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# THE RESTRICTIONS OF REPRESENTATIONS OF SPECIAL LINEAR GROUPS TO SUBSYSTEM SUBGROUPS OF TYPE $A_1 \times A_1$

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Исследуются ограничения неприводимых представлений специальной линейной группы над алгебраически замкнутым полем положительной характеристики $p$ на подсистемные подгруппы типа $A_1 \times A_1$. Доказана симметричность множества старших весов композиционных факторов. Найдены "большие" композиционные факторы таких ограничений для произвольных $p$-ограниченных представлений. Эти результаты будут использованы для полного описания ограничений.

**1. Introduction.** The description of the restrictions of modular representations to subsystem subgroups is one of the main problems in the representation theory of algebraic groups. It is closely connected with the unsolved problem of describing characters and dimensions of modular irreducible representations. Therefore it is reasonable to develop methods of investigating representations, which do not require their characters. Shchigolev in 2009 [6] found the condition under which some Weyl submodules can be embedded in the restrictions of simple modules of the special linear group.

The paper is the part of the study of the restrictions of irreducible modular representations of algebraic groups to subgroups, whose ranks are small enough with respect to a rank of an initial group. Such restrictions for representations of special linear groups to subgroups of type $A_1$ and $A_2$ were described by us earlier in [3] and [5]. Here we describe the restrictions to the subgroups of type $A_1 \times A_1$.

Let $K$ be an algebraically closed field of characteristic $p > 0$, $G$ be a simply connected algebraic group of type $A_r$ over $K$, i.e. $G = SL_{r+1}(K)$, $r \geq 3$; $\alpha_1, \ldots, \alpha_r$ be the simple roots of $G$ numerated in the standard way; $\omega_1, \ldots, \omega_r$ be the corresponding fundamental weights; and let $\varphi(\omega)$ be an irreducible rational representation of $G$ with the highest weight $\omega = a_1\omega_1 + \ldots + a_r\omega_r$. The weight set of a group of type $A_1 \times A_1$ can be identified with the set of pairs of integers with the help of the following map $(x_1\omega_1, x_2\omega_1) \mapsto (x_1, x_2)$ and the set of all dominant weights can be identified with the set $\mathbb{N}^2$ of pairs of nonnegative integers. The symbol $\operatorname{Irr}\psi$ denotes the set of highest weights of composition factors for a representation $\psi$ of $G$ and $\psi|\Pi$ denotes a restriction of $\psi$ to a subgroup $\Pi \subset G$.

If $\beta_1, \ldots, \beta_s$ are roots of $G$, then $G(\beta_1, \ldots, \beta_s)$ is a subgroup in $G$ generated by root subgroups associated with roots $\pm\beta_1, \ldots, \pm\beta_s$. Note that the roots $\beta_1, \ldots, \beta_s$ are chosen in such a way that they constitute a basis of a root system for $G(\beta_1, \ldots, \beta_s)$. Such subgroups $G(\beta_1, \ldots, \beta_s)$ are called subsystem subgroups. Set

$$G(i_1, \ldots, i_s) = G(\alpha_{i_1}, \ldots, \alpha_{i_s}).$$

In what follows $H$ is a subsystem subgroup of $G$ of type $A_1 \times A_1$. All such subgroups are conjugate in $G$. Therefore, we can put, for instance, $H = G(1, n)$. Taking into account the above identification, one can write $\mathrm{Irr}(\varphi(\omega)|H) \subset \mathbb{N}^2$. Recall that the weight $\omega$ is $p$-restricted if $a_i < p$ for $1 \le i \le r$. Set $m = a_1 + \ldots + a_r$, $m' = a_1 + 2a_2 + \ldots + 2a_{r-1} + a_r$, and

$$S(\omega) = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1, x_2 \le m,\ x_1 + x_2 \le m'\}.$$

In [4] the following theorem describing the restrictions for representations of the special linear group with locally small highest weights with respect to characteristic was obtained.

**Theorem 1.** *[4, Theorem 1] Let $r > 6$ and $a_i + a_{i+1} + 1 < p$ for all $1 \le i \le r-1$. Then*

$$\mathrm{Irr}(\varphi(\omega)|H) = S(\omega).$$

Here we examine such restrictions for representations with arbitrary $p$-restricted highest weights. Recall that for $p = 0$ and $r > 3$, $\mathrm{Irr}(\varphi(\omega)|H) = S(\omega)$ (see Theorem 3).

Suppose that $r \ge 5$. Define a series of subsystem subgroups

$$G = P_r \supset P_{r-1} \supset \ldots \supset P_5 \supset P_4 \supset P_3$$

as follows. Let $\mu = b_1\omega_1 + \ldots + b_l\omega_l$ be the restriction of $\omega$ to $P_l$, $3 \le l \le r$. If $b_1 \ge b_l$, put $P_{l-1} = P_k(1, \ldots, l-1)$, otherwise set $P_{l-1} = P_k(2, \ldots, l)$. Eventually we get $P_5 = G(i, i+1, i+2, i+3, i+4) \cong A_5(K)$ for some $1 \le i \le r-4$, $P_4 = G(j, j+1, j+2, j+2)$ with $j = i$ or $i+1$, and $P_3 = G(k, k+1, k+2)$ with $k = j$ or $j+1$. Fix such $i, j, k$ for the next theorem. Our main result is as follows

**Theorem 2.** *Let $r \ge 5$ and $\omega$ be $p$-restricted. Then*

$$S(\omega) \setminus N \subset \mathrm{Irr}(M(\omega)|H)$$

*with*

$$N = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1, x_2 < a_j + a_{j+1} + a_{j+2} + a_{j+3},\quad x_1 + x_2 < a_j + 2a_{j+1} + 2a_{j+2} + a_{j+3}\}.$$

*If $r \ge 6$, then moreover*

$$S(\omega) \setminus N' \subset \mathrm{Irr}(M(\omega)|H)$$

*with*

$$N' = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1, x_2 < a_k + a_{k+1} + a_{k+2},\quad x_1 + x_2 < a_k + 2a_{k+1} + a_{k+2}\}.$$

**2. Some properties of the set of highest weights of composition factors.** First introduce some notation. The symbol $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ is the set of all nonnegative integers. Let $S$ be a subsystem subgroup of $G$ and $V$ be a $G$-module. For the group $G$ denote by $L(\mu)$ the irreducible module with highest weight $\mu$, by $\Delta(\mu)$ the Weyl module with such highest weight, and by $\mathrm{ch}(V)$ the formal character of $V$. Put $\mathrm{ch}(\mu) = \mathrm{ch}(L(\mu))$, $\chi(\mu) = \mathrm{ch}(\Delta(\mu))$, $\mathrm{ch}(\mu|S) = \mathrm{ch}(L(\mu)|S)$, and $\chi(\mu|S) = \chi(\Delta(\mu)|S)$. Denote by $X(V)$ the set of weights of $V$ and by $V^\lambda$ the weight space of a weight $\lambda \in X(V)$. Given a weight vector $v \in V$ denote by $\omega(v)$ and $\omega_S(v)$ its weight with respect to $G$ and $S$.

For a root $\alpha$ of $G$ and a positive integer $k$ the symbols $X_\alpha$, $\mathcal{X}_\alpha$, and $X_{\alpha,k}$ denote the root element of the Lie algebra of $G$ associated with $\alpha$, and the element of the hyperalgebra of $G$ associated with the pair ( $\alpha$, $k$ ), respectively. For $k < p$ one has $X_{\alpha,k} = (X_\alpha)^k/k!$. If $\alpha = \pm\alpha_i$, we use the notation $X_{\pm i}$, $\mathcal{X}_{\pm i}$, and $X_{\pm i,k}$.

