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SEMICLASSICAL ELECTRON MOTION AND NOVIKOV CONJECTURE

§1. Introduction

Electron motion in a crystal grating invariant under the action of the lattice $\mathbb{Z}^3$ is described by the Schrödinger operator commutes with shifts by vectors from $\mathbb{Z}^3$. In the common eigen states an electron has a quasi-momentum $p \in \mathbb{R}^3$ well defined up to a vector of the dual lattice $(\mathbb{Z}^3)^* \cong \mathbb{Z}^3$. The eigen value of the Schrödinger operator is a 3-periodic function of p.

If there is a magnetic field in the crystal the corresponding Schrödinger equation may fail to have periodic coefficients. However, if the magnetic field is weak the eigen states are close to the former and may be assumed under the semiclassical approximation that the electron has a quasi-momentum which now depends on time, $p = p(t)$, in such a way that $\varepsilon(p') = E = \text{const}$, $\dot{p}' \perp H$, where $p' = p - \frac{e}{c}A$, ε is the former energy function, and H is the magnetic field (see [1, 2]).

Let us consider the simplest case when the magnetic field H is constant and uniform. Then electron "trajectories" projected to the quasi-momentum space are the intersection lines of the 3-periodic surface $\varepsilon(p) = E$ and some plane orthogonal to H. The trajectories have a certain asymptotical direction, that is do there exist the limits

$$\lim_{t\to+\infty} \frac{p'(t)}{|p'(t)|} \quad \text{and} \quad \lim_{t\to-\infty} \frac{-p'(t)}{|p'(t)|}, \qquad (*)$$

and whether these limits (if existing) are the same of different:

A. V. Zorich proved in 3 that in the case of a field H sufficiently close to a rational on any non-closed trajectory is contained in a flat stripe of finite width. It is shown in [4] that in Zorich case a non-closed regular trajectory comes through the stripe in one direction, that is the limits $(*)$ are the same.

The Novikov conjecture (see [5]) states that in all cases a non-closed regular trajectory is contained in a finite-wide stripe and comes through it in one direction.

The main idea we use is to consider the conjunction of all the non-closed trajectories lying on the surface $\varepsilon(p) = E$. In the "regular" case this conjunction (if not empty) consists of deformed planes with holes. The non-closed trajectories are in this case straight lines finitely deformed. Such a situation remains if we alter ε, H, and E a little. It is shown in [3] that this situation occurs when H is a rational vector, and, therefore when H is close to rational. Here it is proved that this situation takes place for almost all energy levels if ε and H are fixed.

§2. Definitions and settings

Denote by M_E the surface $M_E = \{p \in \mathbb{T}^3 | \varepsilon(p) = E\}$.

Remark. Sometimes we will consider p as an element of $\mathbb{T}^3$, sometimes as an element of $\mathbb{R}^3$. If some letter denotes a surface in $\mathbb{T}^3$ then the covering surface in $\mathbb{R}^3$ is denoted by the same letter with sign $\widehat{}$. For example, $\widehat{M_E} = \{p \in \mathbb{R}^3 | \varepsilon(p) = E\}$.

Denote by ω_E the restriction of the form $H_1 dp_1 + H_2 dp_2 + H_3 dp_3$ to the surface M_E.

Assume that ε and H satisfy the conditions:

1) the function ε considered in $\mathbb{T}^3$ has a finite number of critical points;

2) the form ω_E has a finite number of critical points for any E;

3) H_1, H_2, H_3 are independent over the field $\mathbb{Q}$.

Level lines of a closed 1-form ω are by definition the level lines of the multivalued function $\int \omega$.

The connected components of the level lines of the form ω_E are called trajectories here.

Denote by K_E the conjunction of all the closed trajectories lying on M_E. The supplement $M_E \backslash K_E$ is denoted by L_E.

