t_n), \forall\nabla_1)^*,\ \Sigma_1,\ \Pi_1^1$. The elementary formulas $E^k(t_i)$ $(1 \leq i \leq n)$ are called an e-main formulas of the operation $(*)$ and the quasi-predicate symbol $E^k$ is a main quasi-predicate symbol of the operation $(*)$. A main quasi-predicate symbol becomes a main predicate symbol of the operation $(*)$ if $k = 0$.*

The operation $(*)$ is nondeterministic, it depends on the choosing of the main quasi-predicate symbol of this operation.

**Example 2.** Let $\Sigma, \Pi^1 = Q(t_1), Q(t_2), P^1(u_1), P^1(u_2), R^1(v)$; $\forall\nabla_1 = \forall x(Q(x) \supset N^1(x)), \forall y(P^1(y) \supset M^2(y))$. Choosing $Q$ as the main predicate symbol we have $(\Sigma, \Pi^1, \forall\nabla_1)^* = Q(t_1), Q(t_2), P^1(u_1), P^1(u_2), R^1(v), N^1(t_1), N^1(t_2), \forall y(P^1(y) \supset M^2(y))$. Choosing $P^1$ as the main quasi-predicate symbol, we get $(\Sigma, \Pi^1, \forall\nabla_1)^* = Q(t_1), Q(t_2), P^1(u_1), P^1(u_2), R^1(v), \forall x(Q(x) \supset N^1(x)), M^2(u_1), M^2(u_2)$.

**Definition 19** (calculi $G^{**}_\omega$, $G^{**}$). *A calculus $G^{**}_\omega$ is obtained from the calculus $G^*_\omega$ replacing the rule $(\forall^* \to)$ by the following rule:*

$$\frac{(\Sigma, \Pi^1, \forall\nabla_1)^*,\ \Box\forall\nabla_2 \to \Box^0 A}{\Sigma, \Pi^1, \forall\nabla_1, \Box\forall\nabla_2 \to \Box^0 A}\ (\forall^{**} \to)$$

*A calculus $G^{**}$ is obtained from the calculus $G^{**}_\omega$ by dropping the rule $(\to \Box_\omega)$.*

**Remark 4.** The rule $(\forall^{**} \to)$ (different from the rule $(\forall^* \to)$) may contain more than one e-main formulas, more than one main formulas and more than one side formulas.

**Lemma 18.** *Let $S = \Sigma, \Pi^1, \forall\nabla_1, \Box\forall\nabla_2 \to A$ be an induction-free D-sequent, $V$ be a derivation (of the sequent $S$ in $G^*$) containing only applications of the rule $(\forall^* \to)$. Then there exists a main quasi-predicate symbol such that $G^* \vdash S \Rightarrow G^{**} \vdash S^* = (\Sigma, \Pi^1, \forall\nabla_1)^*,\ \Box\forall\nabla_2 \to A$.*

**Proof.** We shall use induction on $h(V)$. Let $h(V) = 1$, i.e., $S$ is an axiom. Then by definition of the operation $(*)$, the sequent $S^*$ is also an axiom of the calculus $G^{**}$. Let $h(V) > 1$, then let us consider the last application of the rule $(\forall^* \to)$ in $V$:

$$\frac{P^k(t), \Sigma_1, \Pi^1_1, R^l(t), \forall\nabla_1, \Box\forall\nabla_2 \to A}{P^k(t), \Sigma_1, \Pi^1_1, \forall x(P^k(x) \supset R^l(x)), \forall\nabla_1, \Box\forall\nabla_2 \to A}\ (\forall^* \to)$$

By induction assumption there exists a main quasi-predicate symbol such that $G^{**} \vdash S^+ = (P^k(t), \Sigma_1, \Pi^1_1, R^l(t), \forall\nabla_1)^*, \Box\forall\nabla_2 \to A$. Taking as the main quasi-predicate symbol $P^k$ we get $G^{**} \vdash S^* = (P^k(t), \Sigma_1, \Pi^1_1, \forall x(P^k(x) \supset R^l(x)), \forall\nabla_1)^*, \Box\forall\nabla_2 \to A$.

**Lemma 19.** *Let $S = \Sigma, \Pi^1, \forall\nabla_1, \Box\forall\nabla_2 \to A$ be an induction-free D-sequent, $V$ be a derivation (of the sequent $S$ in $G^*$) containing only applications of the rule $(\forall^* \to)$. Then $G^* \vdash S \Rightarrow G^{**} \vdash S$.*

**Proof.** By using Lemma 18 there exists a main quasi-predicate symbol such that $G^{**} \vdash S^* = (\Sigma, \Pi^1, \forall\nabla_1)^*, \Box\forall\nabla_2 \to A$. Applying the rule $(\forall^{**} \to)$ to the sequent $S^*$ we get $G^{**} \vdash S$.

