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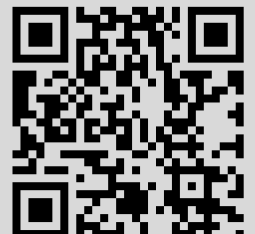
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On two broad classes of heavy-tailed distributions

Since the introduction of Class \mathcal{M} and Class \mathcal{M}^* , they have played important roles in insurance to describe tail equivalence of ruin probability and tail behavior of the deficit at ruin. And in insurance and finance most of heavy-tailed distributions with finite and positive expectations belong to Class \mathcal{M} . So it is important to study tail behaviors of Class \mathcal{M} and Class \mathcal{M}^* . In this paper, we obtain some results on essential tail behaviors of these two classes.

1. Introduction

Let X be a non-negative random variable (r.v.) with distribution function(d.f.) F . X or F is called heavy-tailed if its moment generating function does not exist, i.e.,

$$g(t) := \int_0^\infty e^{tx} dF(x) = \infty, \quad \forall t > 0. \quad (1.1)$$

Otherwise, X or F is light-tailed, i.e., there exists some $t_0 > 0$ such that

$$g(t) < \infty, \quad \forall 0 \leq t < t_0. \quad (1.2)$$

Write \mathcal{K} as the class of heavy-tailed distributions (see [3]) and denote $\overline{F}(x) = 1 - F(x)$.

Several important classes of heavy-tailed distributions are enumerated here for our using, one can see [3] for detail:

$F \in \mathcal{S}$ iff for any $n \in \mathcal{N}$ (or equivalently for $n = 2$),

$$\lim_{x \rightarrow \infty} \frac{\overline{F^{n*}}(x)}{n\overline{F}(x)} = 1,$$

where F^{n*} denotes the n -fold convolution of F with the corresponding tail $\overline{F^{n*}} = 1 - F^{n*}$.

$F \in \mathcal{L}$ iff for any $l \in \mathcal{R}$ (or equivalently for some $l \neq 0$),

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x+l)}{\overline{F}(x)} = 1.$$

$F \in \mathcal{D}$ iff for any $0 < l < 1$ (or equivalently for some $0 < l < 1$)

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(lx)}{\overline{F}(x)} < \infty.$$

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Since the definitions of Class \mathcal{L} and Class \mathcal{D} do not involve convolution and give sensible explanations, they are convenient in applications and widely used in Finance and Insurance. And the two classes include almost all the heavy-tailed distributions which are usually used in applied probability, for instance, the class of regularly varying distributions $\mathcal{R}_{-\alpha}$ ($\alpha > 0$), which plays important roles in classical limit theory, the class of extend regularly varying distributions $\text{ERV}(-\alpha, -\beta)$, Class \mathcal{C} , etc.

$F \in \mathcal{R}_{-\alpha}$ ($\alpha > 0$) iff

$$\overline{F}(x) \sim x^{-\alpha}l(x), \quad x \rightarrow \infty,$$

where $l(x)$ is a slowly varying function as $x \rightarrow \infty$; $F \in \text{ERV}(-\alpha, -\beta)$ ($0 < \alpha \leq \beta < \infty$) iff

$$y^{-\beta} \leq \liminf_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq \limsup_{x \rightarrow \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leq y^{-\alpha}, \quad \forall y > 1;$$

and $F \in \mathcal{C}$ iff

$$\lim_{l \uparrow 1} \limsup_{x \rightarrow \infty} \frac{\overline{F}(lx)}{\overline{F}(x)} = 1.$$

Class \mathcal{C} has been used in different studies of applied probability such as queueing system and ruin theory; see, for example, [12], [5], [10], and references therein.

By the above definitions, one can easily get (see, for example, [3])

$$\begin{aligned} \mathcal{R}_{-\alpha} \subset \text{ERV}(-\alpha, -\beta) \subset \mathcal{C} \subset \mathcal{D} \subset \mathcal{K}; \quad \mathcal{S} \subset \mathcal{L} \subset \mathcal{K}; \\ \mathcal{S} \not\subset \mathcal{D}; \quad \mathcal{D} \not\subset \mathcal{S}; \quad \mathcal{D} \cap \mathcal{L} \neq \emptyset; \quad \mathcal{D} \cap \mathcal{L} \subset \mathcal{S}. \end{aligned}$$

In the literature, as we know, most of the related problems in insurance and finance are considered in Class \mathcal{D} or Class \mathcal{L} .

