ABSTRACTS

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Integrable Geodesic Flows on 2-Surfaces.

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The problem of an integrability of Hamiltonian systems usually reduces to searching for the first integrals of motion. In the talk we will consider geodesic flows on 2-surfaces (including magnetic ones) and will discuss some questions related to existence and nonexistence of such integrals.

Weak and Ultrastrong Coupling Limits of the Quantum Mean Force Gibbs State

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The Gibbs state is widely taken to be the equilibrium state of a system in contact with an environment at temperature T. However, nonnegligible interactions between system and environment can give rise to an altered state. Here we derive general expressions for this mean force Gibbs state, valid for any system that interacts with a bosonic reservoir. First, we derive the state in the weak coupling limit and find that, in general, it maintains coherences with respect to the standard system Hamiltonian. Second, we derive the explicit form of the state in the ultrastrong coupling limit, and show that it becomes diagonal in the basis set by the system-reservoir interaction instead of the system Hamiltonian. Several examples are discussed including a single qubit, a three-level V-system and two coupled qubits all interacting with bosonic reservoirs. The results shed light on the presence of coherences in the strong coupling regime, and provide key tools for nanoscale thermodynamics investigations.

Dynamical properties of differential equations with a discontinuous right-hand side and the principle of quasi-invariance

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Let R^n be an n-dimensional real linear space. Let $(\cdot)'$ be a transpose operation. Let $x = (x_1, x_2, \ldots, x_n)'$ be a vector of R^n. Denote by $|x|$ the vector norm in R^n, $R = (-\infty, \infty)$.

Consider the differential equation

$$\dot{x} = g(t, x),$$

where the right-hand side is the function g defined in some domain $R \times D$, the set $D \subset R^n$ can be represented as $D = D_0 \cup M$, $D_0 = D_1 \cup D_2 \cup \ldots \cup D_l$. The sets D_i $(i = 1, 2, \ldots, l)$ are disjoint subdomains of D. The set M of zero measure consists of the boundaries of D_i $(i = 1, 2, \ldots, l)$. In each subdomain $R \times D_j$ $(j = 1, 2, \ldots, l)$, the function g is continuous.

Assume that for each fixed point $t \in R$, the function g has a finite limit, i.e. $g(t, x_k) \to g_0 = constant$ as $x_k \to x_0 \in M$, where the value of the vector g_0 depends on the choice of the sequence $x_k \to x_0$. Thus, at each time point $t \in R$, M is the set of discontinuity of the function g. Let for each point $(t, x) \in R \times D$, $G(t, x)$ be the smallest convex closed set containing all limit values of the function $g(t, x_k)$, where $x_k \in D_0$, $x_k \to x$ as $k \to \infty$. For equation (1) one can define the differential inclusion [1]

$$\dot{x} \in G(t, x) \tag{1}$$

Assume that the right-hand side of (1) satisfies the conditions: for each compact sets $K \subset D$ and $K_j \subset D_j$ $(j = 1, 2, \ldots, l)$ there exist the constants $m = m(K)$ and $L_j = L_j(K_j)$ $(j = 1, 2, \ldots, l)$ such that $|g(t, x)| \leq m$, $|g(t, x_2) - g(t, x_1)| \leq L_j |x_2 - x_1|$ for all $(t, x) \in R \times K$ and all $(t, x_1), (t, x_2) \in R \times K_j$.

Under these conditions, for equation (1) in each domain $R \times D_j$ one can construct a family of limiting equations [2]

$$\dot{x} = g^*(t,x),\; g^*(t,x) = \frac{d}{dt} \lim_{t_k \to \infty} \int\limits_0^t g(t_k + \tau, x) d\tau. \qquad (2)$$

Let construct a set of limiting equations (2) for each subdomain $D_j \subset D$ $(j = 1, 2, \ldots, l)$ and define the general set of limiting equations for the domain $D_0 = D_1 \cup D_2 \cup \ldots \cup D_l$ according to the following definition.

Definition 1. *Equation (2) defined in the domain* $D_0 = D_1 \cup D_2 \cup \ldots \cup D_l$ *is called limiting one to (1) in relation to the sequence* $t_k \to +\infty$*, if it is defined as the limit to (1) for this sequence in each domain* D_j $(j = 1, 2, \ldots, l)$*.*

Let some equation (2) be a limiting one for (1) with respect to the sequence $t_k \to \infty$. Similarly to the previous one, for each point $(t, x) \in R \times D$ define by $G^*(t, x)$ the smallest convex set containing the sets $\{g(t_k + t, x_k) : \ t_k \to \infty,\ x_k \to x \in D\}$ and $\{g^*(t, x_k) : \ x_k \to x \in D\}$. Define the limiting inclusion as follows

$$\dot{x} \in G^*(t, x). \qquad (3)$$

The following dynamic property of equation (1) is proved.