§3. The structure of K_E and L_E

Regular closed trajectories lying on M_E make up several cylinders, whose bases are singular trajectories or parts of singular trajectories. Any closed trajectory bounds some flat disk orthogonal to H and, therefore, so does any connected component of ∂K_E. L_E is a surface with the same boundary as K_E. Let L'_E be the conjunction of L_E and flat disks bounded by ∂L_E, $L^{(i)_E}$, $i = 1, \ldots m(E)$, be the connected components of L'_E.

Each surface $L^{(i)_E}$ has the genus greater than 0, because covering surface $\widehat{L}^{(i)_E}$ consists of unbounded components by construction.

Denote by $r(E)$ the sum $\sum_{i=1}^{m(E)}(g(L_E^{(i)})-1)$, where $g(L)$ is the genus of a surface L.

Lemma 3.1. *If for some regular value $E_0 r(E_0) > 0$ then for any $E \neq E_0$.*

Proof. Assume for simplicity that the form ω_{E_0} has only non-degenerate critical points, that is saddle points and poles.

If E is sufficiently close to E_0 then K_E contains cylinders close to those K_E. Let K_E be the conjunction of them. Note that K_E may contain and does contain (as we will show) one or more cylinders which do not consist in K_E.

Let $\mathcal{L}_E$ be the supplement $M_E \backslash K_E$, $\mathcal{L}_E'$ be the conjunction of $\mathcal{L}_E$ and the flat disks bounded by $\partial\mathcal{L}_E$, $\mathcal{L}_E^{(i)}$, $i = 1, \ldots, m(E_0)$, be the connected components of $\mathcal{L}_E'$.

The surface $L_{E_0}\backslash\partial L_{E_0}$ contains $\sum_{i=1}^{m(E_0)} 2(g(L_{E_0}^{(i)})-1) = 2r(E_0)$ saddle critical points of the form ω_{E_0}. If E is sufficiently close to E_0 the surface $\mathcal{L}_E\backslash\partial\mathcal{L}_E$ also contains $2r(E_0)$ saddle points of the form ω_E, and these saddle points to those of ω_{E_0}. Let $A_1(E), \ldots, A_{2r(E_0)}(E)$ be these points.

We can assume without loss of generality that $E > E_0$. Denote by N_E the set $\cup_{t\in(E_0,E)}\widehat{\mathcal{L}_t'}$. let us consider the intersection of N_E and some plane $\prod$ orthogonal to H and non-tangent to the surfaces L_{E_0} and $\mathcal{L}_E$. $N_E \cap \prod$ consists of narrow strips, disks attached to them, and crosses. The disks correspond to the intersection of $\prod$ and $\partial\widehat{\mathcal{L}_t}$, $t \in (E_0, E)$. The crosses correspond to the intersection points of $\prod$ and the curves $\pi^{-1}(A_i(t))t \in (E_0, E)$, where π is the projection $\mathbb{R}^3 \longrightarrow \mathbb{T}^3$. There are $CR^2 + 0(R^2)$ crosses in a square of side R, $C > 0$.

It is easy to prove that there exist such constants C_1, C_2, δ that any line segment I of length R in the plane $\prod$ has a δ-deformation $\widetilde{I} \subset \prod$ intersecting $\partial(N_E \cap \prod)$ no more than $C_1R + C_2$ times. The term "δ-deformation" means such a curve $\widetilde{I}$: $I \to \mathbb{R}^3$ that $|\widetilde{I}(x) - x| < \delta \forall x \in$. Let Q_R be a square of side R in the plane $\prod$. $\widetilde{Q}_R$ be a domain in $\prod$ bounded by a δ-deformation of ∂Q_R intersecting $\partial(N_E \cap \prod)$ no more than $4(C_1R + C_2)$ times. The set $N_E \cap \widetilde{Q}_R$ has $O(R)$ connected components, where are the number of crosses in it is equal to $CR^2 + o(R^2)$. The Euler characteristic $\chi(N_E \cap \widetilde{Q}_R)$ equals $-CR^2 + o(R^2)$, and for sufficiently large $R\, \chi(N_E \cap \widetilde{Q}_R) < 0$. It means that the set $N_E \cap \prod$ has non-trivial cycles and its boundary $(\widehat{\mathcal{L}}_E \cup \widehat{L}_{E_0}) \cap \prod$ has closed components.