2. The Class \mathcal{M} and Class \mathcal{M}^*

In view that a risk random variable is usually non-negative and has a finite expectation

$$0 < \mu := EX = \int_0^\infty \overline{F}(x)dx < \infty, \quad (2.1)$$

in this paper we only discuss distributions with support on $[0, \infty)$ and satisfying condition (2.1). Denote by \mathcal{E}_1 the class of these distributions.

For $F \in \mathcal{E}_1$, write

$$V(x) = \int_0^x \overline{F}(u)du, \quad \overline{V}(x) = \int_x^\infty \overline{F}(u)du, \quad F_e(x) = \frac{1}{\mu}V(x), \quad r_e(x) = \frac{\overline{F}(x)}{\overline{V}(x)} := \frac{\epsilon(x)}{x}. \quad (2.2)$$

It is easy to see that $F_e(x)$ is also a distribution on $[0, \infty)$, which is called the integrated tail distribution of F , or its equilibrium distribution. In applied probability, it can be used to describe ruin probabilities of renewal risk model, tail distributions of ladder heights of a random walk, and limiting distributions of waiting time and busy period of queueing model, and so on. See [1], [3], [11] for details.

Let $F \in \mathcal{E}_1$, then for any $x > x_0 > 0$,

$$-\int_{x_0}^x \frac{\epsilon(t)}{t} dt = -\int_{x_0}^x \frac{\overline{F}(t)}{\overline{V}(t)} dt = \int_{x_0}^x \frac{d\overline{V}(t)}{\overline{V}(t)} = \log \frac{\overline{V}(x)}{\overline{V}(x_0)} = \log \frac{\overline{F}_e(x)}{\overline{F}_e(x_0)},$$

which is equivalent to

$$\overline{F}_e(x) = \overline{F}_e(x_0) \exp \left\{ - \int_{x_0}^x \frac{\epsilon(t)}{t} dt \right\}. \quad (2.3)$$

Since the Class $(\mathcal{L} \cup \mathcal{D}) \cap \mathcal{E}_1$ does not contain all important heavy-tailed distribution in Insurance and Finance (please see Example 1 below), we introduced the following two classes of heavy-tailed distributions in [14] and [15]:

$F \in \mathcal{M}$ iff $F \in \mathcal{E}_1$ and

$$\lim_{x \rightarrow \infty} r_e(x) = \lim_{x \rightarrow \infty} \frac{\overline{F}(x)}{\overline{V}(x)} = 0; \quad (2.4)$$

$F \in \mathcal{M}^*$ iff $F \in \mathcal{E}_1$ and

$$\beta_F := \limsup_{x \rightarrow \infty} \epsilon(x) = \limsup_{x \rightarrow \infty} \frac{x \overline{F}(x)}{\overline{V}(x)} < \infty.$$

Clearly $\mathcal{M}^* \subset \mathcal{M}$. It has been proved in [14] and [15], that

$$\mathcal{D} \cap \mathcal{E}_1 \subset \mathcal{M}^*; \quad (\mathcal{L} \cup \mathcal{D}) \cap \mathcal{E}_1 \subset \mathcal{M}.$$

Moreover $F \in \mathcal{M} \iff F_e \in \mathcal{L}$, $F \in \mathcal{M}^* \iff F_e \in \mathcal{C} \iff F_e \in \mathcal{D}$ and

$$\mathcal{M}^* \subset \mathcal{M} \subset \mathcal{K}; \quad \mathcal{M}^* \setminus \mathcal{D} \neq \Phi; \quad \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{D}) \neq \Phi.$$

Therewithal, Class \mathcal{M} not only extends $(\mathcal{L} \cup \mathcal{D}) \cap \mathcal{E}_1$, but also gives us a chance to deeply investigate common properties of Class \mathcal{D} and Class \mathcal{L} . The following interesting example shows the necessity of introducing Classes \mathcal{M} :

Example 1. Assume that the random variable X has a geometry distribution with parameter $p \in (0, 1)$. Let $F(x) = P(X^r \leq x)$ for some $r > 1$, then it is easy to show that $F \in \mathcal{M}$ but $F \notin \mathcal{D} \cup \mathcal{L}$.