**Lemma 20.** *Let $S$ be a D-sequent then $G^*_\omega \vdash S \Rightarrow G^{**}_\omega \vdash S$.*

**Proof.** Follows from the invertibility of the rule $(\to \Box_\omega)$ and Lemmas 4, and 19.

**Lemma 21.** *Let $S$ be a D-sequent then $G^{**}_{\omega} \vdash S \Rightarrow G^{*}_{\omega} \vdash S$.*

**Proof.** Follows from the fact that each application of the rule $(\forall^{**} \to)$ can be step by step replaced by corresponding applications of the rule $(\forall^{*} \to)$.

**Definition 20** (parametrically active/inactive $a\forall$-formula). *An $a\forall$-formula $\forall x(E(x) \supset Q^k(\bar{f}(x)))$ is called parametrically inactive in a D-sequent $S = \Sigma, \Pi^1, \forall x(E(x) \supset Q^k(\bar{f}(x))), \forall\nabla_{12}, \Box\forall\nabla_2 \to B$, if $E(t) \notin \Sigma$. Otherwise, this $a\forall$-formula is called parametrically active in $S$.*

**Theorem 4** (soundness and $\omega$-completeness of the calculus $G^{**}_{\omega}$ for $D$-sequents). *Let $S$ be a D-sequent. Then $\forall M \models S$ if and only if $G^{**}_{\omega} \vdash S$.*

**Proof.** Follows from Theorem 2 and Lemmas 20, 21.

**Lemma 22** (existential invertibility of the rule $(\forall^{**} \to)$ for $a\forall$-formulas). *Let $S = \Sigma, \Pi^1, \forall\nabla_{11}, \forall\nabla_{12}, \Box\forall\nabla_2 \to A$ be an induction-free D-sequent and $\forall\nabla_{11}$ consists of parametrically active $a\forall$-formulas. Then there exists a main predicate symbol of the operation $(*)$ such that if $G^{**} \vdash S$ then $G^{**} \vdash S^* = (\Sigma, \forall\nabla_{11})^*, \Pi^1, \forall\nabla_{12}, \Box\forall\nabla_2 \to A$.*

**Proof.** Let $V$ be a derivation of the sequent $S$ in $G^{**}$. The proof of the lemma can be obtained by induction on $h(V)$. Let $h(V) = 1$, i.e., the sequent $S$ is an axiom. By definition of the operation $(*)$, the sequent $S^*$ is also an axiom. Let $h(V) > 1$ and also let $(j)$ be the last rule applied in $V$. The following cases are possible.

1. $(j) = (\forall^{**} \to)$ and main formulas belong to $\forall\nabla_{12}$, i.e., a main quasi-predicate symbol of the operation $(*)$ belongs to $\Pi^1$:

$$\frac{V_1\{\Sigma, \forall\nabla_{11}, (\Pi^1, \forall\nabla_{12})^*, \Box\forall\nabla_2 \to A}{\Sigma, \Pi^1, \forall\nabla_{11}, \forall\nabla_{12}, \Box\forall\nabla_2 \to A}\,(\forall^{**} \to)$$

Having applied the induction assumption to the derivation $V_1$ we get $G^{**} \vdash S^+ = (\Sigma, \forall\nabla_{11})^*,\ (\Pi^1, \forall\nabla_{12})^*,\ \Box\forall\nabla_2 \to A$. Applying the rule $(\forall^{**} \to)$ to the $D$-sequent $S^+$ we get desired derivation of the $D$-sequent $S^* = (\Sigma, \forall\nabla_{11})^*,\ \Pi^1,\ \forall\nabla_{12},\ \Box\forall\nabla_2 \to A$ in $G^{**}$.

2. $(j) = (\forall^{**} \to)$ and main formulas belong to $\forall\nabla_{11}$, i.e., a main predicate symbol of the operation $(*)$ belongs to $\Sigma$:

$$\frac{V_1\{(\Sigma, \forall\nabla_{11})^*, \Pi^1, \forall\nabla_{12}, \Box\forall\nabla_2 \to A}{\Sigma, \forall\nabla_{11}, \Pi^1, \forall\nabla_{12}, \Box\forall\nabla_2 \to A}\,(\forall^{**} \to)$$

In this case the derivation $V_1$ is desired one.