The vector $v \in V$ is called a primitive vector with respect to $S$ if $v$ is a nonzero weight vector and $\mathcal{X}_\alpha$ fixes $v$ for every positive root $\alpha$ of $S$.

Denote by $M(\mu)$ an indecomposable $G$-module generated by a vector of highest weight $\mu$. It is a quotient of $\Delta(\mu)$ [2, Part II, Lemma 2.13(b)]. Fix a highest weight vector $v^+$ in the module $M(\mu)$. Recall that $L(\mu) = M(\mu) = \Delta(\mu)$ in characteristic 0.

Suppose that $M = M(\omega)$ with $\omega = a_1\omega_1 + \ldots + a_r\omega_r$ and $1 \le i, j \le r$. Assume that $0 < a_j < p$ for some $j$. For an integer $d$ with $0 < d \le a_j$ define the vector $v(i,j,d)$ as follows. Put $d_j = d$. If $i < j$, set $d_k = a_k + d_{k+1}$ for $i \le k < j$. If $i > j$, put $d_k = a_k + d_{k-1}$ for $i \ge k > j$. Now take

$$v(i,j,d) = X_{-i,d_i} \ldots X_{-k,d_k} \ldots X_{-j,d}v^+.$$

For $i = j$ put $v(i,j,d) = X_{-i,d}v^+$. Then $v(i,j,d)$ is primitive with respect to $G(1,\ldots,i-1,i+1,\ldots,r)$ by [8, Lemma 2.46]. Observe that if $a_j \ge p$, then $v(i,j,d)$ is nonzero and primitive with respect to $G(1,\ldots,i-1,i+1,\ldots,r)$ if $\omega - d\alpha_j$ is a weight of $M$.

Let $\mathbb{C}$ be the field of complex numbers, $G_{\mathbb{C}} = SL_{r+1}(\mathbb{C})$, $H_{\mathbb{C}} \subset G_{\mathbb{C}}$ be a subsystem subgroup of type $A_1 \times A_1$, and let $L(\omega)_{\mathbb{C}}$ be the irreducible module over $G_{\mathbb{C}}$ with the highest weight $\omega = a_1\omega_1 + \ldots + a_r\omega_r$.

**Theorem 3.**

*(i) For* $r = 3$

$$\begin{aligned}\mathrm{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}}) = \{(x_1,x_2) \in S(\omega) \mid\ & |a_1 - a_3| \le x_1 + x_2,\\ & |x_1 - x_2| \le a_1 + a_3,\\ & x_1 + x_2 \equiv a_1 - a_3 (\mathrm{mod}\ 2)\}.\end{aligned}$$

*(ii) For* $r > 3$

$$\mathrm{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}}) = S(\omega).$$

**Proof.** The result is proved in [9, Theorems 1.1 and 1.2], but it is a mistake here. Theorem 1.1 from [9] is formulated as follows

**Theorem 4.** *[9, Theorem 1.1] Let* $G = A_3(K)$. *Then*

$$\begin{aligned}\mathrm{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}}) = \{(x_1,x_2) \in \mathbb{N}^2 \mid\ & 0 \le x_1, x_2 \le a_1 + a_2 + a_3,\\ & |a_1 - a_3| \le x_1 + x_2 \le a_1 + 2a_2 + a_3,\\ & x_1 + x_2 \equiv a_1 - a_3 (\mathrm{mod}\ 2)\}.\end{aligned} \tag{1}$$

*For* $a_1 \ge a_3$ *we have*

$$\begin{aligned}\mathrm{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}}) = & \sum_{k=0}^{a_2-1}(k+1)\sum_{j=0}^{a_3}\sum_{i=a_2-k+j}^{a_1+a_2-k+j}(a_1 + 2a_2 + a_3 - 2k - i, i) +\\ & + (a_2+1)\sum_{k=0}^{a_3}\sum_{j=0}^{a_3-k}\sum_{i=j}^{a_1-k+j}(a_1 + a_3 - 2k - i, i).\end{aligned} \tag{2}$$

In [9] the next form of (2) is given as well. It specifies the explicit values of composition factor multiplicities.

$$\mathrm{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}}) = \sum_{k=0}^{a_2+a_3}\ \sum_{i=\max(0,a_3-k)}^{a_1+a_2+a_3-k+\min(0,a_2-k)} \mathbf{C_{ik}}(a_1 + 2a_2 + a_3 - 2k - i, i), \tag{3}$$

where for $0 \le k \le a_2 + a_3$

$$\mathbf{C_{ik}} = \begin{cases} \min(a_2+1, k+1)(i+1-\max(0, a_2-k)) \\ \qquad\qquad \text{if } \max(0, a_2-k) \le i \le a_2+a_3-k, \\ \min(a_2+1, k+1)(a_3+1+\min(0, a_2-k)) \\ \qquad\qquad \text{if } a_2+a_3+1-k \le i \le a_1+a_2-1-k, \\ \min(a_2+1, k+1)(a_1+a_2+a_3-k+\min(0, a_2-k)-i+1) \\ \qquad\qquad \text{if } a_1+a_2-k \le i \le a_1+a_2+a_3-k+\min(0, a_2-k). \end{cases} \tag{4}$$

Formula (1) is incorrect. At the same time, the proof of Theorem 1.1 in [9] (where actually (2) is proven) is true. Formulas (3) and (4) follow from (2) and are true as well. It follows from (3) that for $r = 3$ one should add the inequality $|x_1 - x_2| \le a_1 + a_3$ in the set from (1), thus we get Item (i). Theorem 1.2 in [9] is proved with the help of (1), but when we change (1) to the correct formula, the proof remains valid. This completes the proof.

In characteristic $p$ the composition factors of $L(\omega)|H$ may not coincide with characteristic 0.

**Example.** Let $r = 3$, $\omega = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$, with $a_1 + a_2 + 2 < p$ and $a_2 + a_3 + 2 < p$, but $a_1 + a_2 + a_3 + 3 > p$. Then

$$\operatorname{Irr}(L(\omega)|H) = \{(x_1, x_2) \in \operatorname{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}}) \mid 2p - a_1 - 2a_2 - a_3 - 6 < x_1 + x_2\}.$$

Indeed, in this case $\operatorname{ch}(\omega) = \chi(\omega) - \chi(\omega')$ with $\omega' = (p - a_2 - a_3 - 3)\omega_1 + a_2\omega_2 + (p - a_1 - a_2 - 3)\omega_3$ (see [2, Part II, 8.20]). Restricting $\chi(\omega)$ and $\chi(\omega')$ to $H$ and using Formulas (3) and (4), we get the required.

**Lemma 1.** *Let $p \ge 0$, $r \ge 3$, and let $\omega$ be an arbitrary dominant weight. If $(x_1, x_2) \in \operatorname{Irr}(M(\omega)|H)$, then $(x_2, x_1) \in \operatorname{Irr}(M(\omega)|H)$ as well and composition factors with these highest weights have the same multiplicity in the restriction $M(\omega)|H$.*

**Proof.** First let $r = 3$. If $p = 0$, then $L(\omega) = M(\omega) = \Delta(\omega)$. It follows from (4) that

$$\operatorname{ch}(\omega|H) = \sum\nolimits_{(x_1,x_2)} c(x_1, x_2) \operatorname{ch}(x_1, x_2), \tag{5}$$

where $(x_1, x_2)$ are some dominant weights of $H$, coefficients $c(x_1, x_2) \in \mathbb{N}$, and $c(x_1, x_2) = c(x_2, x_1)$. If $p > 0$, Formula (5) implies that

$$\chi(\omega|H) = \sum\nolimits_{(x_1,x_2)} c(x_1, x_2)\chi(x_1, x_2). \tag{6}$$

On the other hand,

$$\operatorname{ch}(M(\omega)) = \chi(\omega) + \sum_j c_j \chi(\lambda_j), \tag{7}$$

where $\lambda_j$ are some dominant weights of the group $G$ and $c_j \in \mathbb{Z}$. Restricting (7) to $H$ and using (6), we obtain

$$\operatorname{ch}(M(\omega)|H) = \sum\nolimits_{(x_1,x_2)} d(x_1, x_2)\chi(x_1, x_2)$$

with $d(x_1, x_2) \in \mathbb{Z}$ and $d(x_1, x_2) = d(x_2, x_1)$. Observe that if $L(y_1, y_2)$ is a composition factor of $\Delta(x_1, x_2)$, then $L(y_2, y_1)$ is a composition factor of $\Delta(x_2, x_1)$ with the same multiplicity. Taking this into account, finally we get formula

$$\operatorname{ch}(M(\omega)|H) = \sum\nolimits_{(x_1,x_2)} e(x_1, x_2) \operatorname{ch}(x_1, x_2)$$

with $e(x_1, x_2) \in \mathbb{N}$ and $e(x_1, x_2) = e(x_2, x_1)$.

