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New Cases of Logarithmic Equivalence of Welschinger and Gromov–Witten Invariants[1]

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To Vladimir Igorevich Arnold
on his 70th birthday

We consider $\mathbb{P}^1 \times \mathbb{P}^1$ equipped with the complex conjugation $(x, y) \mapsto (\bar{y}, \bar{x})$ and blown up in at most two real or two complex conjugate points. For these four surfaces we prove the logarithmic equivalence of Welschinger and Gromov–Witten invariants.

1. INTRODUCTION

The Welschinger invariants [9, 10] applied to unnodal Del Pezzo surfaces bound from below the number of real rational curves in a given linear system that pass through a real generic collection of points. In our previous papers [1, 2], using the methods of tropical enumerative geometry developed by G. Mikhalkin [4, 5] and E. Shustin [6, 7], we studied the toric unnodal Del Pezzo surfaces with tautological real structure and showed that for these surfaces Welschinger and Gromov–Witten invariants are equivalent in the logarithmic scale if all or almost all fixed points in the generic collection are real. Here we continue such an asymptotic study of Welschinger invariants and consider non-tautological real structures on toric unnodal Del Pezzo surfaces. Up to isomorphisms respecting the real structure, there are only five toric unnodal Del Pezzo surfaces with a non-tautological real structure and nonempty real part. One is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ equipped with the standard (tautological) complex conjugation by blowing up two complex conjugate points, and the four others are obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ equipped with the complex conjugation $(x, y) \mapsto (\bar{y}, \bar{x})$ by blowing up at most two real or two complex conjugate points.

We look at collections of real points on any of the four latter surfaces, apply the tropical formula elaborated in [8] to the multiples nD of a real ample divisor D on such a surface Σ, and prove that the Welschinger and Gromov–Witten invariants, $W_{\Sigma,nD}$ and $GW_{\Sigma,nD}$, are equivalent in the logarithmic scale: $\log W_{\Sigma,nD} = \log GW_{\Sigma,nD} + O(n)$. Recall that, as is shown in [2, 3], $\log GW_{\Sigma,nD} = (c_1(\Sigma) \cdot D)\, n \log n + O(n)$.

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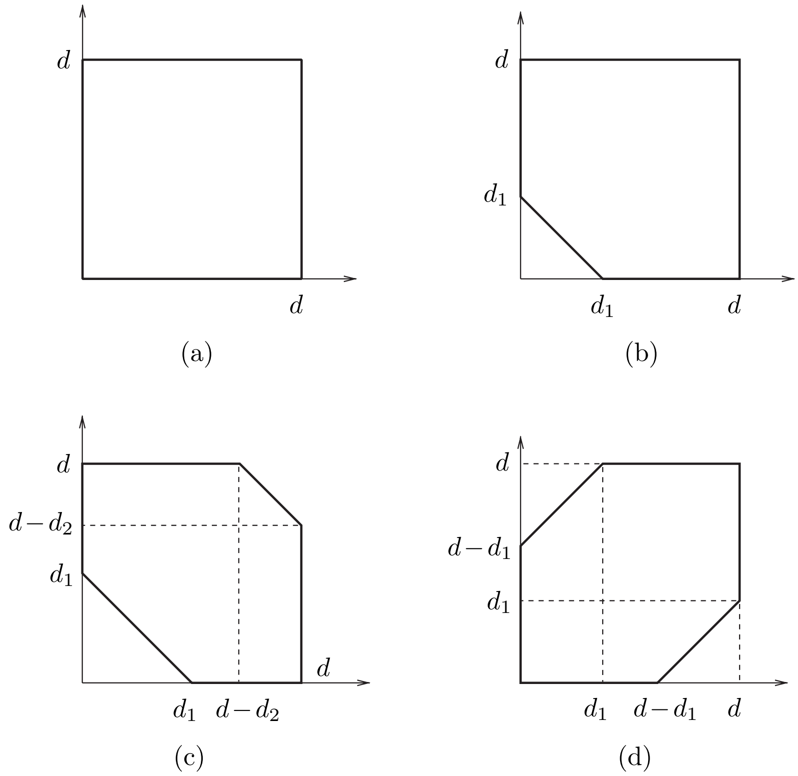


Fig. 1. Polygons defining $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, and $\mathbb{S}^2_{0,2}$