Therefore, $\mathcal{L}_E \cap K_E = \emptyset$, some of the points $A_i(E)$ lie on $\overline{K}_E$, the number of points $A_i(E)$ lying on $L_E \backslash \partial L_E$, which is equal to $2r(E)$, is less than $2r(E_0)$.

Lemma 3.2. *For almost all values E $r(E) = 0$.*

Proof. It immediately follows from lemma 3.1. that the set $\{E | r(E) = k\}$ is countable for any $K > 0$. Therefore the set $\{E | r(E) > 0\}$ is also countable.

§4. Behavior of non-closed trajectories

Lemma 4.1. *Let M be a 2-dimensional surface in $\mathbb{T}^3$, $\widetilde{M}$ be one of the connected components of the covering surface $\widehat{M} \subset \mathbb{R}^3$, and let $\prod$ be a plane orthogonal to H.*

Suppose that image of $H_1(M, \mathbb{Z})$ in the torus homologies under the embedding $i : M \to \mathbb{T}^3$ is 2-dimensional intersection line of $\widetilde{M}$ and $\prod$ is contained in a stripe and comes through it in one direction.

Proof. Since $i_*(H_1(M, \mathbb{Z})) \cong \mathbb{Z}^2$ the surface $\widetilde{M}$ lies between two parallel planes, say, P_1 and P_1, P_2 are defined by the equations $p_3 = \beta$ and $p_3 = -\beta$ respectively. Let l_1 and l_2 be the intersection lines $P_1 \cap \prod$ stripe S bounded by these lines.

Let X and Y be points in l_1 and l_2 respectively, γ be a curve in the stripe S connecting X and Y and intersecting $\widetilde{M}$ the minimal possible number times. Let $B_1, \dots, B_q$ be the intersection points $\gamma \cap \widetilde{M}$. The number of them independent of the choice of X and Y. It is easy to see that γ intersects each connected regular component of $\widetilde{M} \cap \prod$ no more than once.

Let Γ_α be a strip $\Gamma_\alpha = \gamma + I_\alpha$, where I_α is the line segment with the ends points $(-\alpha, 0, 0)$ and $(\alpha, 0, 0)$. For sufficiently small α the strip Γ intersects $\widetilde{M}$ transversely, the intersection consists of curves Φ_i, $i = 1, \dots, q$, where $B_i \in \Phi_i$, no tangent to which is orthogonal to H, and Γ_α does not intersect its shifts by integer vectors.

There are an infinite number of stripes of the shape $\Gamma_\alpha + (p_1, p_2, 0)$, where $p_1, p_2 \in \mathbb{Z}$, which intersect the stripe S. The intersection consists of curves γ_i having the end points X_i and Y_i which belong to l_1 and l_2 respectively. These curves cut the stripe S into an infinite number of bounded parts. Each curve γ_i intersects any connected regular component of $\widetilde{M} \cap \prod$ no more than once, because the number of points in $\gamma_i \cap \widetilde{M}$ equals q, that is the minimal possible number.

Any non-closed regular component of $\widetilde{M} \cap \prod$ intersects at least one of the curves γ_i. On the other hand, it should intersect each of these curves no more than once. It follows from this that it intersects each curve exactly once, which means that a non-closed component of $\widetilde{M} \cap \prod$ comes through the stripe in one direction.

Theorem 4.1. *If some regular value E_0 satisfies $r(E_0) = 0$, and there is at least one non-closed trajectory on M_{E_0} then any non-closed regular trajectory on $\widehat{M}_E$ for any E is contained in a finite-wide stripe and comes through it in one direction. The direction of the stripe is independent of E.*

Proof. Since the intersection of any connected component of $\widehat{L}_E^{(i)}$ and a plane $\prod \perp H$ is unbounded and H_1, H_2, H_3 are independent over the field $\mathbb{Q}$ then $\dim(i_*(H_1(L_E^{(i)}))) \geq 2$.