Now suppose that $r > 3$. Put $\Gamma = G(1,2,3) \cong A_3(K)$. Then

$$\mathrm{ch}(M(\omega)|\Gamma) = \sum\nolimits_i k_i \,\mathrm{ch}(\mu_i)$$

with some dominant weights $\mu_i$ of $\Gamma$ and coefficients $k_i \in \mathbb{N}$. Since $G(1,n)$ and $G(1,3)$ are conjugate, one can assume that $H = G(1,3)$. Restricting $\mathrm{ch}(\mu_i)$ to $H$ and using the result for $r = 3$, we get the assertion of the lemma.

**Lemma 2.** *Let* $\Gamma = SL_2(K)$, $a < 3p-1$.

*(i) If* $a \leq p-1$ *or* $i = 2p-1$, *then* $\mathrm{ch}(a) = \chi(a)$.

*(ii) If* $p \leq a < 2p-1$, *then* $\mathrm{ch}(a) = \chi(a) - \chi(2p-a-2)$.

*(iii) If* $2p \leq a < 3p-1$, *then* $\mathrm{ch}(a) = \chi(a) - \chi(4p-a-2) + \chi(a-2p)$ *for* $p > 2$ *and* $\mathrm{ch}(a) = \chi(a) - \chi(4p-a-2)$ *for* $p = 2$.

**Proof.** Items (i) and (ii) follow from [3, Lemma 3.4].

(iii) Assume that $2p \leq a < 3p-1$. First let $p > 2$. Then, by the Steinberg tensor product theorem [7, Theorem 41], $L(a) \cong L(a-2p) \otimes L(2)^p$. Here $L(2)^p$ denotes the module obtained from $L(2)$ with the help of the field morphism of $\Gamma$ associated with raising the elements of $K$ to the power $p$. The module $L(2)^p$ is the direct sum of one-dimensional weight spaces of weights

$$2p, 0, -2p$$

and $L(a-2p)$ is the direct sum of one-dimensional weight spaces of weights

$$a-2p, a-2p-2, \ldots, -(a-2p-2), -(a-2p).$$

Hence the tensor product $L(a)$ is the direct sum of one-dimensional weight spaces of the following weights

$$a, a-2, \ldots, 4p-a; a-2p, a-2p-2, \ldots, -(a-2p-2), -(a-2p); -(4p-a), \ldots, -(a-2), -a.$$

By [2, Lemma 2.13(b)], we have $L(a) = \Delta(a)/M$ where $M$ is a submodule. Hence $M$ is a direct sum of one-dimensional weight spaces with weights

$$4p-a-2, 4p-a-4, \ldots, a-2p+2; -(a-2p+2), \ldots, -(4p-a-4), -(4p-a-2).$$

We have $p \leq 4p-a-2 < 2p-1$. By Item (ii) $M \cong L(4p-a-2) = \Delta(4p-a-2)/L(a-2p)$.

Now let $p = 2$. Then $a = 4$ and by [7, Theorem 41], $L(4) \cong L(1)^4$. Therefore it is the direct sum of one-dimensional weight spaces of the weights $4$ and $-4$. Hence $\mathrm{ch}(4) = \chi(4) - \chi(2)$. This concludes the proof.

**Lemma 3.** *Let* $r \geq 3$. *Then* $\mathrm{Irr}(M(\omega)|H) \subset \mathrm{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}})$.

**Proof.** By [2, Lemma 2.13(b)], $M(\omega) = \Delta(\omega)/M$, where $M$ is some $G$-module. Therefore it suffices to prove that $\mathrm{Irr}(\Delta(\omega)|H) \subset \mathrm{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}})$. By Theorem 3, $\mathrm{Irr}(L(\omega)_{\mathbb{C}}|H_{\mathbb{C}}) \subset S(\omega)$. For $p > 0$ this implies that

$$\chi(\omega|H) = \sum\nolimits_{(m_1,m_2)} c(m_1,m_2)\chi(m_1,m_2),$$

where $(m_1,m_2) \in S(\omega)$ and $c(m_1,m_2) \in \mathbb{N}$. We need to prove that

$$\chi(m_1,m_2) = \sum\nolimits_{(x_1,x_2)} d(x_1,x_2)\,\mathrm{ch}(x_1,x_2)$$

with $(x_1, x_2) \in S(\omega)$ and $d(x_1, x_2) \in \mathbb{N}$. Since $(x_1, x_2)$ is a weight of $\Delta(m_1, m_2)$, it is obvious that $(x_1, x_2) = (m_1 - 2k_1, m_2 - 2k_2)$ with $k_1$ and $k_2 \in \mathbb{N}$. Therefore $(x_1, x_2) \in S(\omega)$ and the lemma holds for $r > 3$.

It remains to consider $r = 3$. In this case it is clear that $x_1 + x_2 \equiv m_1 + m_2 (\mathrm{mod}\ 2) \equiv a_1 - a_3 (\mathrm{mod}\ 2)$.

We can assume that $a_1 \geq a_3$ (otherwise consider the dual module). To prove that $a_1 - a_3 \leq x_1 + x_2$ and $-a_1 - a_3 \leq x_1 - x_2 \leq a_1 + a_3$ first suppose that $\omega$ is $p$-restricted. Observe that this inequalities apparently hold for $p = 2$. Therefore let $p > 2$. Theorem 3(i) implies that $c(m_1, m_2) \neq 0$ iff $(m_1, m_2) \in \mathrm{Irr}(L(\omega)_{\mathbb{C}} | H_{\mathbb{C}})$.

Since $m_1$ and $m_2 \leq 3p - 3$, it follows from Lemma 2 that

$$\begin{aligned}\chi(m_1, m_2) = {} & \mathrm{ch}(m_1, m_2) + c_1 \,\mathrm{ch}(2p - m_1 - 2, m_2) + c_2 \,\mathrm{ch}(4p - m_1 - 2, m_2) + \\ & + c_3 \,\mathrm{ch}(m_1, 2p - m_2 - 2) + c_4 \,\mathrm{ch}(2p - m_1 - 2, 2p - m_2 - 2) + \\ & + c_5 \,\mathrm{ch}(4p - m_1 - 2, 2p - m_2 - 2) + c_6 \,\mathrm{ch}(m_1, 4p - m_2 - 2) + \\ & + c_7 \,\mathrm{ch}(2p - m_1 - 2, 4p - m_2 - 2) + c_8 \,\mathrm{ch}(4p - m_1 - 2, 4p - m_2 - 2)\end{aligned}$$

with $c_i \in \mathbb{N}$.