2. COMBINATORIAL BOUND

As toric surfaces, the four real Del Pezzo surfaces, $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, and $\mathbb{S}^2_{0,2}$, that we deal with are associated with the following convex lattice polygons in $\mathbb{R}^2$ (see Fig. 1):

- squares with vertices $(0,0)$, $(d,0)$, $(0,d)$, (d,d), where $d \geq 1$;
- pentagons with vertices (d,d), $(0,d)$, $(0,d_1)$, $(d_1,0)$, $(d,0)$, where $1 \leq d_1 < d$;
- hexagons with vertices $(0,d_1)$, $(d_1,0)$, $(d,0)$, $(d,d-d_2)$, $(d-d_2,d)$, $(0,d)$, where $1 \leq d_2 \leq d_1 < d$;
- and hexagons with vertices $(0,0)$, $(d-d_1,0)$, (d,d_1), (d,d), (d_1,d), $(0,d-d_1)$, where $1 \leq d_1 < d$.

For the toric surface Σ associated with such a polygon Δ, the real structure we study is the involution that acts in the principal orbit $(\mathbb{C}^*)^2 \subset \Sigma$ by $\mathrm{Conj}(x,y) = (\bar{y},\bar{x})$. Its natural lift to the ample line bundle $\mathcal{L}_\Delta$ generated by monomials x^iy^j, $(i,j) \in \Delta \cap \mathbb{Z}^2$, acts by $\mathrm{Conj}_*(a_{ij}x^iy^j) = \bar{a}_{ij}x^jy^i$, $(i,j) \in \Delta$, and thus gives rise to the reflection of Δ with respect to the bisectrix $\mathcal{B}$ of the positive quadrant. Denote by D_Δ (or simply D) an ample divisor that defines $\mathcal{L}_\Delta$.

The goal of this section is to deduce from [8, Theorem 1.1] a lower bound for the Welschinger invariant $W_{\Sigma,D}$.[6] To this end, we introduce the following objects.

For each integer point belonging to the boundary of Δ, trace the straight line through this point and its image under the reflection with respect to $\mathcal{B}$. The union of all the traced lines cuts $\mathcal{B} \cap \Delta$ into certain segments; denote their number by m. Identify $\mathcal{B} \cap \Delta$ with the segment $[0,m] \subset \mathbb{R}$ in such a way that the intersection points of $\mathcal{B} \cap \Delta$ with the traced lines are mapped to the integer points of $[0,m]$. With each integer point $i \in [0,m]$ associate a nonnegative integer number $\sigma(i)$ equal to the integer length of the intersection of the corresponding straight line with Δ.

[6]In [8], this invariant is denoted by $W_0(\Sigma,\mathcal{L})$, where $\mathcal{L} = \mathcal{L}_\Delta$.

A finite multiset of closed intervals in $\mathbb{R}$ is called a Δ-*proper system* (or simply *proper system*) if

- each interval is contained in $[0, m]$ and has integer endpoints (intervals reduced to a point are allowed);
- the total number of intervals is $|\partial\Delta| - m - 1$, where $|\partial\Delta|$ is the integer length of the boundary of Δ;
- for any integer $i \in [0, m]$, the number of intervals containing i is equal to $\sigma(i)$.

Given a Δ-proper system, consider the disjoint union g' of the intervals of the system, and complete g' to a graph g introducing m additional vertices indexed by the half-integer points $i+1/2$, $i = 0, \dots, m-1$, and additional edges connecting each point $i + 1/2$ with all the right endpoints i and all the left endpoints $i + 1$ of the intervals in g'.

A Δ-proper system is called *admissible* if its graph g is a tree. An admissible Δ-proper system is *marked* if it is equipped with a *marking* that associates with each interval I of the system an integer point of I; the latter point is called *marked*.

The following statement is an immediate consequence of [8, Theorem 1.1].

Lemma 1. *Let Δ be one of the polygons shown in Fig.* 1 *and Σ the toric surface associated with Δ and equipped with the real structure* Conj (*described above*). *Then, the Welschinger invariant W_{Σ,D_Δ} is greater than or equal to the number of marked admissible Δ-proper systems.*

3. LOGARITHMIC ASYMPTOTICS

3.1. Main theorem.

Theorem 1. *Let Σ be one of the real surfaces $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, and $\mathbb{S}^2_{0,2}$. For any real ample divisor D on Σ,*

$$\log W_{\Sigma,nD} = (c_1(\Sigma) \cdot D) n \log n + O(n). \tag{1}$$

In particular,

$$\lim_{n\to\infty} \frac{\log W_{\Sigma,nD}}{\log GW_{\Sigma,nD}} = 1, \tag{2}$$

where $GW_{\Sigma,nD}$ is the genus zero Gromov–Witten invariant.