Theorem 1. *Let* $x = x(t)$ *be some solution of (1) bounded for all* $t \geq t_0$ *by some compact set* $K \subset D$*,* $\{x(t),\ t \geq t_0\} \subset K$*. Then, the positive limit set* $\omega^+(x(t))$ *of this solution is weakly quasi-invariant, namely, for each point* $p \in \omega^+(x(t))$*, there exists a limiting inclusion (2) and some of its solution* $x = x^*(t)$ *such that* $x^*(0) = p$*,* $\{x^*(t),\ t \in R\} \subset \omega^+(x(t))$*.*

The development of the semi-definite Lyapunov functions method is obtained in the study of stability properties for equations (1).

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Phase Portraits Belonging to Polynomial Dynamic Systems

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A class of polynomial dynamic systems have been studied on an arithmetical plane of their phase variables x and y.

$$\frac{dx}{dt} = X(x,y), \quad \frac{dy}{dt} = Y(x,y),$$

where $X(0,1)Y(0,1) \neq 0$.

Their right parts are considered to be mutually reciprocal polynomial forms of the third and the second degrees. The class includes a limitless set of systems [1]. A task is to provide the full qualitative investigation of this class of systems and reveal all topologically different phase portraits possible for them in the Poincare circle. The total class was subdivided into subclasses of several successive levels. The example of a subclass belonging to the first level of hierarchy is given below [2]:

$$X(x,y) = p_3(y - u_1x)(y - u_2x)(y - u_3x),$$
$$Y(x,y) = c(y - q_1x)(y - q_2x),$$

where $p_3 > 0$, $c > 0$, $u_1 < u_2 < u_3$, $q_1 < q_2$, $u_i \neq q_j$ for every i and j. The first level of subclasses is formed in accordance with numbers and multiplicities of roots of especially introduced for both forms $X(x,y), Y(x,y)$ characteristic polynomials $P(u)$, $Q(u)$, which for the given example look like [3]:

$$P(u) := X(1,u) \equiv p_3(u - u_1)(u - u_2)(u - u_3),$$
$$Q(u) := Y(1,u) \equiv c(u - q_1)(u - q_2).$$

We take into consideration the $RSPQ$ an ascending sequence of all real roots of polynomials $P(u), Q(u)$. For instance, for this example 10 different types of $RSPQ$ are possible. They allow to make further steps of subdivision.

There are 9 different subclasses located at the first hierarchical level. For different branches of subdivision 3 or 4 different numbers of hierarchical levels were found. As a result of alternate study of all subclasses of all possible levels more than two hundred of topologically different and independent phase portraits of the systems were revealed and described in the two ways: with a special characteristic table, as well as in a graphical form in a Poincare circle. Every existing topological type of a portrait is accompanied by criteria of its appearance. E. g., for the above mentioned example of a system of the first level there exist 93 topologically different types of phase portraits. For the aims of these portraits investigation in a Poincare disk, a series of new methodologies and attitudes were specially introduced. Resorting to the Bendixon formula for the index of a singular point, the whole totality of considered dynamic systems have no limit cycles [1, 2]. All the used new methods will find their introductions and applications in further theoretical and applied investigations of polynomial - first of all, cubic and quadratic - dynamic systems, and in their use as mathematical models of physical, biological, economic and sociological processes [1 - 3].

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Ring Isomorphisms of Murray–von Neumann Algebras

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In 1930-s, motivated by the geometry of lattice of the projections of type II_1 factors, von Neumann built the theory on the correspondence between complemented orthomodular lattices and regular rings. He proved that given two regular rings $\mathcal{R}$ and $\mathcal{R}_1$, their lattices of projections are lattice-isomorphic if and only if there exists a ring isomorphism between $\mathcal{R}$ and $\mathcal{R}_1$ which generates the given lattice isomorphism. (J. von Neumann, Continuous geometry. Princeton Mathematical Series, No. 25 Princeton University Press, Princeton, N.J., 1960). Thus, a problem arises to describe ring isomorphisms between regular rings.

The present talk is devoted to study of ring isomorphisms of the algebras of measurable operators affiliated with von Neumann algebras of type II_1, usually referred as Murray–von Neumann algebras, and also of their certain $*$-subalgebras.