1.) If $(x_1, x_2) = (2p - m_1 - 2, m_2)$, then Lemma 2 implies that $c_1 > 0$ for $p \leq m_1 < 2p - 1$. We have $x_1 + x_2 = 2p - 2 - m_1 + m_2 \geq 2p - 2 - a_1 - a_3 \geq a_1 - a_3$ since $a_1 \leq p - 1$, $x_1 - x_2 = 2p - 2 - m_1 - m_2 \geq 2p - 2 - a_1 - 2a_2 - a_3 \geq -a_1 - a_3$ since $a_2 \leq p - 1$, $x_1 - x_2 = x_1 - m_2 \leq m_1 - m_2 \leq a_1 + a_3$.

2.) Now put $(x_1, x_2) = (4p - m_1 - 2, m_2)$. Then $c_2 > 0$ if $2p \leq m_1 < 3p - 1$. We have $x_1 + x_2 = 4p - 2 - m_1 + m_2 \geq 4p - 2 - a_1 - a_3 \geq a_1 - a_3$ since $a_1 \leq p - 1$, $x_1 - x_2 = 4p - 2 - m_1 - m_2 \geq 4p - 2 - a_1 - 2a_2 - a_3 \geq -a_1 - a_3$ since $a_2 \leq p - 1$, $x_1 - x_2 = x_1 - m_2 \leq m_1 - m_2 \leq a_1 + a_3$.

3.) Suppose that $(x_1, x_2) = (m_1, 2p - m_2 - 2)$. Then $c_3 > 0$ if $p \leq m_2 < 2p - 1$. In this case $x_1 + x_2 = 2p - 2 + m_1 - m_2 \geq 2p - 2 - a_1 - a_3 \geq a_1 - a_3$ since $a_1 \leq p - 1$, $x_1 - x_2 = m_1 - x_2 \geq m_1 - m_2 \geq -a_1 - a_3$, $x_1 - x_2 = m_1 + m_2 - 2p + 2 \leq a_1 + 2a_2 + a_3 - 2p + 2 \leq a_1 + a_3$ as $a_2 \leq p - 1$.

4.) Let $(x_1, x_2) = (2p - m_1 - 2, 2p - m_2 - 2)$. Then $c_4 > 0$ if $p \leq m_1 < 2p - 1$ and $p \leq m_2 < 2p - 1$. We have $x_1 + x_2 = 4p - 4 - m_1 - m_2 \geq 4p - 4 - a_1 - 2a_2 - a_3 \geq a_1 - a_3$ as $a_1$ and $a_2 \leq p - 1$. Since $x_1 - x_2 = m_2 - m_1$, we get $-a_1 - a_3 \leq x_1 - x_2 \leq a_1 + a_3$.

5.) Suppose that $(x_1, x_2) = (4p - m_1 - 2, 2p - m_2 - 2)$. The coefficient $c_5 > 0$ if $2p \leq m_1 < 3p - 1$ and $p \leq m_2 < 2p - 1$. Then $x_1 + x_2 = 6p - 4 - m_1 - m_2 \geq 6p - 4 - a_1 - 2a_2 - a_3 \geq a_1 - a_3$ because $a_1$ and $a_2 \leq p - 1$. We have $x_1 - x_2 = 2p - m_1 + m_2 \geq 2p - a_1 - a_3 \geq -a_1 - a_3$ and $x_1 - x_2 \leq m_1 - x_2 = m_1 + m_2 - 2p + 2 \leq a_1 + 2a_2 + a_3 - 2p + 2 \leq a_1 + a_3$ since $a_2 \leq p + 1$.

6.) Put $(x_1, x_2) = (m_1, 4p - m_2 - 2)$. We have $c_6 > 0$ if $2p \leq m_2 < 3p - 1$. Then $x_1 + x_2 = 4p - 2 + m_1 - m_2 \geq 4p - 2 - a_1 - a_3 \geq a_1 - a_3$ since $a_1 \leq p - 1$, $x_1 - x_2 = m_1 - x_2 \geq m_1 - m_2 \geq -a_1 - a_3$, and $x_1 - x_2 = m_1 + m_2 - 4p + 2 \leq a_1 + 2a_2 + a_3 - 4p + 2 \leq a_1 + a_3$ as $a_2 \leq p - 1$.

7.) Let $(x_1, x_2) = (2p - m_1 - 2, 4p - m_2 - 2)$. Then $c_7 > 0$ if $p \leq m_1 < 2p - 1$ and $2p \leq m_2 < 3p - 1$. We have $x_1 + x_2 = 6p - 4 - m_1 - m_2 \geq 6p - 4 - a_1 - 2a_2 - a_3 \geq a_1 - a_3$ since $a_1$ and $a_2 \leq p - 1$, $x_1 - x_2 \geq x_1 - m_2 = 2p - 2 - m_1 - m_2 \geq 2p - 2 - a_1 - 2a_2 - a_3 \geq -a_1 - a_3$ since $a_2 \leq p + 1$, and $x_1 - x_2 = -2p - m_1 + m_2 \leq a_1 + a_3 - 2p \leq a_1 + a_3$.

8.) Finally, let $(x_1, x_2) = (4p - m_1 - 2, 4p - m_2 - 2)$. Then $c_8 > 0$ if $2p \leq m_1 < 3p - 1$ and $2p \leq m_2 < 3p - 1$. We have $x_1 + x_2 = 8p - 4 - m_1 - m_2 \geq 8p - 4 - a_1 - 2a_2 - a_3 \geq a_1 - a_3$ as $a_1$ and $a_2 \leq p - 1$. Since $x_1 - x_2 = m_2 - m_1$, we obtain $-a_1 - a_3 \leq x_1 - x_2 \leq a_1 + a_3$.

Hence for the $p$-restricted highest weight $\omega$, $(x_1, x_2) \in \mathrm{Irr}(L(\omega)_{\mathbb{C}} | H_{\mathbb{C}})$.

Now suppose that $\omega$ is not $p$-restricted. Then $\omega = \omega^0 + p\omega^1 + \ldots + p^k \omega^k$, where all $\omega^j = a_1^j \omega_1 + a_2^j \omega_2 + a_3^j \omega_3$ are $p$-restricted. Therefore $\Delta(\omega) \subset \otimes_{j=0}^{k} p^j \Delta(\omega^j)$ (here $p^j \Delta(\omega^j)$ means a tensor product of $p^j$ copies of $\Delta(\omega^j)$). Hence $\Delta(\omega)|H \subset \otimes_{j=0}^{k} p^j \Delta(\omega^j)|H$. For any factor $(x_1^j, x_2^j) \in \mathrm{Irr}(\Delta(\omega^j)|H)$, $a_1^j - a_3^j$ and $a_3^j - a_1^j \leq x_1^j + x_2^j$ and $-a_1^j - a_3^j \leq x_1^j - x_2^j \leq a_1^j + a_3^j$. Consequently, for $(x_1, x_2) \in \mathrm{Irr}(\Delta(\omega)|H)$, $a_1 - a_3$ and $a_3 - a_1 \le x_1 + x_2$ and $-a_1 - a_3 \le x_1 - x_2 \le a_1 + a_3$ as well. The lemma is proved.

**3. Auxiliary lemmas.**

**Lemma 4.** *Let $r \ge 3$, $a_1 < p$, and $a_r < p$. Then*

$$T = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1, x_2 \le m,\ x_1 + x_2 = m'\} \subset \mathrm{Irr}(M(\omega)|H)$$

*and for any weight from $T$ there exists a primitive with respect to $H$ vector of such weight.*

**Proof.** First let $r = 3$. We can suppose that $H = G(1,3)$. The vector $v_d = v(2,3,d)$ with $0 \le d \le a_3$ is primitive with respect to $H$, $\omega(v_d) = \omega - (a_2 + d)\alpha_2 - d\alpha_3$, and $\omega_H(v_d) = (a_1 + a_2 + d, a_2 + a_3 - d)$. Similarly, $u_e = v(2,1,e)$ with $0 \le e \le a_1$ is primitive with respect to $H$, $\omega(u_e) = \omega - e\alpha_1 - (e + a_2)\alpha_2$, and $\omega_H(u_e) = (a_1 + a_2 - e, a_2 + a_3 + e)$. Thus, for $r = 3$ the lemma is proved.