For our further discussion and use, a common method for constructing distribution $F \in \mathcal{M}$ is given below, where and throughout this paper, $h(x) \downarrow$ means that $h(x)$ is non-increasing.

Proposition 2.1. Assume that $h : [0, \infty) \rightarrow (0, \infty)$ satisfies

$$h(0) = 1; \quad 0 < h(x) \downarrow 0 \quad (0 < x \rightarrow \infty), \quad \int_0^\infty h(u) du = \infty.$$

Denote

$$\overline{F}(x) = 1, \quad x \leq 0; \quad \overline{F}(x) = h(x) \exp \left\{ - \int_0^x h(u) du \right\}, \quad x > 0.$$

Then $F(x) = 1 - \overline{F}(x)$ is a distribution function on $[0, \infty)$, $F \in \mathcal{M}$ and

$$\overline{V}(x) = \int_x^\infty \overline{F}(u) du = \exp \left\{ - \int_0^x h(u) du \right\}, \quad x \geq 0. \quad (2.5)$$

Furthermore, we have

$$\overline{F}_e(x) = \overline{V}(x).$$

The conclusion is clear. In fact, $\overline{F}(x)$ is non-increasing, $\overline{F}(0) = 1$ and $\overline{F}(\infty) = 0$. Thus $F(x) = 1 - \overline{F}(x)$ is a distribution supported on $[0, \infty)$. Since $h(x)$ is monotone, $h(x)$ is continuous almost sure with respect to Lebesgue measure. Hence

$$\frac{d}{dx} \exp \left\{ - \int_0^x h(u) du \right\} = -h(x) \exp \left\{ - \int_0^x h(u) du \right\} = -\overline{F}(x) = \frac{d}{dx} \overline{V}(x), \quad a.s.$$

which yields (2.5) and

$$r_e(x) = \frac{\overline{F}(x)}{\overline{V}(x)} = h(x) \rightarrow 0, \quad x \rightarrow \infty,$$

i.e. $F \in \mathcal{M}$. Furthermore, by (2.5),

$$\int_0^\infty u dF(u) = \int_0^\infty \overline{F}(u) du = \overline{V}(0) = 1,$$

which means that $\overline{F}_e(x) = \overline{V}(x)$. $\#$.

3. Some tail behaviors of heavy-tailed distributions

3.1. General discussion

It is well-known that for $F \in \mathcal{L}$ or $F \in \mathcal{D}$, we have (see, for example, [3])

$$\lim_{x \rightarrow \infty} e^{tx} \overline{F}(x) = \infty, \quad \forall t > 0. \quad (3.1)$$

This fact made some people believe that (3.1) is true for any distribution $F \in \mathcal{K}$ (see, for instance, p47 in [6] and section 2 of [4]). But this is wrong and we can find the following counterexample:

Example 1. Let $a_0 = 0, a_1 = 1, a_{n+1} > a_n$ and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$. Define

$$P(X = a_n) = \exp\{-a_{n-1}\} - \exp\{-a_n\}, \quad n \in \mathcal{N}.$$

Clearly, we have $\lim_{n \rightarrow \infty} a_n = \infty$ and

$$\overline{F}(x) = P(X > x) = P(X \geq a_n) = \exp\{-a_{n-1}\}, \quad a_{n-1} \leq x < a_n.$$

For any $t > 0, \delta > 0$, when n is large, $ta_n > 2a_{n-1}, a_{n-1} < a_n - \delta < a_n$. Denoting $x_n = a_n - \delta$, we have

$$\limsup_{x \rightarrow \infty} e^{tx} \overline{F}(x) \geq \lim_{n \rightarrow \infty} e^{tx_n} \overline{F}(x_n) = \lim_{n \rightarrow \infty} e^{tx_n - a_{n-1}} \geq e^{-t\delta} \lim_{n \rightarrow \infty} e^{a_{n-1}} = \infty,$$