Since $W_{\Sigma,nD} \le GW_{\Sigma,nD}$ and $GW_{\Sigma,nD} = (c_1(\Sigma) \cdot D) n \log n + O(n)$, to prove Theorem 1 it is sufficient to prove the lower bound $W_{\Sigma,nD} \ge (c_1(\Sigma) \cdot D) n \log n + O(n)$. Due to Lemma 1 and the identity $|\partial\Delta| = c_1(\Sigma) \cdot D$, the latter lower bound would follow from the inequality

$$\log S_{n\Delta} \ge |\partial\Delta| \cdot n \log n + O(n), \tag{3}$$

where $S_{n\Delta}$ is the number of marked admissible $n\Delta$-proper systems. This inequality is proved in Subsections 3.3–3.6, where each of the surfaces $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, and $\mathbb{S}^2_{0,2}$ is treated separately.

3.2. Admissibility. Let Γ be a finite set of disjoint horizontal segments with integer endpoints in $\mathbb{R}^2$ (degenerate segments are allowed). For any vertical strip $b = \{i \le x \le i + 1\}$, where i is an integer, denote by $\Gamma^{\mathrm{L}}(b)$ (respectively, $\Gamma^{\mathrm{R}}(b)$) the subset of Γ formed by the segments whose right endpoint belongs to $x = i$ (respectively, left endpoint belongs to $x = i + 1$).

Lemma 2. *Assume that Γ can be represented as the disjoint union of two subsets Γ_{L} and Γ_{R} satisfying the following properties*:

(i) *for any vertical strip $b = \{i \le x \le i + 1\}$ such that i is an integer, the union of $\Gamma_{\mathrm{R}} \cap \Gamma^{\mathrm{R}}(b)$ and $\Gamma_{\mathrm{L}} \cap \Gamma^{\mathrm{L}}(b)$ contains at most one element*;

(ii) *if the union of $\Gamma_{\mathrm{R}} \cap \Gamma^{\mathrm{R}}(b)$ and $\Gamma_{\mathrm{L}} \cap \Gamma^{\mathrm{L}}(b)$ contains an element s, no element of $\Gamma^{\mathrm{L}}(b) \cup \Gamma^{\mathrm{R}}(b)$ lies below s*;

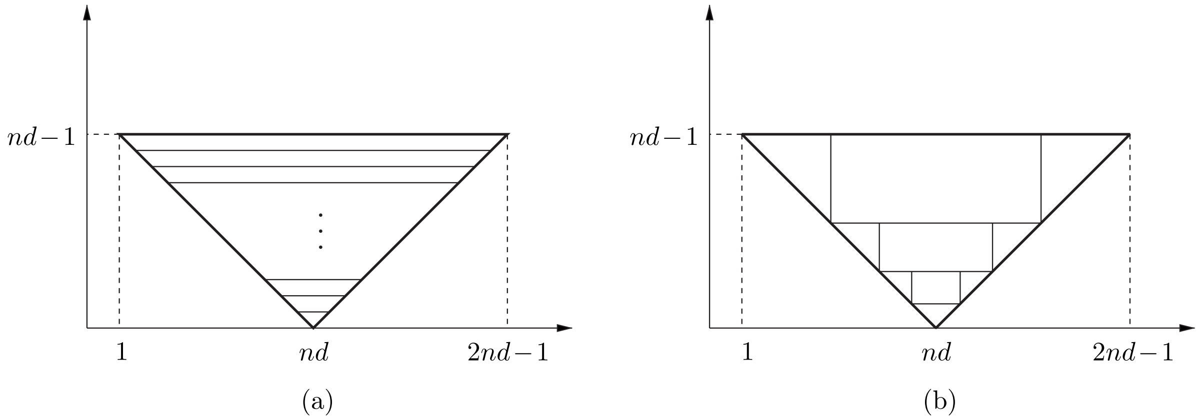


Fig. 2. First steps in the construction of admissible systems for $\mathbb{S}^2$

(iii) *there exists exactly one vertical strip* $b = \{i \le x \le i+1\}$ *such that* i *is an integer, at least one of the sets* $\Gamma^{\mathrm{L}}(b)$ *and* $\Gamma^{\mathrm{R}}(b)$ *is nonempty, and the union of* $\Gamma_{\mathrm{R}} \cap \Gamma^{\mathrm{R}}(b)$ *and* $\Gamma_{\mathrm{L}} \cap \Gamma^{\mathrm{L}}(b)$ *is empty.*