If $r > 3$, choose $\Gamma = G(\alpha_1, \alpha_2 + \ldots + \alpha_{r-1}, \alpha_r) \cong A_3(K)$. Then

$$M = K\Gamma v^+ = M(a_1\omega_1 + (a_2 + \ldots + a_{r-1})\omega_2 + a_r\omega_3)$$

is a submodule of $M(\omega)|\Gamma$ and its highest weight satisfies the assertion of the lemma. Restricting $M$ to $H$, we get the required primitive vectors.

**Lemma 5.** *Let $r \ge 4$ and $a_i < p$ for $i \in \{1, r-1, r\}$. Suppose that $a_1 \ge a_r$. Then*

$$S \subset \mathrm{Irr}(M(\omega)|H)$$

*with*

$$S = \{(x_1, x_2) \in \mathbb{N}^2 \mid m - a_1 - a_{r-1} - a_r \le x_1, x_2 \le m,\ m' - a_{r-1} - a_r \le x_1 + x_2 \le m'\}.$$

**Proof.** First assume that $r = 4$. Set $\Gamma_1 = G(1,2,3)$, $\lambda_1 = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$, and $H = G(1,3) \subset \Gamma_1$. Then $M_1 = K\Gamma_1 v^+ = M(\lambda_1)$ is a submodule of the restriction $M(\omega)|\Gamma_1$. By Lemma 4, $w_j \in M(\lambda_1)$, where $w_j$ is primitive with respect to $H$ and

$$\omega_H(w_j) = (a_2 + j, a_1 + a_2 + a_3 - j),$$

$0 \le j \le a_1 + a_3$. Moreover, $\omega_{G(4)}(w_j) = a_4$ if $\omega_{G(3)}(w_j) \ge a_2 + a_3$ and $\omega_{G(4)}(w_j) = a_4 + k$ if $\omega_{G(3)}(w_j) = a_2 + a_3 - k$, $0 \le k \le a_3$. Denote $\omega_{G(4)}(w_j)$ by $d$.

By [6, Theorem A(i)] applied to the group $G(3,4)$ and the module $KG(3,4)v^+$, $\Delta(a_4 + k) = KG(4)X_{-3,k}v^+$ for $0 \le k \le a_3$. Hence the vectors $w'_j = X_{-4,a}w_j$ with $0 \le a \le d$ are primitive with respect to $H$ in all cases and

$$\omega_H(w'_j) = (a_2 + j, a_1 + a_2 + a_3 - j + a).$$

Therefore

$$\mathrm{Irr}(M(\omega)|H) \supset S_1 = \{(x_1, x_2) \in \mathbb{N}^2 \mid a_2 \le x_1 \le a_1 + a_2,\ a_1 + 2a_2 + a_3 \le x_1 + x_2 \le a_1 + 2a_2 + a_3 + a_4\}$$

and

$$\mathrm{Irr}(M(\omega)|H) \supset S_2 = \{(x_1, x_2) \in \mathbb{N}^2 \mid a + 1 + a_2 \le x_1 \le a_1 + a_2 + a_3,\ x_2 \le a_2 + a_3 + a_4,\ a_1 + 2a_2 + a_3 \le x_1 + x_2\}.$$

By Lemma 1,

$$\mathrm{Irr}(M(\omega)|H) \supset S_3 = \{(x_1, x_2) \in \mathbb{N}^2 \mid a_2 \le x_2 \le a_1 + a_2,\ a_1 + 2a_2 + a_3 \le x_1 + x_2 \le a_1 + 2a_2 + a_3 + a_4\}$$

and

$$\mathrm{Irr}(M(\omega)|H) \supset S_4 = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1 \le a_2 + a_3 + a_4,\\ a + 1 + a_2 \le x_2 \le a_1 + a_2 + a_3,\\ a_1 + 2a_2 + a_3 \le x_1 + x_2 \le a_1 + 2a_2 + a_3 + a_4\}.$$

Put

$$S' = \{(x_1, x_2) \in \mathbb{N}^2 \mid a_2 \le x_1, x_2 \le a_1 + a_2 + a_3 + a_4,\\ a_1 + 2a_2 + a_3 \le x_1 + x_2 \le a_1 + 2a_2 + a_3 + a_4\}.$$

If $a_1 + a_3 \ge a_4$, then $S' \subset S_1 \cup S_2 \cup S_3 \cup S_4 \subset \mathrm{Irr}(M(\omega)|H)$.

To find remaining factors one can suppose that $H = G(1,4)$. Put $G_1 = G(1,2)$ and $v_1 = v(3,1,d_1)$ with $0 \le d_1 \le a_1$. The vector $v_1$ is primitive with respect to $G_1$ and

$$\omega_{G_1}(v_1) = (a_1 + a_2 - d_1)\omega_1 + a_3\omega_2.$$

For $0 \le s_3 \le a_3$ and a subgroup $G_1$ put $w_1 = X_{-2,s_3}v_1$. The vector $w_1$ is primitive with respect to $H$ and

$$\omega_H(w_1) = (a_1 + a_2 - d_1 + s_3, d_1 + a_2 + a_3 + a_4).$$

Hence $\mathrm{Irr}(M(\omega)|H) \supset T_1$ with

$$T_1 = \{(x_1, x_2) \in \mathbb{N}^2 \mid a_2 + a_3 + a_4 \le x_2 \le a_1 + a_2 + a_3 + a_4,\\ a_1 + 2a_2 + a_3 + a_4 \le x_1 + x_2 \le a_1 + 2a_2 + 2a_3 + a_4\}.$$

It follows from Lemma 1 that $\mathrm{Irr}(M(\omega)|H) \supset T_2$ with

$$T_2 = \{(x_1, x_2) \in \mathbb{N}^2 \mid a_2 + a_3 + a_4 \le x_1 \le a_1 + a_2 + a_3 + a_4,\\ a_1 + 2a_2 + a_3 + a_4 \le x_1 + x_2 \le a_1 + 2a_2 + 2a_3 + a_4\}.$$

We get $\mathrm{Irr}(M(\omega)|H) \supset T \setminus T' = T_1 \cup T_2$ with

$$T = \{(x_1, x_2) \in \mathbb{N}^2 \mid a_2 \le x_1, x_2 \le a_1 + a_2 + a_3 + a_4,\\ a_1 + 2a_2 + a_3 + a_4 \le x_1 + x_2 \le a_1 + 2a_2 + 2a_3 + a_4\}$$

and

$$T' = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1, x_2 \le a_2 + a_3 + a_4,\\ a_1 + 2a_2 + a_3 + a_4 \le x_1 + x_2 \le a_1 + 2a_2 + 2a_3 + a_4\}.$$

As $a_1 \ge a_4$, we get $T' \subset S_2$. Consequently, $T \subset \mathrm{Irr}(M(\omega)|H)$. Now, since $S = S' \cup T$, we get the required for $r = 4$.

If $r > 4$, take $\Gamma = G(\alpha_1, \alpha_2 + \ldots + \alpha_{r-2}, \alpha_{r-1}, \alpha_r) \cong A_4(K)$. Then

$$M = K\Gamma v^+ = M(a_1\omega_1 + (a_2 + \ldots + a_{r-2})\omega_2 + a_{r-1}\omega_3 + a_r\omega_4)$$

is a submodule of $M(\omega)|\Gamma$ and its highest weight satisfies the assertion of the lemma. Restricting $M$ to $H$, we obtain the required factors.