Thus for any $t > 0, \int_0^\infty e^{tx} dF(x) = \infty$, which means that $F \in \mathcal{K}$. But it is easy to see that for any $0 < t < 1$, we have

$$\liminf_{x \rightarrow \infty} e^{tx} \overline{F}(x) \leq \lim_{n \rightarrow \infty} e^{ta_n} \overline{F}(a_n) = \lim_{n \rightarrow \infty} e^{(t-1)a_n} = 0,$$

i.e. (3.1) fails. $\#$

The example motivates us try to find some criterions under which (3.1) hold. At first, we give a proposition:

Proposition 3.1. *Assume that*

$$d_F(y) = \limsup_{x \rightarrow \infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty, \quad \forall y > 0, \quad (3.2)$$

then for any

$$t > \nu_F := \inf_{y>0} \frac{\log d_F(y)}{y}, \quad (3.3)$$

we have

$$\lim_{x \rightarrow \infty} e^{tx} \overline{F}(x) = \infty.$$

P r o o f. For $t > \nu_F$, there is $y_0 > 0$ such that $t > \frac{\log d_F(y_0)}{y_0}$. Since $d_F(y_0) < \infty$, for any fixed $\varepsilon > 0$, there exists $0 < x_0 = x_0(\varepsilon) < \infty$ such that

$$\sup_{x \geq x_0} \frac{\overline{F}(x)}{\overline{F}(x+y_0)} < d_F(y_0) + \varepsilon.$$

Especially, for any $n \in \mathcal{N}$,

$$\frac{\overline{F}(x_0)}{\overline{F}(x_0 + ny_0)} < (d_F(y_0) + \varepsilon)^n, \quad R(x_0 + ny_0) \leq n \log(d_F(y_0) + \varepsilon) + R(x_0),$$

where $R(x) := -\log \overline{F}(x)$. Since $R(x)$ is non-decreasing, for any $x \in [x_0 + ny_0, x_0 + (n+1)y_0)$, we get

$$\frac{R(x)}{x} \leq \frac{(n+1) \ln(d_F(y_0) + \varepsilon) + R(x_0)}{x_0 + ny_0}, \quad (3.4)$$

Hence $\limsup_{x \rightarrow \infty} \frac{R(x)}{x} \leq \frac{1}{y_0} \ln(d_F(y_0) + \varepsilon)$. By the randomness of ε , we get $\mu_F := \limsup_{x \rightarrow \infty} \frac{R(x)}{x} \leq \frac{1}{y_0} \ln d_F(y_0)$. Thus $t > \frac{1}{y_0} \ln d_F(y_0) \geq \mu_F$, i.e. there exists $\delta > 0$ such that $\mu_F < t - 2\delta$. So for large x , $tx - R(x) \geq \delta x$, which shows

$$\lim_{x \rightarrow \infty} e^{tx} \overline{F}(x) = \lim_{x \rightarrow \infty} \exp\{tx - R(x)\} \geq \lim_{x \rightarrow \infty} \exp\{\delta x\} = \infty. \quad \#$$

Clearly, $\nu_F = 0$ for $F \in \mathcal{L}$. Thus

Corollary 1. (3.1) holds for $F \in \mathcal{L}$.

3.2. The case of Class \mathcal{M}

Noting that in Example 1, we have $\lim_{n \rightarrow \infty} \overline{F}(a_n)/\overline{V}(a_n) = 1$, which means $F \notin \mathcal{M}$. A problem will be put forward that ‘‘whether (3.1) holds for all $F \in \mathcal{M}$?’’ At first, we introduce two symbols which are useful in next discussion:

$$\Gamma_F := \left\{ \gamma \left| \int_0^\infty x^\gamma dF(x) < \infty \right. \right\}, \quad \gamma_F := \sup \{ \gamma \mid \gamma \in \Gamma_F \}. \quad (3.5)$$

From the definitions in section 2, we have that $\gamma_F \geq 1$ holds for $F \in \mathcal{M}$, and $1 \leq \gamma_F < \infty$ for $F \in \mathcal{M}^*$.