If the projections of the segments of Γ *on the horizontal axis form a proper system, then this proper system is admissible.*

Proof. For a proper system as in the lemma, identify Γ with the disjoint union g' of the intervals of the system and consider the graph g as in Section 2. Orient the segments of Γ_{L} to the left, the segments of Γ_{R} to the right, and orient each additional edge of g by extending the orientation of the adjacent horizontal segment. Conditions (i) and (ii) give a deformation retraction of g to a finite set of vertices, and condition (iii) guarantees that the latter set has only one element. □

3.3. Case $\Sigma = \mathbb{S}^2$. Let Δ be the square shown in Fig. 1a. In this case, the required inequality (3) reads as

$$\log S_{n\Delta} \ge 4dn \log n + O(n). \tag{4}$$

To construct an appropriate number of marked admissible $n\Delta$-proper systems, consider the triangle $T(n,d)$ with vertices $(1, nd-1)$, $(nd, 0)$, and $(2nd-1, nd-1)$ (see Fig. 2a). At each integer level $y = j$, $0 \le j \le nd-1$, consider the maximal horizontal segment contained in $T(n,d)$. If $j \ne 0$, make a *hole* in the considered segment by removing an open unit interval with integer endpoints. This *perforation procedure* gives rise to a set of $2nd-1$ horizontal segments whose projections form an $n\Delta$-proper system.

Inscribe in $T(n,d)$ a sequence of maximum size rectangles R_i satisfying the following properties: each rectangle is symmetric with respect to the vertical line $x = nd$, and the length of the horizontal edges of each rectangle is twice the length of its vertical edges (see Fig. 2b). The right upper vertices (x_i, y_i), $i \ge 1$, of these rectangles are given by

$$x_1 = nd + \left[\frac{nd-1}{2}\right], \quad y_1 = nd - 1, \qquad y_{i+1} = y_i - \left[\frac{y_i}{2}\right], \quad x_{i+1} = nd + y_{i+1} - \left[\frac{y_{i+1}}{2}\right].$$

Let k be the number of rectangles. Notice that $y_k = 2$, and put $y_{k+1} = y_k - \left[\frac{y_k}{2}\right] = 1$.

Restrict the choice of holes in the perforation procedure in the following way:

- all the holes are contained in the half-plane $x \ge nd$;
- for any integer $1 \le i \le k$, all the holes at the levels $y_{i+1} + 1 \le y \le y_i$ are contained in R_i;
- for any integer $1 \le i \le k-1$, no two holes at the levels $y_{i+1} + 1 \le y \le y_i$ have the same projection on the horizontal axis.

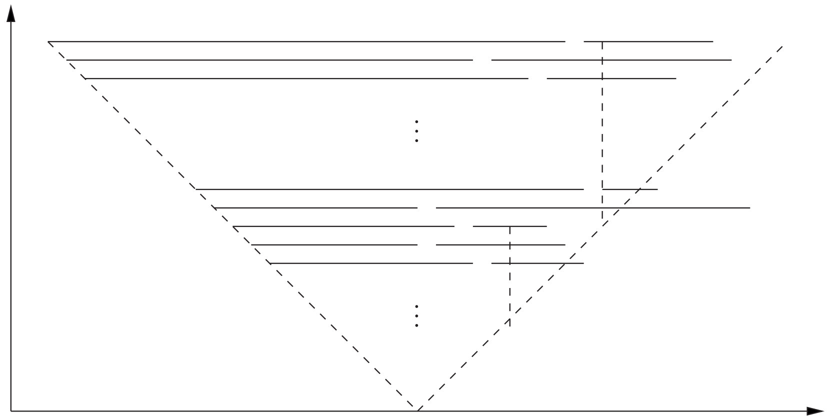


Fig. 3. A permuted perforated (n,d)-collection

The set of segments obtained via such a perforation procedure is called a *perforated* (n,d)-*collection*. The number $M(n,d)$ of perforated (n,d)-collections is equal to

$$(y_1-y_2)!\,(y_2-y_3)!\ldots(y_k-y_{k+1})!\,.$$

According to the Stirling formula,

$$\log M(n,d)=\big((y_1-y_2)+(y_2-y_3)+\ldots+(y_k-y_{k+1})\big)\log n+O(n)=dn\log n+O(n).$$