Let $\mathcal{M}$, $\mathcal{N}$ be von Neumann algebras of type II_1, and let $S(\mathcal{M})$, $S(\mathcal{N})$ be the $*$-algebras of all measurable operators affiliated with $\mathcal{M}$ and $\mathcal{N}$, respectively. Suppose that $\mathcal{A} \subset S(\mathcal{M})$, $\mathcal{B} \subset S(\mathcal{N})$ are their $*$-subalgebras such that $\mathcal{M} \subset \mathcal{A}$, $\mathcal{N} \subset \mathcal{B}$. We prove that for every ring isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ there exist a positive invertible element $a \in \mathcal{B}$ with $a^{-1} \in \mathcal{B}$ and a real $*$-isomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ (which extends to a real $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$) such that $\Phi(x) = a\Psi(x)a^{-1}$ for all $x \in \mathcal{A}$. In particular, Φ is automatically real-linear and continuous in the measure topology.

In particular, noncommutative Arens algebras and noncommutative $\mathcal{L}_{log}$-algebras associated with von Neumann algebras of type II_1 satisfy the above conditions and, therefore, the main result implies the automatic continuity of their ring isomorphisms in the corresponding metrics. We also present an example of a $*$-subalgebra in $S(\mathcal{M})$, which shows that the condition $\mathcal{M} \subset \mathcal{A}$ is essential in the above mentioned result.

Three-Dimensional Model for the Steady-State Fluid Flow in a Pipe Network: Existence Analysis

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We study a mathematical model for the steady-state flow of a fluid with shear-dependent viscosity in a netlike domain $\widetilde{P} = P \cup Q$, where $P \stackrel{\text{def}}{=} \cup_{i=1}^{N} P_i$, $Q \stackrel{\text{def}}{=} \cup_{j=1}^{M} Q_j$, P_i and Q_j are locally Lipschitz bounded domains in $\mathbf{R}^3$ such that $P_i \cap Q_j = \emptyset$ for any $i \in \{1, 2, \dots, N\}$, $j \in \{1, 2, \dots, M\}$, $\overline{P}_i \cap \overline{P}_k = \emptyset$ for any $i, k \in \{1, 2, \dots, N\}$, $i \neq k$, and $\overline{Q}_j \cap \overline{Q}_l = \emptyset$ for any $j, l \in \{1, 2, \dots, M\}$, $j \neq l$. Note that $\widetilde{P}$ can be considered as a network of pipes: $P_1, \dots, P_N$ model pipes, while $Q_1, \dots, Q_M$ represent junctions in which pipes are connected.

Assume that the boundary of any junction Q_j has only one connected component and there exist exactly m_j domains P_{j_1}, $P_{j_2}, \dots, P_{j_{m_j}}$, $1 \leq j_1 < j_2 < \dots < j_{m_j} \leq N$, $1 \leq m_j \leq N$, such that $\overline{Q}_j \cap \overline{P}_{j_k} \neq \emptyset$ for any $k \in \{1, 2, \dots, m_j\}$.

If $m_j \geq 2$, then we say that the junction Q_j is *interior*; in the case $m_j = 1$, the junction Q_j is called *open*. Let Q_{int} be the union of all interior junctions and $\mathcal{J} \stackrel{\text{def}}{=} \{j : m_j \geq 2\}$.

Suppose $S_{jk} \stackrel{\text{def}}{=} \overline{Q}_j \cap \overline{P}_{j_k}$ is a flat surface and, for each P_i, there exist exactly two junctions Q_{i_1} and Q_{i_2} such that $\overline{P}_i \cap \overline{Q}_{i_1} \neq \emptyset$ and $\overline{P}_i \cap \overline{Q}_{i_2} \neq \emptyset$. Obviously, for each $i \in \{1, 2, \dots, N\}$, there exists a uniquely determined pair (i_1', i_2') such that $\overline{P}_i \cap \overline{Q}_{i_1} = S_{i_1 i_1'}$ and $\overline{P}_i \cap \overline{Q}_{i_2} = S_{i_2 i_2'}$. Let $\Gamma_i \stackrel{\text{def}}{=}$ $\stackrel{\text{def}}{=} \partial P_i \setminus (S_{i_1 i_1'} \cup S_{i_2 i_2'})$ and $\Gamma \stackrel{\text{def}}{=} \cup_{i=1}^{N} \Gamma_i$.