We discover an interesting property for $F \in \mathcal{M}$:

Theorem 1. *For $F \in \mathcal{M}$ with $\gamma_F > 1$, (3.1) holds; but for F with $\gamma_F = 1$, (3.1) doesn't always hold.*

A counterexample for $F \in \mathcal{M}$ with $\gamma_F = 1$ (even an example $F \in \mathcal{M}^*$) is as follows:

Example 2. *The example is constructed by the method in Proposition 2.1. Let*

$$a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{n}e^{a_n}, \quad n \in \mathcal{N},$$

then $a_n > 0$ is strictly increasing and $a_n \rightarrow \infty$. Assume

$$h(x) = 1, \quad 0 \leq x < a_1; \quad h(x) = e^{-a_n}, \quad a_n \leq x < a_{n+1}, \quad n \in \mathcal{N}.$$

Clearly $\int_0^\infty h(u)du = 1 + \sum_{n=1}^\infty \frac{1}{n} = \infty$ and $h(x)$ satisfies the conditions in Proposition 2.1. If we define non-negative distribution F as that in Proposition 2.1, then $F \in \mathcal{M}$ and $\frac{\overline{F}(x)}{\overline{V}(x)} = r_\epsilon(x) = h(x)$. For $0 < t < 1$, we have

$$\liminf_{x \rightarrow \infty} e^{tx} \overline{F}(x) \leq \liminf_{x \rightarrow \infty} e^{tx} h(x) \leq \lim_{n \rightarrow \infty} e^{ta_n} h(a_n) = \lim_{n \rightarrow \infty} e^{(t-1)a_n} = 0,$$

which shows that (3.1) doesn't hold. On the other hand,

$$\limsup_{x \rightarrow \infty} \frac{x \overline{F}(x)}{\overline{V}(x)} = \limsup_{x \rightarrow \infty} x h(x) \leq \lim_{n \rightarrow \infty} a_{n+1} h(a_n) = \lim_{n \rightarrow \infty} (a_n e^{-a_n} + \frac{1}{n}) = 0,$$

which means that $F \in \mathcal{M}^*$. By a conclusion in [13], $\gamma_F = 1$. #

This example shows that the condition " $\gamma_F > 1$ " in Theorem 1 can't be cancelled.

Remark. Theorem 1 shows that distributions F with $\gamma_F = 1$ behave strangely. This has never been noticed before.

In order to prove the conclusion in Theorem 1 concerning $F \in \mathcal{M}$ with $\gamma_F > 1$, we first give the following lemma.

Lemma 3.1. *If $\gamma_F > 1$, then for any $0 < p < 1 - \frac{1}{\gamma_F}$, we have*

$$\lim_{x \rightarrow \infty} \frac{\overline{F}^p(x)}{\overline{V}(x)} = \lim_{x \rightarrow \infty} \frac{\overline{F}^p(x)}{\int_x^\infty \overline{F}(u)du} = \infty. \quad (3.6)$$

P r o o f. For any $0 < p < 1 - \frac{1}{\gamma_F}$, there exists some $r < \gamma_F$ such that $r(1-p) > 1$. Choose $\delta = 1 - p$ and $\rho := 2^{1-r\delta} < 1$. It is well-known that for distribution F supported on $[0, \infty)$, its r th moment exists if and only if $\sum_{k=1}^\infty 2^{kr} \overline{F}(2^k) < \infty$. Then for large k , we have $2^{kr} \overline{F}(2^k) < 1$, which shows that $2^{\frac{\delta}{k}} \overline{F}(2^k) < 2^{1-r\delta} = \rho$. Hence for large n and $2^{n-1} \leq x < 2^n$, we have