For any perforated (n,d)-collection c and any permutations $\sigma_1,\ldots,\sigma_{k-1}$, where σ_i is a permutation of $\{y_{i+1}+1,y_{i+1}+2,\ldots,y_i\}$, $i=1,\ldots,k-1$, consider the set of segments $c_{\sigma_1,\ldots,\sigma_{k-1}}$ obtained from c in the following way: for each integer $1\le i\le k-1$, cut along the vertical line $x=x_i$ the segments of c lying on the levels $y_{i+1}+1\le y\le y_i$ and intersecting the line $x=x_i$, permute according to σ_i the right-hand parts of the segments we have cut, and glue together the adjacent parts in order to form new segments (see Fig. 3). The set $c_{\sigma_1,\ldots,\sigma_{k-1}}$ is called a *permuted perforated* (n,d)-*collection*. It consists of the point $(nd,0)$ and two segments at each integer level $1\le y\le nd-1$.

The number $\widetilde{M}(n,d)$ of permuted perforated (n,d)-collections $c_{\sigma_1,\ldots,\sigma_{k-1}}$, where c runs over all perforated (n,d)-collections and σ_i runs over all permutations of $\{y_{i+1}+1,y_{i+1}+2,\ldots,y_i\}$, $i=1,\ldots,k-1$, is equal to

$$M(n,d)(y_1-y_2)!\,(y_2-y_3)!\ldots(y_k-y_{k+1})!\,.$$

Thus, $\log\widetilde{M}(n,d)=2dn\log n+O(n)$.

The projection of any permuted perforated (n,d)-collection on the horizontal axis is an $n\Delta$-proper system. The restriction imposed above on the choice of holes guarantees that the projection of all permuted perforated (n,d)-collections produces $\widetilde{M}(n,d)$ pairwise distinct $n\Delta$-proper systems. All the resulting systems are admissible, as follows from Lemma 2 applied to any permuted perforated (n,d)-collection represented as the disjoint union of the segments lying on the left-hand side of the holes (the subset Γ_{R}) and the segments lying on the right-hand side of the holes (the subset Γ_{L}).

Mark each of $\widetilde{M}(n,d)$ admissible $n\Delta$-proper systems as above in such a way that

- no marked point of the projection of a segment at level 1 coincides with the point nd;
- for any integer i between 1 and k, the marked points of the projections of segments at any level $y_{i+1}+1\le y\le y_i$ are placed outside the projection of R_i.

For each system, this can be done in $((y_1-y_2)!\,(y_2-y_3)!\ldots(y_k-y_{k+1})!)^2$ different ways. Thus, the logarithm of the number of marked admissible $n\Delta$-proper systems obtained is $4dn\log n+O(n)$. This proves Theorem 1 in the case $\Sigma=\mathbb{S}^2$.