For an algebraic group $\Gamma$ and a $\Gamma$-module $M$ put $m(\lambda) = \dim M^\lambda$.

**Lemma 6.** *Let $\Gamma = A_2(K)$, $\mu = b_1\omega_1 + b_2\omega_2$ be a dominant weight of $\Gamma$, and $b_1 < p$. Then there exist primitive with respect to $\Gamma(1)$ vectors $v(k_1, k_2) \in \Delta(\mu)$, $0 \le k_1 \le b_1$, $0 \le k_2 \le b_2$, with weights $\omega(v(k_1, k_2)) = \mu - k_1\alpha_1 - (k_1 + k_2)\alpha_2$.*

**Proof.** For any weight $\lambda$ of $\Gamma$, $m(\lambda) = \dim L(\mu)^\lambda_{\mathbb{C}}$ since the characters of modules $\Delta(\mu)$ and $L(\mu)_{\mathbb{C}}$ coincide. It follows from [1, Ch. VIII, §7, Proposition 10] that

$$X(L(\mu)_{\mathbb{C}}) = b_1 X(L(\omega_1)_{\mathbb{C}}) + b_2 X(L(\omega_2)_{\mathbb{C}})$$

(the sum of $b_1$ copies of $X(L(\omega_1)_{\mathbb{C}})$ and $b_2$ copies of $X(L(\omega_2)_{\mathbb{C}})$. By [1, Ch. VIII, §13.1],

$$X(L(\omega_1)_{\mathbb{C}}) = \{\omega_1, \omega_1 - \alpha_1, \omega_1 - \alpha_1 - \alpha_2\}$$

and

$$X(L(\omega_2)_{\mathbb{C}}) = \{\omega_2, \omega_2 - \alpha_2, \omega_2 - \alpha_1 - \alpha_2\}.$$

Hence

$$X(\Delta(\mu)) = X(L(\mu)_{\mathbb{C}}) = \{\mu - r_1\omega_1 - r_2\omega_2 \mid 0 \le r_1, r_2 \le b_1 + b_2,\ -b_1 \le r_2 - r_1 \le b_2\}.$$

Applying the formula for weight multiplicities [1, Ch. VIII, §9.3], we get that for $\lambda \in X(\Delta(\mu))$

$$m(\lambda) = m(\lambda + \alpha_1) + m(\lambda + \alpha_2) - m(\lambda + 2\alpha_1 + \alpha_2) - m(\lambda + \alpha_1 + 2\alpha_2) + m(\lambda + 2\alpha_1 + 2\alpha_2).$$

One can easily deduce that

$$m(\lambda) = \begin{cases} \min(r_1, r_2) + 1 & \text{if } 0 \le r_1 \le b_1,\ 0 \le r_2 \le b_2, \\ \min(r_1, r_2) - \min(r_1 - b_1 - 1, r_2) & \text{if } r_1 > b_1,\ 0 \le r_2 \le b_2,\ r_1 - r_2 \le b_1, \\ \min(r_1, r_2) - \min(r_1, r_2 - b_2 - 1) & \text{if } 0 \le r_1 \le b_1,\ r_2 > b_2,\ r_2 - r_1 \le b_2, \\ \min(r_1, r_2) - \min(r_1 - b_1 - 1, r_2) - & \\ - \min(r_1, r_2 - b_2 - 1) - 1 & \text{if } b_1 < r_1 \le b_1 + b_2,\ b_2 < r_2 \le b_1 + b_2. \end{cases}$$

The latter formula yields that $m(\lambda) = m(\lambda + \alpha_1) + 1$ if and only if $0 \le r_1 \le b_1$ and $0 \le r_2 - r_1 \le b_2$. Hence weights of the form $\mu - k_1\alpha_1 - (k_1 + k_2)\alpha_2 \in X(\Delta(\mu))$ for $0 \le k_1 \le b_1$, $0 \le k_2 \le b_2$ and there exist weight vectors $v(k_1, k_2) \in \Delta(\mu)$ with these weights such that $X_1 v(k_1, k_2) = 0$. Using formula from [7, §3] for $x_1(t)$ and the fact that $b_1 < p$, we get that $v(k_1, k_2)$ are primitive with respect to $\Gamma(1)$.

**Lemma 7.** *Let $\mu = m_1\omega_1 + \ldots + m_n\omega_n$ be a $p$-restricted dominant weight of $A_n(K)$, $n \ge 3$, and $S \subset A_n(K)$ be a subsystem subgroup of type $A_1$. Then $M(\mu)$ contains primitive with respect to $S$ vectors with weights $i$, $0 \le i \le m_1 + m_2 + \ldots + m_n$.*

**Proof.** Put $S = G(n)$ and $v = v(n-1, k, d_k)$ with $1 \le k \le n-1$, $0 \le d_k \le m_k$. Then

$$\omega_S(v) = d_k + m_{k+1} + \ldots + m_n.$$

Therefore it remains to get primitive vectors with weights $d_n$, where $0 \le d_n < m_n$.

If $m_{n-1} + m_n \le p - 2$, or $m_{n-1} = p - 1$, or $m_n = p - 1$, then by [3, Lemma 3.1], $KG(n-1, n)v^+ = L(m_{n-1}\omega_1 + m_n\omega_2) = \Delta(m_{n-1}\omega_1 + m_n\omega_2)$. Lemma 6 implies that in this module one can find primitive with respect to $G(n)$ vectors $v(d_n) \in M(\mu)^{\mu - d_n(\alpha_{n-1} + \alpha_n)}$ with weights $m_n - d_n$.

Let $m_1 + \ldots + m_{n-2} + m_{n-1} \ge p - 1$. Put $w = v(n-2, j, d)$ with $1 \le j \le n-2$ and $d + m_{j+1} + \ldots + m_{n-1} = p - 1$. By [3, Lemma 3.1], $KG(n-1, n)w = L((p-1)\omega_1 + m_n\omega_2) = \Delta((p-1)\omega_1 + m_n\omega_2)$. As above Lemma 6 implies that in such module there exist primitive with respect to $G(n)$ vectors with weight $m_n - d_n$.

Finally suppose that $m_1 + \ldots + m_{n-2} + m_{n-1} < p - 1$, $m_n < p - 1$, and $m_{n-1} + m_n \ge p - 1$. Choose $1 \le b \le m_{n-1}$ such that $b + m_n = p - 1$ and put $u = v(n-2, n-1, b)$. Then $u$ is primitive with respect to $G(n-1, n)$ and $KG(n-1, n)u = L((m_{n-2} + m_{n-1} - b)\omega_1 + (p-1)\omega_2) =$

$\Delta((m_{n-2}+m_{n-1}-b)\omega_1+(p-1)\omega_2)$ as above. Applying Lemma 6, we get primitive with respect to $G(n)$ vectors with required weights. The lemma is proved.

**Lemma 8.** *Let* $r\geq 5$ *and let* $a_i<p$ *for* $2\leq i\leq r-1$. *Then*

$$R\cup R'\subset \mathrm{Irr}(M(\omega)|H),$$

*where*

$$R=\{(x_1,x_2)\in\mathbb{N}^2\mid x_1=m,\ x_2\leq m'-m\},$$
$$R'=\{(x_1,x_2)\in\mathbb{N}^2\mid x_1\leq m'-m,\ x_2=m\},$$

*and for any weight from* $R\cup R'$ *there exists a primitive with respect to* $H$ *vector of such weight.*

**Proof.** We can assume that $H=G(1)\times G(n)$. Take a vector $v=v(2,r,a_r)$. It is primitive with respect to $G_1=G(1)\times G(3,\dots,r)$ and its weight with respect to this subgroup equals $(m,a_2\omega_1+a_3\omega_2+\ldots+a_{r-1}\omega_{r-2})$. Applying Lemma 7 to the group $G(3,\dots,r)$ and $\mu=a_2\omega_1+a_3\omega_2+\ldots+a_{r-1}\omega_{r-2}$, we get primitive vectors for all weights from $R$.