We consider a nonlinear system that describes 3D flows of an incompressible fluid with shear-dependent viscosity in $\widetilde{P}$:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} - \operatorname{div} \mathbf{T} = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } P, \tag{1}$$

$$\mathbf{T} = \mu(|\mathbf{D}(\mathbf{u})|)\mathbf{D}(\mathbf{u}) - \pi\mathbf{I} \quad \text{in } P, \tag{2}$$

$$(\mathbf{v}\cdot\nabla)\mathbf{v} - \operatorname{div}\mathbf{T} = \mathbf{f}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_{\text{int}}, \tag{3}$$

$$\mathbf{T} = \mu(|\mathbf{D}(\mathbf{v})|)\mathbf{D}(\mathbf{v}) - \pi\mathbf{I} \quad \text{in } Q_{\text{int}}, \tag{4}$$

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad (\mathbf{T}\mathbf{n})_\tau = -\alpha(\mathbf{x}, |\mathbf{u}_\tau|)\mathbf{u}_\tau \quad \text{on } \Gamma_i \quad \forall\, i \in \{1, 2, \dots, N\}, \tag{5}$$

$$\mathbf{u}_\tau = \mathbf{0}, \quad |\mathbf{u}|^2/2 + \pi = h_i \quad \text{on } S_{i_1 i_1'} \cup S_{i_2 i_2'} \quad \forall\, i \in \{1, 2, \dots, N\}, \tag{6}$$

$$\mathbf{v} = \mathbf{u} \text{ on } S_{jk} \;\; \forall j \in \mathcal{J} \text{ and } k \in \{1, 2, \dots, m_j\}, \tag{7}$$

$$\mathbf{v} = \mathbf{0} \text{ on } \partial Q_j \setminus \left(\cup_{k=1}^{m_j} S_{jk}\right) \;\; \forall j \in \mathcal{J}, \tag{8}$$

$$\sum_{k=1}^{m_j} \int_{S_{jk}} \mathbf{u}\cdot\mathbf{n}\, d\sigma = 0 \quad \forall\, j \in \mathcal{J}, \tag{9}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x})$ is the flow velocity in the pipes, $\mathbf{v} = \mathbf{v}(\mathbf{x})$ is the flow velocity in the junctions, $\mathbf{T} = (\mathrm{T}_{ij}(\mathbf{x}))$ is the Cauchy stress tensor, $\pi = \pi(\mathbf{x})$ is the pressure, $\mathbf{f} = \mathbf{f}(\mathbf{x})$ is the external force field, $\mathbf{D}(\mathbf{u})$ is the deformation rate tensor, $\mathbf{D}(\mathbf{u}) = \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top\right)/2$, $\mu(|\mathbf{D}(\mathbf{u})|) > 0$ is the viscosity coefficient, $\alpha(\mathbf{x}, |\mathbf{u}_\tau|) > 0$ is the slip coefficient, the function $h_i = h_i(\mathbf{x})$ describes the dynamic pressure on $S_{i_1 i_1'}$ and $S_{i_2 i_2'}$, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the outer (with respect to P_i) unit normal to ∂P_i, and $\mathbf{w}_\tau \stackrel{\text{def}}{=} \mathbf{w} - (\mathbf{w}\cdot\mathbf{n})\mathbf{n}$.

Assume that the following conditions hold:

(H1) $\mathbf{f} \in \mathbf{L}^2(\widetilde{P})$ and $h_i \in \mathrm{L}^2(S_{i_1 i_1'} \cup S_{i_2 i_2'})$ for any $i = 1, \dots, N$;

(H2) the functions α and μ are continuous and there exist constants α_0, α_1, μ_0, μ_1 such that $0 < \alpha_0 \le \alpha(\mathbf{x}, y) \le \alpha_1$ and $0 < \mu_0 \le \mu(y) \le \mu_1$ for any $\mathbf{x} \in \Gamma$ and $y \in [0, +\infty)$;

(H3) $(\mu(|\mathbf{A}|)\mathbf{A} - \mu(|\mathbf{B}|)\mathbf{B}) : (\mathbf{A} - \mathbf{B}) \ge 0$ for any symmetric 3×3-matrices $\mathbf{A}$ and $\mathbf{B}$.

Our main result is the following existence theorem.

Theorem. *Under conditions* **(H1)**–**(H3)**, *problem* (1)–(9) *has at least one weak solution* $(\mathbf{u}, \mathbf{v})$ *and the energy equality holds:*

$$\sum_{i=1}^{N} \int_{P_i} \mu(|\mathbf{D}(\mathbf{u})|)|\mathbf{D}(\mathbf{u})|^2 \, \mathbf{dx} + \sum_{i=1}^{N} \int_{\Gamma_i} \alpha(\mathbf{x}, |\mathbf{u}|)|\mathbf{u}|^2 \, d\sigma$$

$$+ \sum_{i=1}^{N} \int_{S_{i_1 i_1'} \cup S_{i_2 i_2'}} h_i \mathbf{u} \cdot \mathbf{n} \, d\sigma = \sum_{i=1}^{N} \int_{P_i} \mathbf{f} \cdot \mathbf{u} \, \mathbf{dx}.$$

The proof of this theorem is given in [1].