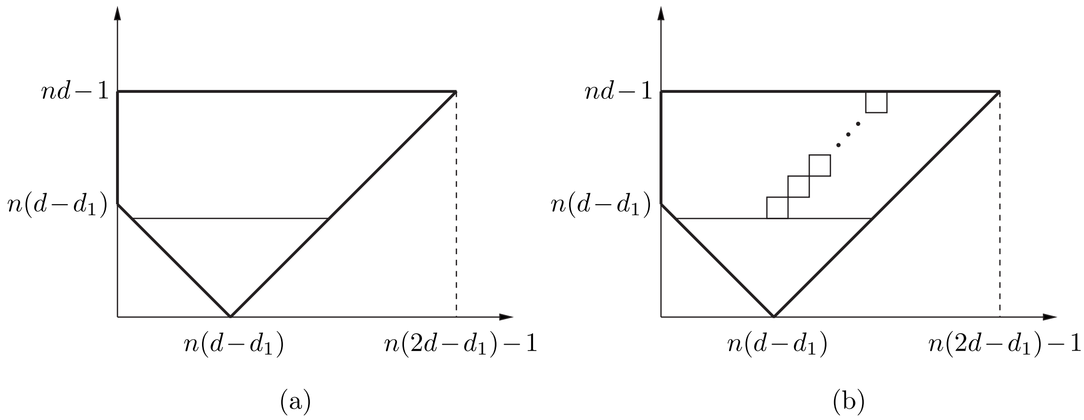


Fig. 4. Construction of admissible systems for $\mathbb{S}^2_{1,0}$

3.4. Case $\Sigma = \mathbb{S}^2_{1,0}$. Let Δ be the pentagon shown in Fig. 1b. In this case, the required inequality (3) reads as

$$\log S_{n\Delta} \ge (4d - d_1)n \log n + O(n). \tag{5}$$

To construct an appropriate number of marked admissible $n\Delta$-proper systems, consider the quadrangle $Q(n,d,d_1)$ with vertices

$$(0, nd-1), \quad (0, n(d-d_1)), \quad (n(d-d_1), 0), \quad \text{and} \quad (n(2d-d_1)-1, nd-1)$$

(see Fig. 4a). For the triangle $T(n,d-d_1) \subset Q(n,d,d_1)$, use the construction described in Subsection 3.3. To complete the resulting permuted perforated $(n,d-d_1)$-collections, we proceed in the following way.

Consider the upward right staircase E formed by squares of size $n \times n$ such that E starts at the middle point $(n(d-d_1), n(d-d_1)-1)$ of the upper side of $T(n,d-d_1)$ (see Fig. 4b). At each integer level $y = j$, $n(d-d_1) \le j \le nd-1$, consider the maximal horizontal segment contained in $Q(n,d,d_1)$ and use the perforation procedure (that is, make a hole in each segment considered) choosing holes in such a way that all these holes are contained in E, no hole is taken on the lower sides of the squares forming E, and no two holes have the same projection on the horizontal axis. This gives $(n!)^{d_1}$ sets of segments. For any of these sets and any permuted perforated $(n,d-d_1)$-collection, their union is called a *perforated* (n,d,d_1)*-collection*.

The projection of any perforated (n,d,d_1)-collection on the horizontal axis is an $n\Delta$-proper system. Due to Lemma 2, any resulting $n\Delta$-proper system is admissible. For any such system, there are at least $(nd_1)!\,(nd_1)!$ choices of marking for the projections of segments lying above $T(n,d-d_1)$. Thus, the logarithm of the number of marked admissible $n\Delta$-proper systems is at least

$$4(d-d_1)n \log n + O(n) + d_1 \log n! + 2\log(nd_1)! = (4d-d_1)n\log n + O(n).$$

This proves Theorem 1 in the case $\Sigma = \mathbb{S}^2_{1,0}$.

3.5. Case $\Sigma = \mathbb{S}^2_{2,0}$. Let Δ be the hexagon shown in Fig. 1c. In this case, the required inequality (3) reads as

$$\log S_{n\Delta} \ge (4d - d_1 - d_2)n \log n + O(n). \tag{6}$$

To construct an appropriate number of marked admissible $n\Delta$-proper systems, consider the pentagon $P(n,d,d_1,d_2)$ with vertices