3. $(j) = (\Box \to)$ and the main formula of this application is the following formula $\Box\forall x(E^k(x) \supset P^l(x))$, where $k < l$ and $k > 0$, i.e., $\Box\forall\nabla_2 = \Box\forall x(E^k(x) \supset P^l(x))$, $\Box\forall\nabla_{21}$:

$$\frac{V_1\{\Sigma, \forall\nabla_{11}, \Pi^1, \forall\nabla_{12}, \forall x(E^k(x) \supset P^l(x)), \Box C^1, \Box\forall\nabla_{21} \to A}{\Sigma, \forall\nabla_{11}, \Pi^1, \forall\nabla_{12}, \Box\forall x(E^k(x) \supset P^l(x)), \Box\forall\nabla_{21} \to A}\,(\Box \to),$$

where $C^1 = \forall x(E^{k+1}(x) \supset P^{l+1}(x))$.

Having applied the induction assumption to the derivation $V_1$ we get $G^{**} \vdash S^+ = (\Sigma, \forall\nabla_{11})^*, \Pi^1, \forall\nabla_{12}, \forall x(E^k(x) \supset P^l(x)), \Box C^1, \Box\forall\nabla_2 \to A$. Applying the rule $(\Box \to)$ to the $D$-sequent $S^+$ we get desired derivation of the $D$-sequent $S^* = (\Sigma, \forall\nabla_{11})^*, \Pi^1, \forall\nabla_{12}, \Box C, \Box\forall\nabla_2 \to A$ in $G^{**}$.

4. $(j) = (\Box \to)$ and the main formula of this application is the following formula $\Box\forall x(E(x) \supset P^l(x))$, i.e., $\Box\forall\nabla_2 = \Box\forall x(E(x) \supset P^l(x))$, $\Box\forall\nabla_{21}$:

$$\frac{V_1\{\Sigma, \forall\nabla_{11}, \Pi^1, \forall\nabla_{12}, \forall x(E(x) \supset P^l(x)), \Box C^1, \Box\forall\nabla_{21} \to A}{\Sigma, \forall\nabla_{11}, \Pi^1, \forall\nabla_{12}, \Box\forall x(E(x) \supset P^l(x)), \Box\forall\nabla_{21} \to A}\,(\Box \to),$$

where $C^1 = \forall x(E^1(x) \supset P^{l+1}(x))$. Having applied the induction assumption to the derivation $V_1$ we get $G^{**} \vdash S^+ = (\Sigma, \forall\nabla_{11}, \forall x(E(x) \supset P^l(x)))^*, \Pi^1, \forall\nabla_{12}, \Box C^1, \Box\forall\nabla_{21} \to A$. Let us consider two cases.

4.1. The main predicate symbol of the operation $(*)$ is different from $E$. In this case $(\Sigma, \forall\nabla_{11}, \forall x(E(x) \supset P^l(x)))^* = (\Sigma, \forall\nabla_{11})^*$, $\forall x(E(x) \supset P^l(x))$. Applying the rule $(\Box \to)$ to the $D$-sequent $S^+$ we get desired derivation of $D$-sequent $S^* = (\Sigma, \forall\nabla_{11})^*$, $\Pi^1, \forall\nabla_{12}, \Box\forall x(E(x) \supset P^l(x))$, $\Box\forall\nabla_{21} \to A$ in $G^{**}$.

4.2. The main predicate symbol of the operation $(*)$ is $E$. In this case $(\Sigma, \forall\nabla_{11}, \forall x(E(x) \supset P^l(x)))^* = (\Sigma, \forall\nabla_{11})^*, P^l(t_1), \dots, P^l(t_n)$, where $n \geq 1$. Applying the rule $(\forall^{**} \to)$ to the $D$-sequent $S^+$ we get $G^{**} \vdash S' = (\Sigma, \forall\nabla_{11})^*$, $\forall x(E(x) \supset P^l(x)), \Pi^1, \forall\nabla_{12}$, $\Box C^1 \to A$. Applying the rule $(\Box \to)$ to the $D$-sequent $S'$ we get desired derivation of the $D$-sequent $S^* = (\Sigma, \forall\nabla_{11})^*, \Pi^1, \forall\nabla_{12}, \Box C, \Box\forall\nabla_{21} \to A$ in $G^{**}$.

**Remark 5.** Lemma 22 is not valid if a quasi-main predicate symbol is chosen instead of main predicate symbol. Indeed, let $S = R^1(u), P(t), \forall x(P(x) \supset R^1(x)), \forall y(R^1(y) \supset N^2(y)) \to N^2(t)$. It is easy to verify that bottom-up applying the rule $(\forall^{**} \to)$ to the $D$-sequent $S$ with $P$ as the main predicate symbol and after then the rule $(\forall^{**} \to)$ with $R^1$ as the main quasi-predicate symbol, we get $G^{**} \vdash S$. However $G^{**} \nvdash R^1(u), P(t), \forall x(P(x) \supset R^1(x)), N^2(u) \to N^2(t)$.