Similarly, one can get primitive vectors with weights from $R'$.

**Lemma 9.** *Let* $r=4$ *and let* $a_2$, $a_3<p$. *Then*

$$R_1\cup R_1'\subset \mathrm{Irr}(M(\omega)|H),$$

*where*

$$R_1=\{(x_1,x_2)\in\mathbb{N}^2\mid x_1=m,\ \min(a_2,a_3)\leq x_2\leq a_2+a_3\},$$
$$R_1'=\{(x_1,x_2)\in\mathbb{N}^2\mid \min(a_2,a_3)\leq x_1\leq a_2+a_3,\ x_2=m\},$$

*and for any weight from* $R\cup R'$ *there exists a primitive with respect to* $H$ *vector of such weight.*

**Proof.** The arguments are similar to the same of Lemma 8.

First assume that $H=G(1)\times G(4)$ and take a vector $v=v(2,4,a_4)$. It is primitive with respect to $G_1=G(1)\times G(3,4)$ and its weight with respect to this subgroup equals $(m,a_2\omega_1+a_3\omega_2)$. Now for $0\leq d\leq a_2$ put $w_d=X_{-3,d}v$. Such vectors are primitive with respect to $H$ as well, and $\omega_H(w_d)=(m,a_3+d)$. If we assume that $H=G(1)\times G(3)$ and for $0\leq c\leq a_3$ put $u_c=X_{-4,d}v$, then $\omega_H(u_c)=(m,a_2+c)$. Hence $R_1\subset\mathrm{Irr}(M(\omega)|H)$. Similarly, $R_1'\subset\mathrm{Irr}(M(\omega)|H)$.

**4. Proof of the main theorem.**

**Lemma 10.** *Let* $r\geq 5$ *and* $\omega$ *be* $p$*-restricted. Suppose that* $a_1\geq a_r$. *Then*

$$S(\omega)\setminus M\subset \mathrm{Irr}(M(\omega)|H),$$

*where*

$$M=\{(x_1,x_2)\in\mathbb{N}^2\mid x_1,x_2<m-a_r,\ x_1+x_2<m'-a_{r-1}-a_r\}.$$

**Proof.** Set $\Gamma=G(1,2,\dots,r-1)$, $\lambda=a_1\omega_1+a_2\omega_2+\ldots+a_{r-1}\omega_{r-1}$, and $H=G(1)\times G(r-1)\subset\Gamma$. Then $L=K\Gamma_1v^+=M(\lambda)$ is a submodule of the restriction $M(\omega)|\Gamma$.

1) First suppose that $r\geq 6$. By Lemma 8, $w_j\in M(\lambda)$, where $w_j$ are primitive with respect to $H$ and

$$\omega_H(w_j)=(j,m-a_r),$$

$0\leq j\leq a_2+\ldots+a_{r-2}$. It follows from the proof of Lemma 8 that one can choose $w_j$ so that $\omega_{G(r)}(w_j)=a_r$. Hence the vectors $w_j'=X_{-r,a}w_j$ with $0\leq a\leq a_r$ are nonzero and primitive with respect to $H$ and

$$\omega_H(w_j')=(j,m-a_r+a).$$

Therefore

$$\mathrm{Irr}(M(\omega)|H)\supset M_1=\{(x_1,x_2)\in\mathbb{N}^2\mid x_1\leq m-a_1-a_{r-1}-a_r,\\ m-a_r\leq x_2\leq m\}.$$

By Lemma 1 we obtain that

$$\mathrm{Irr}(M(\omega)|H) \supset M_2 = \{(x_1,x_2) \in \mathbb{N}^2 \mid m - a_r \le x_1 \le m, \\ x_2 \le m - a_1 - a_{r-1} - a_r\}.$$

Now, applying Lemma 5, we deduce that $M_1 \cup M_2 \cup S = S(\omega) \setminus M \subset \mathrm{Irr}(M(\omega)|H)$.

2) Let $r = 5$. Put $\Gamma' = G(1) \times G(3,4,5)$ and $v_d = v(2,5,d)$ with $0 \le d \le a_5$. Then

$$\omega_{\Gamma'}(v_d) = (a_1 + a_2 + a_3 + a_4 + d, a_2\omega_1 + a_3\omega_2 + (a_4 + a_5 - d)\omega_3).$$

If $a_4 + a_5 - d < p$, then, applying Theorem 1.2 from [3], we get

$$\omega_H(v_d) = (a_1 + a_2 + a_3 + a_4 + d, k)$$

with $0 \le k \le a_2 + a_3 + a_4 + a_5 - d$. For $a_4 + a_5 - d \ge p$ using Theorems 1.2 and 1.3 from [3], we obtain the same weights with $0 \le k \le a_2+a_3+a_4+a_5-d-p$. Since $a_2+a_3+a_4+a_5-d \ge a_2+a_3$ and $a_2+a_3+a_4+a_5-d-p \ge a_2+a_3$, in both cases we get $\mathrm{Irr}(M(\omega)|H) \supset M_2$. Lemma 1 implies that $\mathrm{Irr}(M(\omega)|H) \supset M_1$. As in Item 1, one can get from Lemma 5 that $M_1 \cup M_2 \cup S = S(\omega) \setminus M \subset \mathrm{Irr}(M(\omega)|H)$.

Now recall a series of subgroups $G = P_r \supset P_{r-1} \supset \ldots \supset P_5 \supset P_4 \supset P_3$ from Introduction. We have $P_5 = G(i, i+1, i+2, i+3, i+4) \cong A_5(K)$ for some $1 \le i \le r-4$, $P_4 = G(j, j+1, j+2, j+2)$ with $j = i$ or $i+1$, and $P_3 = G(k, k+1, k+2)$ with $k = j$ or $j+1$. Fix such $i, j, k$.

**Lemma 11.**

*(i) Let $r = 5$, $\omega$ be $p$-restricted and $k = 1$ or $3$. Then*

$$S(\omega) \setminus M' \subset \mathrm{Irr}(M(\omega)|H),$$

*where*

$$M' = \{(x_1,x_2) \in \mathbb{N}^2 \mid x_1, x_2 < a_k + a_{k+1} + a_{k+2},\ x_1 + x_2 < a_k + 2a_{k+1} + a_{k+2}\}.$$

*(ii) Let $r = 6$, $\omega$ be $p$-restricted and $k = j+1$. Then*

$$S(\omega) \setminus M' \subset \mathrm{Irr}(M(\omega)|H).$$

**Proof.** (i) We can assume that $a_1 \ge a_5$ and $j = 1$. This implies $k = 1$ and $a_1 \ge a_4$. By Lemma 10, $S(\omega) \setminus M \subset \mathrm{Irr}(M(\omega)|H)$, where

$$M = \{(x_1,x_2) \in \mathbb{N}^2 \mid x_1, x_2 < a_1 + a_2 + a_3 + a_4, \\ x_1 + x_2 < a_1 + 2a_2 + 2a_3 + a_4\}.$$

Using Lemma 5 for $r = 4$, we get $N' \subset \mathrm{Irr}(M(\omega)|H)$ with

$$N' = \{(x_1,x_2) \in \mathbb{N}^2 \mid a_{k+1} \le x_1, x_2 < a_1 + a_2 + a_3 + a_4, \\ a_k + 2a_{k+1} + a_{k+2} \le x_1 + x_2 < a_1 + 2a_2 + 2a_3 + a_4\}.$$