$$(0, nd-1), \quad (0, n(d-d_1)), \quad (n(d-d_1), 0),$$

$$(n(2d-d_1-d_2), n(d-d_2)), \quad \text{and} \quad (n(2d-d_1-d_2), nd-1)$$

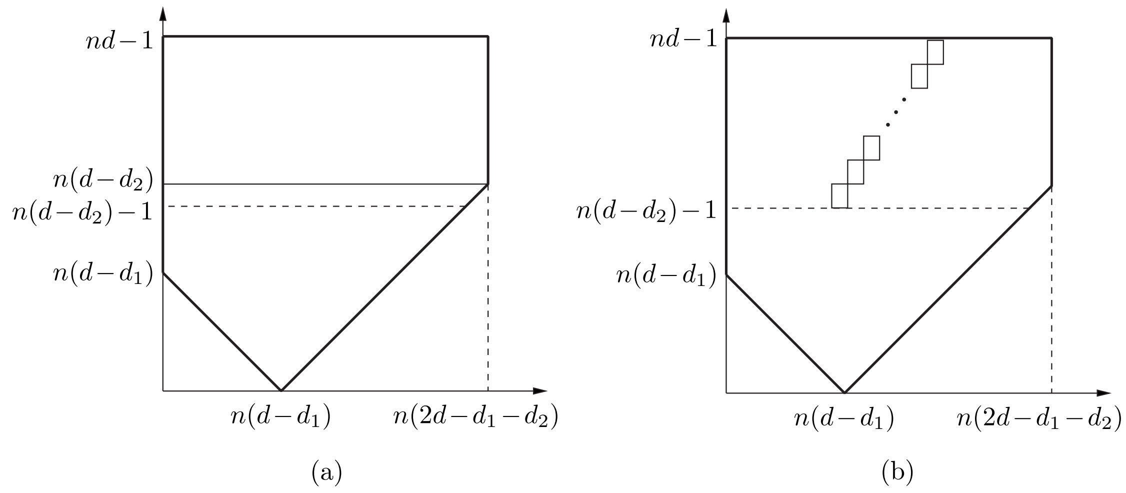


Fig. 5. Construction of admissible systems for $\mathbb{S}^2_{2,0}$

(see Fig. 5a). For the quadrangle $Q(n, d-d_2, d-d_1) \subset P(n, d, d_1, d_2)$, use the construction described in Subsection 3.4. To complete the resulting perforated $(n, d-d_2, d_1-d_2)$-collections, we proceed in the following way.

The remaining part of $P(n, d, d_1, d_2)$ is formed by a horizontal strip of height 1 and a rectangle of width $n(2d-d_1-d_2)$ and height $nd_2 - 1$ (see Fig. 5a). Consider an upward right staircase

- starting at a point $(x_0, n(d-d_2)-1)$ with $x_0 \geq [n(2d-d_1-d_2)/4]$,
- ending at a point $(x_1, nd-1)$ with $x_1 \leq [3n(2d-d_1-d_2)/4] - 1$,
- and formed by rectangles such that each rectangle is of width 1 and of positive height smaller than or equal to $a = \left[\frac{2d_2}{2d-d_1-d_2}\right] + 1$

(see Fig. 5b). At each integer level $y = j$, $n(d-d_2) \leq j \leq nd-1$, consider the maximal horizontal segment contained in $P(n, d, d_1, d_2)$ and use the perforation procedure choosing holes in the rectangles of the staircase in such a way that no hole is taken on the lower sides of the rectangles. For any perforated $(n, d-d_2, d_1-d_2)$-collection, its union with the constructed set of segments is called a *perforated* (n, d, d_1, d_2)*-collection.*

The projection of any perforated (n, d, d_1, d_2)-collection on the horizontal axis is an $n\Delta$-proper system. Due to Lemma 2, any resulting $n\Delta$-proper system is admissible. For any such system, there are at least $[n(2d-d_1-d_2)/4]^{2nd_2}(a!)^{-2nd_2}$ choices of marking for the projections of segments lying above $Q(n, d-d_2, d_1-d_2)$. Thus, the logarithm of the number of marked admissible $n\Delta$-proper systems is at least

$$(4(d-d_2)-(d_1-d_2))n\log n + O(n) + 2d_2 n\log n + O(n) = (4d-d_1-d_2)n\log n + O(n).$$

This proves Theorem 1 in the case $\Sigma = \mathbb{S}^2_{2,0}$.

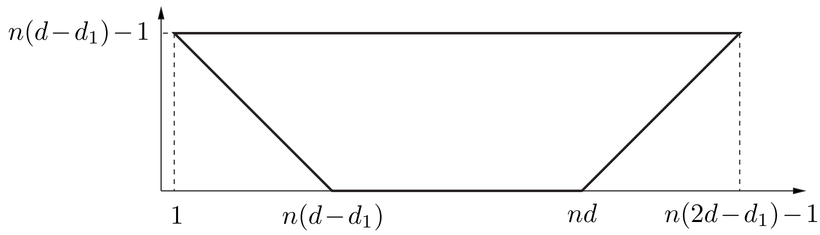
3.6. Case $\Sigma = \mathbb{S}^2_{0,2}$. Let Δ be the hexagon shown in Fig. 1d. In this case, the required inequality (3) reads as