**Lemma 23** (elimination of parametrically inactive $a\forall$-formulas). *Let $S$ be an induction-free D-sequent of the form $\Sigma, \Pi^1, \forall\nabla_{11}, \forall\nabla_{12}, \Box\forall\nabla_2 \to B$, where $\forall\nabla_{12}$ consists of parametrically inactive $a\forall$-formulas. Then the condition $G^{**} \vdash S = \Sigma, \Pi^1, \forall\nabla_{11}, \forall\nabla_{12}, \Box\forall\nabla_2 \to B$ implies $G^{**} \vdash S^* = \Sigma, \Pi^1, \forall\nabla_{11}, \Box\forall\nabla_2 \to B$.*

**Proof.** Let $V$ be a derivation of the sequent $S$ in $G^{**}$. The proof of the lemma is conducted by induction on $h(V)$. If $h(V) = 1$, i.e., $S$ is an axiom, then $S^*$ is also an axiom.

Let $h(V) > 1$ and let $(i)$ be the last rule applied in $V$. All possible cases can be classified as follows.

1. $(i) = (\forall^{**} \to)$. The expression $\forall\nabla_{12}$ consists of parametrically inactive $a\forall$-formulas; therefore, $(\Sigma, \Pi^1, \forall\nabla_{11}, \forall\nabla_{12})^* = (\Sigma, \Pi^1, \forall\nabla_{11})^*, \forall\nabla_{12}$ and the end of derivation $V$ is as follows:

$$\frac{V_1\{(\Sigma, \Pi^1, \forall\nabla_{11})^*, \forall\nabla_{12}, \Box\forall\nabla_2 \to B}{\Sigma, \Pi^1, \forall\nabla_{11}, \forall\nabla_{12}, \Box\forall\nabla_2 \to B}\,(\forall^{**} \to).$$

Having applied, first, the induction assumption to the derivation $V_1$ and afterwards the rule $(\forall^{**} \to)$, we have the required derivation of the $D$-sequent $\Sigma, \Pi^1, \forall\nabla_{11}, \Box\forall\nabla_2 \to B$ in $G^{**}$.

2. $(i) = (\Box \to)$. Let $\Box\forall\nabla_2 = \Box\forall x A(x), \Box\forall\nabla_{21}$. Then the end of this derivation is as follows:

$$\frac{V_1\{\Sigma, \Pi^1, \forall\nabla_{11}, \forall\nabla_{12}, \forall x A(x), \Box\forall x A^1(x), \Box\forall\nabla_{21} \to B}{\Sigma, \Pi^1, \forall\nabla_{11}, \forall\nabla_{12}, \Box\forall x A(x), \Box\forall\nabla_{21} \to B}\,(\Box \to).$$

2.1. The formula $\forall x A(x)$ is a parametrically active $a\forall$-formula. Thus, having applied, first, the induction hypothesis and afterwards the rule $(\Box \to)$ to the derivation $V_1$, we get the derivation of the sequent $\Sigma, \Pi^1$, $\forall\nabla_{11}, \Box\forall x A(x), \Box\forall\nabla_{21} \to B$.

2.2. The formula $\forall x A(x)$ is a parametrically inactive $a\forall$-formula. Then, having applied the induction assumption to the derivation $V_1$, we get the derivation $V_2$ of the sequent $\Sigma, \Pi^1, \forall\nabla_{11}, \Box\forall x A^1(x), \Box\forall\nabla_{21} \to B$. Having applied the structural rule weakening (which is admissible in the calculus $G^{**}$) and then the rule $(\Box \to)$ to the derivation $V_2$, we have the required derivation of the sequent $\Sigma, \Pi^1, \forall\nabla_{11}, \Box\forall x A(x), \Box\forall\nabla_{21} \to B$.

**Definition 21** (temporal-primary induction-free $D$-sequent, in short: $t$-primary induction-free $D$-sequent; temporal $D$-sequent, in short: $t$-$D$-sequent). *An induction-free D-sequent $S$ is called t-primary, if $S = \Sigma, \Pi^1, \Box\Omega^1 \to B$, where $\Sigma$ consists of atomic formulas, $\Pi^1$ either is empty* *or consists of elementary formulas of the form* $E^{l+1}$ *and* $\Box\Omega^1$ *consists of* $\forall$*-formulas of the form* $\Box\forall x(P^{n+1}(x) \supset Q^{m+1}(\bar{f}(x)))$, $n < m$, $n \geq 0$. *A D-sequent S is called t-D- sequent, if* $S = \Sigma, \Pi^1, \Box\Omega \rightarrow \Box^0 B$ *and* $\Box\Omega$ *consists of the formulas of the form* $\Box\forall x(E(x) \supset P^l(\bar{f}(x)))$, $l > 0$.