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The Effective Hamiltonian as a Necessary Basis of the Quantum Open System Theory

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Based on a model open system, the kinetic equation obtained from the initial exact Hamiltonian with anti-rotating terms is shown to differ from the kinetic equation from the approximate effective Hamiltonian without anti-rotating terms. The paper proves the necessity of transition from the exact Hamiltonian to the effective Hamiltonian, with the main idea of its construction being the requirement for the absence of rapidly varying terms in the interaction representation. This requirement is necessary to obtain the correct kinetic equation. The method for obtaining the effective

Hamiltonian is the algebraic perturbation theory. The first order on coupling constant of the algebraic perturbation theory gives the rotating wave approximation with the rigorous conditions of its validity. The second order of the algebraic perturbation theory takes into account the Stark interaction of the quantum vacuum (the fact that is ignored in other theories) and generalizes various relaxation dynamics of atomic ensemble. Namely, it states the effect of stabilization of the excited state related to collective decay (superradiance) in case of a certain number of atoms in ensemble. The algebraic perturbation theory (including the second order) is shown to be effectively supplemented by the quantum stochastic differential equation method. The stochastic equation is shown to be governed by both the quantum creation-annihilation and counting processes, with their Ito's differentials obeyed by the Hudson-Parthasarathy algebra.

On the Existence of Bounded Soliton Solutions in the Problem of Longitudinal Vibrations of an Elastic Infinite Rod in a Field with a Nonlinear Potential of General Form[1]

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In the theory of plastic deformation, the following infinite-dimensional dynamical system is studied

$$m\ddot{y}_i = y_{i-1} - 2y_i + y_{i+1} + \phi(y_i), \quad i \in \mathbb{Z}, \quad y_i \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (1)$$

where the potential $\phi(\cdot)$ is given by a smooth periodic function. The equation (1) is a system with the Frenkel-Kontorova potential.

Definition 1. *We say that the solution $\{y_i(\cdot)\}_{-\infty}^{+\infty}$ of the system (1), defined for all $t \in \mathbb{R}$, has the type of a traveling wave (is a soliton solution) if there exists $\tau > 0$, independent of t and i, that for all $i \in \mathbb{Z}$ and $t \in \mathbb{R}$ the following equality holds*

$$y_i(t + \tau) = y_{i+1}(t).$$

The constant τ is called the characteristic of a traveling wave.

For such a system with a general nonlinear potential, the existence of a family of bounded soliton solutions is established. Previously, such a system was studied in the case of a quadratic potential [?]. The proof is carried out within the framework of a formalism establishing a one-to-one correspondence between soliton solutions of an infinite-dimensional dynamical system and solutions of a family of functional differential equations of pointwise type. For the considered class of equations, the presence of a number of symmetries is also important.

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Regular and Singular Systems of Nonlinear Forward Kolmogorov Equations

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We consider systems of nonlinear parabolic equations which describe distributions of interacting populations and the underlying stochastic processes. Namely, we derive stochastic differential equations for stochastic processes with distribution densities satisfying systems of nonlinear parabolic equations. Since coefficients of these equations depend on the densities we add the Feynman-Kac formula as a relation which allows to obtain a closed system. The resulting stochastic system allow to define the required densities in a weak sense and thus we are looking for a more tractable definition.

This leads to necessity to deal with regularised systems of equations using a suitable mollification. As a result we derive a stochastic counterpart of a system of regular parabolic equations of the McKean-Vlasov type. At the end we discuss the limiting behaviour of both the stochastic system and the PDE system as a a family of mollifying functions converges to the Dirac delta-function. In one dimensional case we derive as well a system of nonlinear PDE equations for cumulative distribution functions of the above SDE solutions.

Multipliers on Periodic Bessel Potential Spaces: Description Theorems and Applications to Uniform Resolvent Approximation Problem for Laplace Type Operators

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Our main aim is to establish constructive description theorems for multipliers acting in the scale of periodic Bessel potential spaces and to apply these results to the study of elliptic operators' singular perturbations under quite general assumptions on the perturbation potential. More specifically, we prove that under natural assumptions on a positive index α, which guarantee that the distribution $u \in H_2^{-\alpha}(\mathbb{T}^n)$ acts as a multiplier from $H_2^{\alpha}(\mathbb{T}^n)$ to $H_2^{-\alpha}(\mathbb{T}^n)$, the singular perturbation of Laplacian's fractional power $(-\Delta)^{\alpha}$ by perturbation potential u is well–defined. We also consider the problem of approximating (in terms of uniform resolvent convergence) the general perturbation with multiplier potential by smooth perturbations.