$$\log S_{n\Delta} \geq (4d - 2d_1)n\log n + O(n). \qquad (7)$$

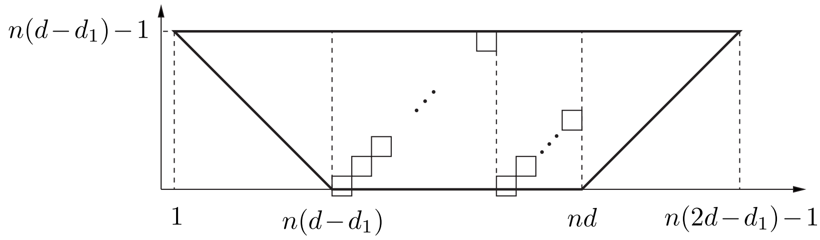
To construct an appropriate number of marked admissible $n\Delta$-proper systems, consider the trapezium $K(n, d, d_1)$ with vertices

$$(1, n(d-d_1)-1), \quad (n(d-d_1), 0), \quad (nd, 0), \quad \text{and} \quad (n(2d-d_1)-1, n(d-d_1)-1)$$

(see Fig. 6a).



(a)



(b)

Fig. 6. Construction of admissible systems for $\mathbb{S}^2_{0,2}$

Consider the sequence C of upward right staircases formed by squares of size $n \times n$ such that

- all the staircases of C are contained in the vertical strip

$$\mathcal{B} = \{n(d - d_1) \le x \le nd\};$$

- each staircase starts at the level $y = -1$;
- each staircase ends at the upper side of $K(n, d, d_1)$, the only possible exception being the last staircase;
- the first staircase starts at the point $(-1, n(d - d_1))$;
- for each staircase except the first one, the vertical line where the staircase starts coincides with the vertical line where the preceding staircase ends

(see Fig. 6b; in the case $d_1 \le d - d_1$, there is only one staircase in C). At each integer level $y = j$, $0 \le j \le n(d - d_1) - 1$, consider the maximal horizontal segment contained in $\mathcal{B}$ and use the perforation procedure (this time we authorize several holes at the same level) by choosing holes in such a way that all these holes are contained in C, no hole is taken on the lower sides of the squares forming the staircases, and there is exactly one hole in each integer vertical strip $i \le x \le i + 1$ contained in $\mathcal{B}$. This gives $(n!)^{d_1}$ sets of segments.

Pick a permuted perforated $(n, d - d_1)$-collection π in $T(n, d - d_1)$, cut π along the vertical line $x = n(d-d_1)$, keep the left half of π at its place, and shift the right half by the vector $(nd_1, 0)$. The result of gluing the obtained collection to a set of segments constructed in $\mathcal{B}$ as described above is called a *perforated* $K(n, d, d_1)$*-collection.* The projection of any perforated $K(n, d, d_1)$-collection on the horizontal axis is an $n\Delta$-proper system.

Any resulting $n\Delta$-proper system is admissible. Indeed, let γ be a perforated $K(n, d, d_1)$-collection. Identifying γ with the disjoint union g' of the intervals of the projection of γ to the horizontal axis, consider the graph g as in Section 2. In each integer vertical strip $i \le x \le i + 1$ contained in $\mathcal{B}$, there is exactly one pair of additional edges of g, and this pair fills up the only hole in $i \le x \le i + 1$. Once the holes in $\mathcal{B}$ are filled up, Lemma 2 applies. This proves the admissibility of the projection of γ.

Consider a perforated $K(n, d, d_1)$-collection γ obtained by gluing a permuted perforated $(n, d-d_1)$-collection π to a set of segments constructed in $\mathcal{B}$ as above. Any marking of the projection of π can be extended to a marking of the projection of γ via a choice of an integer point on each segment entering under a staircase. The latter choice can be done in at least $(nb_1)!\dots(nb_k)!$ ways, where $b_1,\dots,b_k$ are the numbers of stairs in the staircases (in fact, $b_1=\dots=b_{k-1}$). Thus, the logarithm of the number of marked admissible $n\Delta$-proper systems is at least

$$4(d-d_1)n\log n+O(n)+d_1\log n!+n(b_1+\dots+b_k)\log n+O(n)=(4d-2d_1)n\log n+O(n).$$

This proves Theorem 1 in the case $\Sigma=\mathbb{S}^2_{0,2}$.

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