$$\begin{aligned} \frac{\overline{V}(x)}{\overline{F}^p(x)} &= \frac{\int_x^\infty \overline{F}(u)du}{\overline{F}^{1-\delta}(x)} \leq \int_x^\infty \overline{F}^\delta(u)du \leq \sum_{k=n-1}^\infty \int_{2^k}^{2^{k+1}} \overline{F}^\delta(u)du \\ &\leq \sum_{k=n-1}^\infty 2^k \overline{F}^\delta(2^k) = \sum_{k=n-1}^\infty \left\{ 2^{\frac{\delta}{k}} \overline{F}(2^k) \right\}^k < \sum_{k=n-1}^\infty \rho^k = \frac{\rho^n}{\rho(1-\rho)}. \end{aligned}$$

Let $x \rightarrow \infty$ (so $n \rightarrow \infty$), then (3.6) holds. #

Remark. Combining the discussion in [13], we obtain

$$(3.6) \text{ holds for any } 0 < p < 1 \iff \gamma_F = \infty.$$

P r o o f of Theorem 1 (the case $\gamma_F > 1$): Let $\gamma_F > 1$ and $0 < p < 1 - \frac{1}{\gamma_F}$, then (3.6) holds by Lemma 3.1. For any $t > 0$, write $u = \frac{p t}{1-p}$ and

$$\frac{e^{tx}\overline{F}(x)}{\overline{V}(x)} = \left\{ \frac{\overline{F}^p(x)}{\overline{V}(x)} \right\}^{1/p} \cdot e^{tx\overline{V}^{\frac{1-p}{p}}(x)} = \left\{ \frac{\overline{F}^p(x)}{\overline{V}(x)} \right\}^{1/p} \{e^{ux}\overline{V}(x)\}^{\frac{1-p}{p}}. \quad (3.7)$$

By the fact that $F \in \mathcal{M} \Leftrightarrow F_e \in \mathcal{L}$ (see [14] or [15]), and noticing the definition of $\overline{V}(x)$, it follows from the property of Class \mathcal{L} that $\lim_{x \rightarrow \infty} e^{ux}\overline{V}(x) = \infty$, which together with (3.6) and (3.7) yields

$$\lim_{x \rightarrow \infty} \frac{e^{tx}\overline{F}(x)}{\overline{V}(x)} = \infty, \quad \forall t > 0. \quad (3.8)$$

Writing $0 < t = 2s$, by (3.8), we have $e^{sx}\overline{F}(x) \geq \overline{V}(x)$ for large x . By the property of Class \mathcal{L} again,

$$\lim_{x \rightarrow \infty} e^{tx}\overline{F}(x) = \lim_{x \rightarrow \infty} e^{2sx}\overline{F}(x) \geq \lim_{x \rightarrow \infty} e^{sx}\overline{V}(x) = \infty, \quad \forall t > 0.$$

Thus (3.1) holds. #

4. The tail behaviors of Class \mathcal{M}^*

Except the misunderstanding on tail behaviors of heavy-tailed distributions, which is mentioned in the last section, another one is the problem that for $F \in \mathcal{D}$, whether there exists some $t > 0$ such that

$$\lim_{x \rightarrow \infty} x^t \overline{F}(x) = \infty? \quad (4.1)$$

For instance, In order to obtained results on rates of approximations to ruin probabilities, Mikosch and Nagaev [7] assume that

“ $F \in \mathcal{D}$ has non-zero and finite expectation μ , and there exists some $\alpha > 0$ such that

$$\liminf_{x \rightarrow \infty} x^\alpha \overline{F}(x) > 0.” \quad (4.2)$$

In fact, according to the following Proposition 4.1, their assumption is unnecessary since all distributions in Class \mathcal{D} satisfy the condition showed in (4.1) for some $t > 0$.

Proposition 4.1. *For any $F \in \mathcal{D}$, there exists some $t > 0$ such that (4.1) holds.*

The detailed proof of Proposition 4.1 can be found in [13].

As for Class \mathcal{M}^* , we also have the following theorem.