The equation $r(E_0) = 0$ means that $g(L_{E_0}^{()}) = 1$, $\dim(i_*(H_1(L_{E_0}^{(j)}))) = 2$ for any $j = 1, \ldots, m(E_0)$.

Let $\widetilde{L}_{E_0}^{(1)}$ be a connected of $\widehat{L}_{E_0}^{(1)}$. It follows from $\dim(i_*(H_1(L_{E_0}^{(1)}))) = 2$ that $\widetilde{L}_{E_0}^{(1)}$ lies between two parallel planes, say, P_1 and P_2. Topologically, $\widetilde{L}_{E_0}^{(1)}$ is a deformed plane. Therefore, $\widehat{L}_{E_0}^{(1)}$ cuts our space $\mathbb{R}^3$ into pieces, each of which lies between two planes parallel to P_1.

It easy to see that if $E \neq E'$ then $L_{E'} \cap L_E = \emptyset$. Any component of $\widehat{L}_E^{(j)}$ for any $j = 1, \ldots, m(E)$, $i_*(H_1(L_E^{(j)})) = i_*(H_1(L_E^{(1)}))$. The theorem statement now follows from lemma 4.1. The direction of the trajectories is the direction of the straight line $l = P_1 \cap \prod$.

Theorem 4.2. *Under the conditions of theorem 4.1. the direction of the trajectories is the direction of the intersection line of two planes, the first is orthogonal to H, the other is orthogonal to some integer vector $k \in \mathbb{Z}^3$ which is well-defined up to a multiplication constant. This direction is the same for any pair (ε', H') sufficiently close to (ε, H).*

Proof. It is evident that the plane P_1 from the proof of theorem 4.1. is orthogonal to some integral vector. If there were two non-collinear integer vectors orthogonal to the direction of the trajectories this direction of the trajectories this direction would be rational and H_1, H_2, H_3 would not be independent over $\mathbb{Q}$.

If a pair (ε', H') is close to (ε, H) the surfaces $K_{E_0}(\varepsilon', H')$, $L_{E_0}(\varepsilon', H')$ are close to $K_{E_0}(\varepsilon, H)$ and $L_{E_0}(\varepsilon, H)$ respectively. Therefore, the planes $P_1(\varepsilon', H')$ and $P_1(\varepsilon, H)$ have the same slope.

Lemma 4.2. *If the surfaces M_{E_1} and M_{E_2}, $E_1 < E_2$, both contains non-closed trajectory then so does M_E for any $E \in (E_1, E_2)$.*

Proof. Suppose that the surface M_E contains no non-closed trajectory. Let $\prod$ be a plane orthogonal to H. Since the connected components of $\widehat{M}_E \cap \prod$ are all compact then one of the sets $N_+ = \prod \cap \{p \in \mathbb{R}^3 | \varepsilon(p) \geq E\}$ consists of compact components only.

If N_- consists of compact components then so does $M_{E'} \cap \prod$ for any $E' < E$ and therefore, $E < E_1$. Otherwise, $E > E_2$.

Lemma 4.3. *There is at least one level E such that M_E contains non-closed trajectories.*

Proof. First note that the set of such values E that N_- (or N_+) consists of compact components only is opened. The sets $N_\pm$ can not consist of compact components simultaneously. Therefore, for some E sets N_- and N_+ both have unbounded components. It means that M_E contains non-closed trajectories.

Theorem 4.3. *There may be two cases:*

1) *M_E contains non-closed trajectories for $E \in [E_1, E_2]$, $E_1 < E_2$, and they are such as those described in theorems 4.1. and 4.2.*

2) *M_E contains non-closed trajectories for exactly one value E.*

In the second case nothing is proved about behavior of non-closed trajectories. But no example of ε and H satisfying our conditions and such that some non-closed trajectory is not contained in a stripe is known now. S. P. Tsarev proposed an example of surface M non-closed trajectories on which are not contained in stripes. But in his case only two numbers H_1, H_2, H_3 are independent over the field $\mathbb{Q}$, and the trajectorieshave an asymptotical direction though they are not contained in stripes.

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