Put $\Gamma = G(1) \times G(3,4,5)$, $H = G(1) \times G(4) \subset \Gamma$, and $v_d = v(2,4,d)$ with $0 \le d \le a_4$. Then

$$\omega_\Gamma(v_d) = (a_1 + a_2 + a_3 + d, a_2\omega_1 + (a_3 + a_4 - d)\omega_2 + (a_5 + d)\omega_3).$$

If $a_3 + a_4 - d < p$ and $a_5 + d < p$, then, applying Theorem 1.2 from [3], we get

$$\omega_H(v_d) = (a_1 + a_2 + a_3 + d, k)$$

with $0 \leq k \leq a_2+a_3+a_4+a_5$. For $a_3+a_4-d \geq p$ and $a_5+d < p$ or $a_3+a_4-d < p$ and $a_5+d \geq p$ using Theorems 1.2 and 1.3 from [3], we obtain the same weights with $0 \leq k \leq a_2+a_3+a_4+a_5-p$. For $a_3+a_4-d \geq p$ and $a_5+d \geq p$ the same theorems imply that $0 \leq k \leq a_2+a_3+a_4+a_5-2p$. In all cases $0 \leq k \leq a_2$ are present. Hence

$$\operatorname{Irr}(M(\omega)|H) \supset M_1 = \{(x_1, x_2) \in \mathbb{N}^2 \mid a_1+a_2+a_3 \leq x_1 \leq a_1+a_2+a_3+a_4, \\ x_2 \leq a_2\}.$$

Lemma 1 implies that

$$\operatorname{Irr}(M(\omega)|H) \supset M_2 = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1 \leq a_2, \\ a_1+a_2+a_3 \leq x_2 \leq a_1+a_2+a_3+a_4\}.$$

Now $M \cup N' \cup M_1 \cup M_2 = S(\omega) \setminus M' \subset \operatorname{Irr}(M(\omega)|H)$.

(ii) Here we can suppose that $a_1 \geq a_6$. If $j = 2$, then $k = 3$ and the assertion follows from Item (i) and Lemma 10. Hence it suffices to consider the case $j = 1$ and $k = 2$. By Lemma 10 for $r = 6$ and $5$, $S(\omega) \setminus M \subset \operatorname{Irr}(M(\omega)|H)$ and by Lemma 5 for $r = 4$, $N' \subset \operatorname{Irr}(M(\omega)|H)$.

Put $\Gamma' = G(2) \times G(4,5,6)$, $H = G(2) \times G(5) \subset \Gamma'$, and $u_b = X_{-1,b}v(3,4,a_4)$ with $0 \leq b \leq a_1$. Then

$$\omega_{\Gamma'}(u_b) = (b+a_2+a_3+a_4, a_3\omega_1 + (a_4+a_5)\omega_2 + a_6\omega_3).$$

If $a_4+a_5 < p$, then by [3, Theorem 1.2]

$$\omega_H(u_b) = (b+a_2+a_3+a_4, k)$$

with $0 \leq k \leq a_3+a_4+a_5+a_6$. For $a_4+a_5 \geq p$, using Theorems 1.2 and 1.3 from [3], we obtain the same weights with $0 \leq k \leq a_3+a_4+a_5+a_6-p$. In both cases $0 \leq k \leq a_3$ are present. Hence

$$\operatorname{Irr}(M(\omega)|H) \supset N_1 = \{(x_1, x_2) \in \mathbb{N}^2 \mid a_2+a_3+a_4 \leq x_1 \leq a_1+a_2+a_3+a_4, \\ x_2 \leq a_3\}.$$

Lemma 1 implies that

$$\operatorname{Irr}(M(\omega)|H) \supset N_2 = \{(x_1, x_2) \in \mathbb{N}^2 \mid x_1 \leq a_3, \\ a_2+a_3+a_4 \leq x_2 \leq a_1+a_2+a_3+a_4\}.$$

Now $M \cup N' \cup N_1 \cup N_2 = S(\omega) \setminus M' \subset \operatorname{Irr}(M(\omega)|H)$.

**Proof of Theorem 2.** The assertion follows from Lemmas 10 and 11.

The condition that $\omega$ is $p$-restricted is essential in the proof of Theorem 2.

**Lemma 12.** *Let $r \geq 3$ and $\omega = \lambda_0 + p\lambda_1 + \ldots + p^k\lambda_k$, where all $\lambda_s = a_{1,s}\omega_1 + \ldots + a_{r,s}\omega_r$ are $p$-restricted. Then $(x_1, x_2)$ with $x_1 + x_2 = m'$ lies in $\operatorname{Irr}(L(\omega)|H)$ if and only if $(x_1, x_2) = (m'-m+i, m-i)$ and $i$ can be represented in a form $i = i_0 + i_1p + \ldots + i_kp^k$, $0 \leq i_s \leq a_{1,s} + a_{r,s}$.*

**Proof.** By the Steinberg tensor product theorem [7, Theorem 41], $L(\omega) = L(\lambda_0) \otimes L(\lambda_1)^p \otimes \ldots \otimes L(\lambda_k)^{p^k}$. Here $L(\lambda_s)^{p^s}$ denotes the module obtained from $L(\lambda_s)$ with the help of the field morphism of $G$ associated with raising the elements of $K$ to the power $p^s$. It follows from Lemmas 3 and 4 that $(x_{1,s}, x_{2,s})$ with $x_{1,s} + x_{2,s} = a_{1,s} + 2a_{2,s} + \ldots + 2a_{r-1,s} + a_{r,s}$ lies in $\operatorname{Irr}(L(\lambda_s)|H$ if and only if $(x_{1,s}, x_{2,s}) = (a_{2,s} + \ldots + a_{r-1,s} + i_s, a_{1,s} + \ldots + a_{r,s} - i_s) \in \operatorname{Irr}(L(\lambda_s)|H$ with $0 \leq i_s \leq a_{1,s} + a_{r,s}$.

Observe that for small $r$ the composition factors of $L(\omega)|H$ may not coincide with characteristic 0.

**Example.** Let $r \geq 3$ and $\omega = p\omega_1$. Observe that for $r = 3$ it follows from Lemmas 3 and 4 that $\operatorname{Irr}(L(\omega_1)|H) = \{(1,0),(0,1)\}$. If $r > 3$, then $(0,0) \in \operatorname{Irr}(L(\omega_1)|H)$ since $\omega_{G(2)\times G(4)}(v^+) = (0,0)$. Now Lemmas 3 and 4 imply that

$$\operatorname{Irr}(L(\omega_1)|H) = \{(0,0),(1,0),(0,1)\}$$

in this case. By the Steinberg tensor product theorem, $L(\omega) = L(0) \otimes L(\omega_1)^p$. Restricting this module to $H$, we obtain that for $r = 3$

$$\mathrm{Irr}(L(p\omega_1)|H) = \{(p,0),(0,p)\}$$

and for $r > 3$

$$\mathrm{Irr}(L(p\omega_1)|H) = \{(0,0),(p,0),(0,p)\}.$$

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**A. A. Osinovskaya**

**The restrictions of representations of special linear groups to subsystem subgroups of type $A_1 \times A_1$**


## Summary

The restrictions of irreducible representations of the special linear group over an algebraically closed field of positive characteristic $p$ to subsystem subgroups of type $A_1 \times A_1$ are studied. The symmetry of the set of highest weights of composition factors is proven. The "big" composition factors of such restrictions for the arbitrary $p$-restricted representations are found. These results will be used for the complete description of such restrictions.