**Lemma 24** reduction to a $t$-primary induction-free $D$-sequent). *Let* $S = \Sigma, \Pi^1, \Box\Omega, \Box\Omega^1_1 \rightarrow B$ *be an induction-free D-sequent. Then the condition* $G^{**} \vdash S$ *implies* $G^{**} \vdash S^* = \Sigma, \Pi^1, \Theta^1, \Box\Omega^1, \Box\Omega^1_1 \rightarrow B$, *where the expressions* $\Pi^1$ *and* $\Theta^1$ *consist of elementary formulas of the form* $E^{l+1}$.

**Proof.** The lemma is proved in three steps. In the first step, by invertibility of the rule ($\Box \rightarrow$), the initial induction-free $D$-sequent $S$ is reduced to the $D$-sequent $S_1 = \Sigma, \Pi^1, \forall\nabla_1, \forall\Delta, \Box\Omega^1, \Box\Omega^1_1 \rightarrow B$, where $\forall\nabla_1(\forall\Delta)$ consists of parametrically active (respectively, inactive) $a\forall$-formulas such that $G^{**} \vdash S_1$. In the second step, by Lemma 22, in the sequent $S_1$, parametrically active $a\forall$-formulas are replaced by the respective conclusions of these $a\forall$-formulas. Hence, $G^{**} \vdash S_{11} = \Sigma, \Pi^1, \Theta^1, \forall\Delta, \Box\Omega^1, \Box\Omega^1_1 \rightarrow B$, where $\Theta^1$ consists of conclusions of parametrically active $a\forall$-formulas. In the third step, by Lemma 23, we can remove parametrically inactive $a\forall$-formulas from the sequent $S_{11}$, thus obtaining the derivation $G^{**} \vdash S^* = \Sigma, \Pi^1, \Theta^1, \Box\Omega^1, \Box\Omega^1_1 \rightarrow B$, i.e., the desired derivation of a $t$-primary $D$-sequent.

**Lemma 25** (separation property). *Let S be an induction-free D-sequent of the form* $\Sigma, \Pi^1, \forall\Delta^1, \Box\Omega^1 \rightarrow B$, *where* $\Sigma$ *consists of atomic formulas;* $\Pi^1$ *either is empty or consists of elementary formulas of the form* $E^{l+1}$; $\forall\Delta^1$ *either is empty or consists of formulas of the form* $\forall x(E^{n+1}(x) \supset P^{m+1}(\bar{f}(x)))$ $(n < m)$, $B = \bigvee_{i=1}^{m} \exists y_i E_i^{k_i}(y_i)$. *Let S is not an axiom with a main formula from* $\Sigma$. *Then the condition* $G^{**} \vdash S$ *implies* $G^{**} \vdash \Pi, \forall\Delta, \Box\Omega \rightarrow B^{-1}$, *where* $B^{-1}$ *denotes the formula which is obtained from B, replacing an elementary formula* $E_i^{k_i}(y_i)$ *by* $E_i^{k_i-1}(y_i)$, *moreover, if* $k_i - 1 < 0$ *then the i-th disjunctive component is omitted.* $\Pi$ $(\forall\Delta, \Box\Omega)$ *is obtained from* $\Pi^1$ $(\forall\Delta^1, \Box\Omega^1$, *respectively) having reduced the indices of elementary formulas by* 1.

**Proof.** Let $V$ be a derivation of the sequent $S$ in $G^{**}$. The lemma is proved by induction on $h(V)$. The basis of the lemma is trivial. Let $h(V) > 1$ and let $(i)$ be the last rule applied in the derivation of $V$. All possible cases can be classified as follows.

1. $(i) = (\forall^{**}\to)$. Then the end of the derivation $V$ is as follows:

$$\frac{V_1\{\Sigma, (\Pi^1, \forall\Delta^1)^*, \Box\Omega^1 \to B}{\Sigma, \Pi^1, \forall\Delta^1, \Box\Omega^1 \to B}\,(\forall^{**}\to).$$

Having applied the induction assumption to the derivation $V_1$ we have $G^{**} \vdash S^+ = (\Pi, \forall\Delta)^*, \Box\Omega \to B^{-1}$. Having applied the rule $(\forall^{**}\to)$ to the $D$-sequent $S^+$ we get the required derivation of the $D$-sequent $\Pi, \forall\Delta$, $\Box\Omega \to B^{-1}$.