Using the definition of periodic Bessel potential spaces $H_r^{\gamma}(\mathbb{T}^n)$ (where $\gamma \in \mathbb{R}$ and $r > 1$), which is based on the existence of the smooth partition of unity on n-dimensional torus $\mathbb{T}^n$, we establish multiplicative estimates, analogous to the ones obtained in [1] and [2] for the Bessel potential spaces $H_r^{\gamma}(\mathbb{R}^n)$. Namely, if $p > 1,\ q > 1,\ s \geq 0,\ t \geq 0$ and either $p \leq q,\ s - \frac{n}{p} \geq t - \frac{n}{q}$, or $p \geq q,\ s \geq t$, then there exists a positive constant $C_{s,\,t,\,p,\,q}$ such that the following multiplicative estimate holds true:

$$\forall\, f \in \mathcal{D}(\mathbb{T}^n),\ \forall\, g \in \mathcal{D}(\mathbb{T}^n)$$

$$\|f \cdot \mathbf{g}\|_{H_q^t(\mathbb{T}^n)} \leq C_{s,\,t,\,p,\,q} \cdot \|\mathbf{f}\|_{H_p^s(\mathbb{T}^n)} \cdot \|\mathbf{g}\|_{H_q^t(\mathbb{T}^n)},$$

where by $\mathbf{h} \in \mathcal{D}^{'}(\mathbb{T}^n)$ we denote a regular distribution generated by function h.

This estimate allows us to prove the following result.

Theorem 1. *Let $s \geq 0$, $t \geq 0$ and $p > 1$, $q > 1$. Let also either 1) $s \geq t$, $s > n/p$ and $p \leq q$, $s - n/p \geq t - n/q$, or 2) $t \geq s$, $t > n/q$ and $q \leq p$, $t - n/q \geq s - n/p$. Then the multiplier space $M[H_p^s(\mathbb{T}^n) \to H_q^{-t}(\mathbb{T}^n)]$ coincides with the intersection of periodic Bessel potential spaces $H_{p'}^{-s}(\mathbb{T}^n) \cap H_q^{-t}(\mathbb{T}^n)$.*

If $p = 2$, then for arbitrary $\gamma \in \mathbb{R}$ the space $H_2^{\gamma}(\mathbb{T}^n)$ can be characterized in terms of Fourier coefficients. Let us define a linear operator $J_s \colon \mathcal{D}'(\mathbb{T}^n) \to \mathcal{D}'(\mathbb{T}^n)$. If $\{f_k \mid k \in \mathbb{Z}^n\}$ is an orthonormal basis in $L_2(\mathbb{T}^n)$, then for $u = \sum\limits_{k \in \mathbb{Z}^n} C_k(u) \cdot \mathbf{f}_k$ we have

$$J_s(u) \overset{def}{=} \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^{\frac{s}{2}} \cdot C_k(u) \cdot \mathbf{f}_k.$$

It can be demonstrated that $u \in \mathcal{D}'(\mathbb{T}^n)$ belongs to the space $H_2^{\gamma}(\mathbb{T}^n)$ if and only if $J_{\gamma}(u)$ is a regular functional generated by a function from $L_p(\mathbb{T}^n)$.

Then it can be shown that the scale $H_r^{\gamma}(\mathbb{T}^n)$ coincides with the scale of Hilbert spaces, associated with the self-adjoint operator $A = Id + (-\Delta)^{\alpha}$, acting on the space $\mathbf{L_2}(\mathbb{T}^n)$, such that $A - Id$ is a positive definite operator (for the definition and basic properties of the latter scale see, e.g., [3; 1.2.2]).

Combination of this fact with Theorem 1 allows us to obtain the following result.

Theorem 2. *Let $\alpha > n/2$. Let q be a distribution from $H_2^{-\alpha}(\mathbb{T}^n)$ and $M_q \colon u \mapsto q \cdot u$ be a multiplication operator, which acts from $H_2^{\alpha}(\mathbb{T}^n)$ to $H_2^{-\alpha}(\mathbb{T}^n)$. Then the perturbation $(-\Delta)^{\alpha} + M_q$ by singular potential q is well-defined on the whole space $H_2^{\alpha}(\mathbb{T}^n)$ and its restriction on $\{u \in H_2^{\alpha}(\mathbb{T}^n) \mid ((-\Delta)^{\alpha} + M_q)(u) \in \mathbf{L}_2(\mathbb{T}^n)\}$ is a sectorial densely defined operator in $\mathbf{L_2}(\mathbb{T}^n)$.*

Moreover, we can approximate this perturbation with the general multiplier potential by the perturbations with smooth potentials in the following sense.