Theorem 1. *For $F \in \mathcal{M}^*$ with $\gamma_F > 1$, we have*

$$\lim_{x \rightarrow \infty} x^\gamma \overline{F}(x) = \infty, \quad \forall \gamma > \frac{\gamma_F \beta_F}{\gamma_F - 1}, \quad (4.3)$$

where γ_F is defined in (3.5).

Remark. It is similar to Theorem 1 that the conclusion doesn't always hold for $F \in \mathcal{M}^*$ with $\gamma_F = 1$. We shall give an example later, which is another illustration of the abnormality of non-negative distribution with $\gamma_F = 1$.

P r o o f of Theorem 1: For $0 < p < 1 - \frac{1}{\gamma_F}$, (3.6) holds by Lemma 3.1. Choose $u > \beta_F$ and write $t = \frac{(1-p)u}{p}$, then

$$\frac{x^t \overline{F}(x)}{\overline{V}(x)} = \left\{ \frac{\overline{F}^p(x)}{\overline{V}(x)} \right\}^{1/p} \cdot x^t \overline{V}^{\frac{1-p}{p}}(x) = \left\{ \frac{\overline{F}^p(x)}{\overline{V}(x)} \right\}^{1/p} \{x^u \overline{V}(x)\}^{\frac{1-p}{p}}.$$

Noticing $F_e = \frac{1}{\overline{V}(0)}V \in \mathcal{D}$ and using (3.6) and Proposition 4.1, we get

$$\lim_{x \rightarrow \infty} \frac{x^t \overline{F}(x)}{\overline{V}(x)} = \infty, \quad (4.4)$$

which shows that $x^t \overline{F}(x) > \overline{V}(x)$ holds for large x . Thus if $\gamma > \frac{\beta_F}{p}$, denoting $\gamma = \frac{(1-p)u}{p} + u = t + u$, then we have $u > \beta_F$ and

$$\lim_{x \rightarrow \infty} x^\gamma \overline{F}(x) = \lim_{x \rightarrow \infty} x^{t+u} \overline{F}(x) \geq \lim_{x \rightarrow \infty} x^u \overline{V}(x) = \infty.$$

To summarize, (4.3) holds for $\gamma = t + u > \frac{\gamma_F \beta_F}{\gamma_F - 1}$. #

Remark. By [13], we have $\gamma_F \leq \beta_F + 1$. When $\gamma_F = \beta_F + 1$, Theorem 1 holds for all $\gamma > \gamma_F$, which shows that the bound of γ is precise. And it is easy to get $\frac{\gamma_F \beta_F}{\gamma_F - 1} \geq \beta_F + 1$.

As follows, we give a counterexample to explain that the conclusion of Theorem 1 doesn't always hold for " $F \in \mathcal{M}^*$ with $\gamma_F = 1$."

Example 1. Let us use the method in Proposition 2.1 to construct counterexample. Let $a_1 = 2$, $a_{n+1} = a_n + \frac{1}{n}a_n^n$, $n \geq 1$ and define

$$h(x) = 1, \quad x \in [0, a_1]; \quad h(x) = a_n^{-n}, \quad x \in [a_n, a_{n+1}), \quad n \geq 1.$$

Clearly $0 < h(x) \downarrow 0$, $\int_{a_n}^{a_{n+1}} h(x) dx = \frac{1}{n}$, which yields $\int_0^\infty h(x) dx = \infty$. Thus $h(x)$ satisfies the conditions in Proposition 2.1. If we define $F \in \mathcal{M}$ by the method in Proposition 2.1, then F has expectation and $\frac{\overline{F}(x)}{\overline{V}(x)} = r_e(x) = h(x)$. Since $\limsup_{x \rightarrow \infty} xh(x) \leq \lim_{n \rightarrow \infty} a_{n+1}h(a_n) = 0$, i.e.

$$\beta_F = \limsup_{x \rightarrow \infty} \frac{x \overline{F}(x)}{\overline{V}(x)} = \limsup_{x \rightarrow \infty} x r_e(x) = 0,$$

we obtain $F \in \mathcal{M}^*$. And from [13], we get $\gamma_F = 1$. But for any $\gamma > 0$,

$$\liminf_{x \rightarrow \infty} x^\gamma \overline{F}(x) \leq \liminf_{x \rightarrow \infty} x^\gamma h(x) \leq \lim_{n \rightarrow \infty} a_n^r h(a_n) = \lim_{n \rightarrow \infty} a_n^{r-n} = 0. \quad \#$$