2. $(i) = (\Box\to)$. Then the end of the derivation $V$ is of the form

$$\frac{V_1\{\Sigma, \Pi^1, \forall\Delta^1, A^1, \Box A^2, \Box\Omega_1^1 \to B}{\Sigma, \Pi^1, \forall\Delta^1, \Box A^1, \Box\Omega_1^1 \to B}\,(\Box\to),$$

where $A = \forall x(E^{n+1}(x) \supset P^{m+1}(\bar{f}(x)))$ $(n < m)$. Having applied the induction hypothesis to the derivation $V_1$, we have $G^{**} \vdash S^* = \Pi$, $\forall\Delta$, $A$, $\Box A^1$, $\Box\Omega_1 \to B^{-1}$. Having applied the rule $(\Box\to)$ to the sequent $S^*$, we get the required derivation of the sequent $\Pi$, $\forall\Delta$, $\Box A$, $\Box\Omega_1 \to B^{-1}$.

**Lemma 26** ($t$-primary separation property)**.** *Let $S = \Sigma, \Pi^1, \Box\Omega^1 \to B$ be a $t$-primary $D$-sequent and $S$ is not an axiom with a main formula from $\Sigma$. Then $G^{**} \vdash \Sigma, \Pi^1, \Box\Omega^1 \to B$ implies $G^{**} \vdash \Pi, \Box\Omega \to B^{-1}$.*

**Proof.** The lemma follows from Lemma 25 (take $\forall\Delta^1 = \emptyset$).

**Lemma 27.** *In the calculus $G^{**}$ the rule*

$$\frac{\Gamma \to B}{\Pi, \Gamma^1 \to B^1}\,(+1)$$

*is admissible.*

**Proof.** The lemma is proved by induction on the height of the derivation of the $D$-sequent $\Gamma \to B$.

**Remark 6.** The separation properties are inverted rule $(+1)$ that corresponds to a traditional rule for the temporal operator "next".

## 5. Infinitary calculus without loop rules

In this section an infinitary calculus for $t$-$D$-sequents, containing the $\omega$-type rule $(\to \Box_\omega)$ and only one so-called loop-free integrated separation rule $(IS)$, is constructed. The rule $(IS)$ includes the rules $(\Box\to)$, $(\forall^{**}\to)$ and $t$-primary separation property. First we introduce an operation $(+)$.

**Definition 22** (operation (+), $\Omega$-consistent atomic formula). *Let* $S = \Sigma, \Pi^1, \Box\Omega \to B$ *be an induction-free t-D-sequent and* $E(t) \in \Sigma$, *and* $\Box\forall x(Q(x) \supset P^l(\bar{f}(x))) \in \Box\Omega$. *Then* $(E(t), \Box\forall x(Q(x) \supset P^l(\bar{f}(x))))^+ = \Box\forall x(Q(x) \supset P^l(\bar{f}(x)))$, *if* $E \neq Q$. *Let* $E = Q$ *then* $(E(t), \Box\forall x(E(x) \supset P^l(\bar{f}(x))))^+ = P^{l-1}(\bar{f}(t)), \Box\forall x(E(x) \supset P^l(\bar{f}(x)))$; *in this case* $E(t)$ *is an* $\Omega$*-consistent atomic formula. Let* $\Box\Omega = \Box\forall x_1(Q_1(x_1) \supset P_1^{l_1}(\bar{f}_1(x_1))), \dots, \Box\forall x_n(Q_n(x_n) \supset P_n^{l_n}(\bar{f}_n(x_n)))$, *then* $(E(t), \Box\Omega)^+ = (E(t), \Box\forall x_1(Q_1(x_1) \supset P_1^{l_1}(\bar{f}_1(x_1))))^+, \dots, (E(t), \Box\forall x_n(Q_n(x_n) \supset P_n^{l_n}(\bar{f}_n(x_n))))^+$. *Let* $\Sigma = E_1(t_1), \dots, E_n(t_n)$, *then* $(\Sigma, \Box\Omega)^+ = (E_1(t_1), \Box\Omega)^+, \dots, (E_n(t_n), \Box\Omega)^+$.

**Example 3.** Let $\Sigma = R(u), Q(t_1), Q(t_2), P(v)$; $\Box\Omega = \Box\forall x(P(x) \supset N_1^1(f_1(x))), \Box\forall y(P(y) \supset N_2^2(f_2(y))), \Box\forall z(Q(z) \supset M^1(g(z)))$. Then $(\Sigma, \Box\Omega)^+ = M(g(t_1)),\ M(g(t_2)),\ N_1(f_1(v)),\ N_2^1(f_2(v)),\ \Box\Omega$.