Theorem 3. *Let* $\alpha > n/2$ *and let* q *be a distribution from* $H_2^{-\alpha}(\mathbb{T}^n)$*. Then there exists a sequence* $(q_n \in D(\mathbb{T}^n) \mid n \in \mathbb{N})$*, such that the sequence of operators* $(-\Delta)^\alpha + M_{\mathbf{q}_n}$ *converges to* $(-\Delta)^\alpha + M_q$ *with respect to the uniform resolvent convergence.*

This talk is based on joint work with Professor A. Shkalikov.

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Differentiability in Measure of the Flow Associated to a Nearly Incompressible BV Vector Field

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We study the regularity of the flow $X(t, y)$ which represents (in the sense of Smirnov or as regular Lagrangian flow of Ambrosio) a solution $\rho \in L^\infty(\mathbb{R}^{d+1})$ of the transport PDE

$$\partial_t \rho + \mathrm{div}(\rho b) = 0,$$

with $b \in L^1_t \mathrm{BV}_x$. We prove that X is differentiable in measure in the sense of Ambrosio-Malý, i.e.

$$\frac{X(t, y + rz) - X(t, y)}{r} \to_{r \to 0} W(t, y)z \quad \text{in measure},$$

where derivative $W(t, y)$ is a BV function satisfying the ODE

$$\frac{d}{dt} W(t, y) = \frac{(Db)_y(dt)}{J(t-, y)} W(t-, y),$$

where $(Db)_y(dt)$ is the disintegration of the measure $\int Db(t, \cdot)dt$ with respect to the partition given by the trajectories $X(t, y)$ and the Jacobian $J(t, y)$ solves

$$\frac{d}{dt} J(t, y) = (\mathrm{div}\, b)_y(dt) = \mathrm{Tr}(Db)_y(dt).$$

The proof of this regularity result is based on the theory of Lagrangian representations and proper sets introduced by Bianchini and Bonicatto (2019), on the construction of explicit approximate tubular neighborhoods of trajectories, and on estimates that take into account the local structure of the derivative of a BV vector field.

Trace Inequalities For Rickart C^*-Algebras

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Rickart C^*-algebras are unital and satisfies polar decomposition. We proved that if a C^*-algebra $\mathcal{A}$ satisfies polar decomposition and admits a "good" faithful tracial state then $\mathcal{A}$ is a Rickart C^*-algebra. Via polar decomposition, we characterized tracial states among all states on a Rickart C^*-algebras. We presented triangle inequality for Hermitian elements and a traces on such algebras.

For a block projection operator and a trace on a Rickart C^*-algebra we proved a new inequality. As a corollary, we obtain an estimate for trace of a commutator of Hermitian element and projection. We give a characterization of traces in a wide class of weights on von Neumann algebra.

Dynamics of Torus Piecewise Isometries

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By now, we have learned reasonably well how to study hyperbolic (locally expanding/contracting or both) chaotic dynamical systems, thanks to a large extent to the development of the so called operator approach. Contrary to this almost nothing is known about piecewise isometries, except for a special case of one-dimensional interval exchange mappings. The last case is fundamentally different from the general situation in the obvious presence of an invariant measure (Lebesgue measure), which helps a lot in the analysis. We will show that already the restriction of the rotation of the plane to a torus demonstrates a number of rather unexpected properties. Our main results give sufficient conditions for the existence/absence of invariant measures of general piecewise isometries of the torus.

The analysis of simple ergodic properties of these measures is also carried out.

On Bifurcations of Essential Spectrum under Singular Geometric Perturbation

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Let $x' = (x_1, \ldots, x_{d-1})$, $x = (x', x_d)$ be Cartesian coordinates in $\mathbb{R}^{d-1}$ and $\mathbb{R}^d$, where $d \geqslant 2$, and $\Omega \subseteq \mathbb{R}^{d-1}$ be an arbitrary domain, which can be both bounded and unbounded. If the boundary of Ω is non-empty, it is assumed to have the smoothness C^2. By $A_{ij} = A_{ij}(x')$, $A_j = A_j(x')$, $A_0 = A_0(x')$, $i, j = 1, \ldots, d-1$, we denote real function defined on $\overline{\omega}$ and possessing the following smoothness: $A_{ij}, A_j \in C^1(\overline{\omega})$, $A_0 \in C(\overline{\omega})$. The functions A_{ij} are assumed to satisfy the usual ellipticity condition.