From this example, we have

Proposition 4.2. *There exists $F \in \mathcal{M}^*$ with $\gamma_F = 1$ such that $\lim_{x \rightarrow \infty} x^\gamma \overline{F}(x) = \infty$ doesn't hold for any $\gamma > 0$.*

Список литературы

1. *Asmussen, S.* Ruin Probabilities. World Scientific, Singapore, 2000.
2. *Bingham N. H., Goldie C. M., Teugels J. L.* Regular Variation. Cambridge: Cambridge University Press, 1987.

3. *Embrechts P., Klüppelberg C., Mikosch T.* Modelling Extremal Events for Insurance and Finance, Springer, Berlin, 1997.
4. *Greiner M, Jobmman M & Klüppelberg C.* (1999): Telecommunication Traffic, Queueing Models and Subexponential Distributions. Queueing Systems, 1999, 33. P. 125–152.
5. *Jelenković, P.R., Lazar, A.A.* Asymptotic results for multiplexing subexponential on-off processes. Adv. Appl. Prob. 31, 1999. P 394-421.
6. *Kalashnikov V.* Geometric Sums Bounds for Rare Events with Applications: Risk Analysis, Reliability, Queueing, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.
7. *Mikosch T., Nagaev A.* Rates in Approximations to Ruin Probabilities for Heavy-Tailed Distributions, Extremes, 2001, 4(1): P. 67–78.
8. *Ng, K., Tang, Q., Yang, H.* Maxima of sums of heavy-tailed random variables. Astin Bull. 32, № 1, 2002. P 43–55.
9. *Ng K., Tang Q., Yan J., Yang H.* 2003_a. Precise large deviations for the prospective-loss process. J. Appl. Probab. 40, № 2. P 391–400.
10. *Ng K., Tang Q., Yan J., Yang, H.* 2003_b. Precise large deviations for sums of random variables with consistently varying tails. Adv. in Appl. Probab., to appear.
11. *Rolski, T., Schmidli, H., Schmidt, V., Teugels, J.* Stochastic processes for insurance and finance. John Wiley & Sons, Chichester, England, 1999.
12. *Schlegel, S.* Ruin probabilities in perturbed risk models. The interplay between insurance, finance and control (Aarhus, 1997). Insurance Math. Econom. 22. № 1,1998. P. 93–104.
13. *Su C., Hu Z. S., Tang Q. H.* Characterizations on heaviness of distribution tails of non-negative variables, Adv. Math.(China). 2003, 32, P. 606–614.
14. *Su Chun, Tang Qihe.* Characterizations on Heavy-tailed Distributions by Means of Hazard Rate, Acta Mathematicae Applicatae Sinica, English Series, 2003, 19(1). P. 135–142.
15. *Su Chun, Tang Qihe, Chen Yu, Liang Hanying* Two Broad Classes of Heavy-Tailed Distributions and Their Applications to Insurance. To appear.
16. *Tang, Q. H.* (2002) Moments of the deficit at ruin in the renewal model with heavy-tailed claims. Science in China (Series A), to appear.

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Chun Su, Zhishui Hu On two broad classes of heavy-tailed distributions. Far Eastern Mathematical Journal. 2004. V. 5. № 2. P. 195–204.

ABSTRACT

Since the introduction of Class \mathcal{M} and Class \mathcal{M}^* , they have played important roles in insurance to describe tail equivalence of ruin probability and tail behavior of the deficit at ruin. And in insurance and finance most of heavy-tailed distributions with finite and positive expectations belong to Class \mathcal{M} . So it is important to study tail behaviors of Class \mathcal{M} and Class \mathcal{M}^* . In this paper, we obtain some results on essential tail behaviors of these two classes.

Key words: *class \mathcal{M} , class \mathcal{M}^* , class \mathcal{D} , tail behaviors*