**Definition 23** (integrated separation rule). *The integrated separation rule, when applied to an arbitrary induction-free t-D-sequent, is of the form:*

$$\frac{(\Sigma, \Box\Omega)^+, \Pi \to B^{-1}}{\Sigma, \Pi^1, \Box\Omega \to B}\ (IS),$$

*where* $\Sigma, \Pi^1, \Box\Omega \to B$ *is not an axiom with a main formula from* $\Sigma$; $B = \bigvee_{i=1}^{m} \exists y_i E_i^{k_i}(y_i)$; $B^{-1}$ *denotes the formula which is obtained from* $B$, *replacing an elementary formula* $E_i^{k_i}(y_i)$ *by* $E_i^{k_i-1}(y_i)$, *and if* $k_i - 1 < 0$ *then the* $i$*-th disjunctive component is omitted.* $\Pi$ *is obtained from* $\Pi^1$ *having reduced the indices of elementary formulas by* 1.

**Definition 24** (index complexity of induction-free $t$-$D$-sequent). *Let* $S = \Sigma,\ \Pi^1,\ \Box\Omega \to B$ *be an induction-free t-D-sequent. Then an index complexity of the sequent* $S$ *(in short* $IC(S)$*) is defined as maximal index in* $B$.

**Remark 7.** Let $S$ be a conclusion and $S_1$ be a premise of the rule $(IS)$. Then from the form of the rule $(IS)$ it follows that $IC(S_1) < IC(S)$. Thus, the rule $(IS)$ is not a loop rule with respect to index complexity.

**Definition 25** (calculus $G^+$). *The calculus* $G^+$ *is obtained from the calculus* $G^{**}$ *replacing the rules* $(\forall^{**} \to)$, $(\Box \to)$ *by the rule* $(IS)$.

Now we prove the equivalence of the calculi $G^{**}$ and $G^+$ with respect to induction-free $t$-$D$-sequents. First we prove the following lemma.

**Lemma 28.** *The rule* $(IS)$ *is admissible in the calculus* $G^{**}$.

**Proof.** Let $\Box\Omega = \Box\Omega_1, \Box\Omega_2$, where $\Box\Omega_1 = \Box\forall x_1(E_1(x_1) \supset P_1^{k_1}(\bar{f}_1(x_1)))$, $\dots, \Box\forall x_n(E_n(x_n) \supset P_n^{k_n}(\bar{f}_n(x_n)))$; $\Box\Omega_2 = \Box\forall y_1(Q_1(y_1) \supset R_1^{l_1}(\bar{\rho}_1(y_1)))$, $\dots, \Box\forall y_m(Q_m(y_m) \supset R_m^{l_m}(\bar{\rho}_m(y_m)))$ and let $\Sigma = \Sigma_1, \Sigma_2$, where $\Sigma_1$ consists of atomic formulas that are $\Omega_1$-consistent, whereas $\Sigma_2$ consists of atomic formulas that are not $\Omega$-consistent. Let $G^{**} \vdash S_1 = (\Sigma, \Box\Omega)^+, \Pi \to B^{-1}$. Having applied the rule $(+1)$ (which by Lemma 27 is admissible in the calculus $G^{**}$) to the sequent $S_1$, we have $G^{**} \vdash S_2 = (\Sigma_1, \forall x_1(E_1(x_1) \supset P_1^{k_1}(\bar{f}_1(x_1))), \dots, \forall x_n(E_n(x_n) \supset P_n^{k_n}(\bar{f}_n(x_n))))^*$, $\Box\Omega_1^1$, $\forall y_1(Q_1(y_1) \supset R_1^{l_1}(\bar{\rho}_1(y_1))), \dots, \forall y_m(Q_m(y_m) \supset R_m^{l_m}(\bar{\rho}_m(y_m))), \Box\Omega_2^1, \Pi^1 \to B$. Having applied the rule $(\forall^{**} \to)$ and afterwards the rule $(\Box \to)$ to the sequent $S_2$, we have $G^{**} \vdash S_3 = \Sigma_1, \Sigma_2, \Pi^1, \Box\Omega_1, \Box\Omega_2 \to B$, i.e. the conclusion of the rule $(IS)$.

**Lemma 29.** *Let $S$ be an induction-free t-D-sequent. Then the condition $G^{**} \vdash S$ implies $G^+ \vdash S$.*

**Proof.** Let $S = \Sigma, \Pi^1, \Box\Omega \to B$, where $\Sigma = \Sigma_1, \Sigma_2$ and $\Box\Omega = \Box\Omega_1, \Box\Omega_2$, where $\Omega_1$ consists of parametrically active $a\forall$-formulas and $\Omega_2$ consists of parametrically inactive $a\forall$-formulas. Let $\Box\Omega_1 = \Box\forall x_{11}(E_1(x_{11}) \supset P_{11}^{k_1}(\bar{f}_1(x_{11}))), \dots, \Box\forall x_{1i}(E_1(x_{1i}) \supset P_{1i}^{k_i}(\bar{f}_i(x_{1i}))), \dots, \Box\forall x_{n1}(E_n(x_{n1}) \supset P_{n1}^{l_1}(\bar{g}_1(x_{n1}))), \dots, \Box\forall x_{nj}(E_n(x_{nj}) \supset P_{nj}^{l_j}(\bar{g}_j(x_{nj})))$; $\Box\Omega_2 = \Box\forall y_1(Q_1(y_1) \supset R_1^{l_1}(\bar{\rho}_1(y_1))), \dots, \Box\forall y_m(Q_m(y_m) \supset R_m^{l_m}(\bar{\rho}_m(y_m)))$. Since $\Omega_1$ consists of parametrically active $a\forall$-formulas, we have $\Sigma_1 = E_1(p_{11}), \dots, E_1(p_{1i})$, $\dots, E_n(p_{n1}), \dots, E_n(p_{nj})$. We prove Lemma 29 by induction on $IC(S)$, i.e., on the index complexity of the sequent $S$. Let $IC(S) = 0$, then the sequent $S$ is an axiom and, therefore, $G^+ \vdash S$. Let $IC(S) > 0$. If $S$ is an axiom then $G^+ \vdash S$. If $S$ is not an axiom then, by Lemma 24, the sequent $S$ can be reduced to the $t$-primary $D$-sequent $S_1 = \Sigma_1$, $\Sigma_2$, $\Pi^1$, $P_{11}^{k_1}(\bar{f}_1(p_{11})), \dots, P_{nj}^{l_j}(\bar{g}_j(p_{nj}))$, $\Box\Omega_1^1, \Box\Omega_2^1 \to B$, where $G^{**} \vdash S_1$. The sequent $S_1$ satisfies the condition of Lemma 26, because we consider the case that $S$ is not an axiom. Having applied Lemma 26 to the sequent $S_1$, we get that $G^{**} \vdash S_2 = \Pi, P_{11}^{k_1-1}(\bar{f}_1(p_{11})), \dots, P_{nj}^{l_j-1}(\bar{g}_j(p_{nj}))$, $\Box\Omega_1, \Box\Omega_2 \to B^{-1}$. According to the induction hypothesis, $G^+ \vdash S_2$. Applying the rule $(IS)$ to the sequent $S_2$, we get that $G^+ \vdash S$, i.e., the desired derivation of the sequent $S$ in the calculus $G^+$.

**Lemma 30.** *If $S$ is an induction-free t-D-sequent then $G^{**} \vdash S$ if and only if $G^+ \vdash S$.*

**Proof.** Follows from Lemmas 28, and 29.

**Lemma 31.** *The rule $(IS)$ is invertible in the calculus $G^+$, i.e., if $G^+ \vdash S = \Sigma, \Pi^1, \Box\Omega \to B$ and the sequent $S$ is not an axiom with a main formula from $\Sigma$ then $G^+ \vdash S^* = (\Sigma, \Box\Omega)^+, \Pi \to B^{-1}$.*

**Proof.** Let $V$ be a derivation of the sequent $S$ in $G^+$. The lemma is proved by induction on $h(V)$. The case when $h(V) = 1$ is trivial. Let $h(V) > 1$. Since $G^+$ consists of a single rule $(IS)$, the last applied rule in $V$ is the rule $(IS)$, the premise of which is the desired derivation.

**Lemma 32.** *The calculus $G^+$ is decidable for induction-free t-D-sequents.*

**Proof.** Let $S = \Sigma, \Pi^1, \Box\Omega \to B$ be an induction-free $t$-$D$-sequent. The decision procedure is quite simple. Since the axiom of the calculus $G^+$ is decidable, we can check whether $S$ is an axiom or not. If $S$ is not an axiom then by Lemma 31 we have that if $G^+ \vdash \Sigma, \Pi^1, \Box\Omega \to B$ then $G^+ \vdash (\Sigma, \Box\Omega)^+, \Pi \to B^{-1}$. Continuing in the same manner, we get either an axiom (and then $G^+ \vdash S$) or a sequent of the form $\Sigma^*, \Pi^1_1, \Box\Omega \to B_1$, which is not an axiom (and then $G^+ \nvdash S$).

**Definition 26** (infinitary calculus $G^+_\omega$ without loop rules)**.** *The calculus $G^+_\omega$ is obtained from the calculus $G^+$ by attaching the rule $(\to \Box_\omega)$.*

**Theorem 5** (soundness and $\omega$-completeness of the calculus $G^+_\omega$)**.** *The calculus $G^+_\omega$ is sound and $\omega$-complete for t-D-sequents $S$, i.e., $\forall M \models S$ if and only if $G^+_\omega \vdash S$.*

**Proof.** Follows from Theorem 4, Lemma 30 and the invertibility of the rule $(\to \Box_\omega)$.

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