On the domain $\Pi := \Omega \times \mathbb{R}$ we consider the operator

$$\mathcal{H} = -\sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} - \frac{\partial^2}{\partial x_d^2} +$$

$$+\mathrm{i} \sum_{j=1}^{d-1} \left(A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j \right) + A_0 \quad \text{in} \quad \Pi \tag{1}$$

subject to the Dirichlet condition or Robin condition:

$$u = 0 \quad \text{on} \quad \partial\Pi \qquad \text{or} \qquad \frac{\partial u}{\partial \boldsymbol{\nu}} - a(x')u = 0 \quad \text{on} \quad \partial\Pi, \tag{2}$$

where

$$\frac{\partial u}{\partial \boldsymbol{\nu}} := \sum_{i,j=1}^{d-1} A_{ij} \nu_i \frac{\partial u}{\partial x_j} - \mathrm{i} \sum_{j=1}^{d-1} A_j \nu_j u + \nu_d \frac{\partial u}{\partial x_d}, \tag{3}$$

$\nu = (\nu_1, \ldots, \nu_d)$ is the unit normal to $\partial\Omega$, and $a = a(x')$ is a real function continuous and uniformly bounded on $\partial\Omega$.

We assume that the lower part of the spectrum of the self-adjoint operator

$$\mathcal{H}' = -\sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \mathrm{i} \sum_{j=1}^{d-1} \left(A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j \right) + A_0 \quad \text{in} \quad \Omega$$

subject to the same boundary condition as in (3) consists of discrete eigenvalues $\Lambda_1 \leqslant \Lambda_2 \leqslant \ldots \leqslant \Lambda_m < c_0$, while its essential spectrum is located above c_0.

In the work we consider a singular geometric perturbation of the operator $\mathcal{H}$, which is a small hole cut out in the domain Π. This hole is introduced as $\omega^\varepsilon := \{x : (x - x_0)\varepsilon^{-1} \in \omega\}$, where $x_0 \in \Pi$ is some fixed point, ε is a small parameter and $\omega \subset \mathbb{R}^d$ is a bounded domain with a C^4-boundary. Then we consider the operator $\mathcal{H}_\varepsilon$ in $\Pi \setminus \omega^\varepsilon$ with differential expression (1) subject to boundary conditions (2) and to the condition

$$\frac{\partial u}{\partial \nu^\varepsilon} - \alpha_\varepsilon u = 0 \quad \text{on} \quad \partial\omega^\varepsilon,$$

where the first term is a conormal derivative similar to that in (3) and α_ε is an arbitrary sufficiently smooth complex-valued function defined on $\partial\omega^\varepsilon$, which can be expanded into a power asymptotic series in ε.

The essential spectrum of the operator $\mathcal{H}_\varepsilon$ reads as $\sigma_{ess}(\mathcal{H}_\varepsilon) = [\Lambda_1, +\infty)$, while Λ_j serve as the thresholds in this essential spectrum. The main result of our work is a detailed description of the phenomenon of the bifurcation of the mentioned thresholds into the eigenvalues and resonances of the operator $\mathcal{H}_\varepsilon$ under the presence of the hole ω^ε. We obtain necessary and sufficient conditions for the existence of such eigenvalues and resonances and describe their asymptotic behavior.

This is a joint work with D.A. Zezyulin.

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Movement Characteristics of Two Models with Closed Curve Equilibrium

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The aim of this report is to analyze the dynamical properties of two models with closed curve equilibrium. The first model, introduced by Gotthans and Petržela [1], is motivated by an electronic circuit. The second model [2] is a generalisation of the previous model with a right hand side that is not a C^1 function. The corresponding three-variable models are given as a set of nonlinear ordinary differential equations. The dynamics of the models are studied depending on several parameters (see [3-4]). For this purpose, mainly new methods, as the approximate entropy and the 0-1 test for chaos, are applied. Using these tools, the dynamics are quantified and qualified. It is shown that depending on the system's parameters, the system exhibits both irregular (chaotic) and regular (periodic) character.

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Finitely Additive Measure on Hilbert Space

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We will consider a new construction of a finitely additive measure on a real separable Hilbert space, which makes it possible to calculate the measure of a ball of arbitrary radius. The mentioned construction is a natural extension of a finitely additive measure constructed earlier by V. Zh. Sakbaev and is obtained as a direct limit of a family of measures. It will be shown that the space of square-integrable functions with respect to this measure intersects with the Smolyanov-Shamarov space only by zero element.

Collision Models Can Efficiently Simulate Any Multipartite Markovian Quantum Dynamics

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We introduce the multipartite collision model, defined in terms of elementary interactions between subsystems and ancillas, and show that it can simulate the Markovian dynamics of any multipartite open quantum system. We develop a method to estimate an analytical error bound for any repeated interactions model, and we use it to prove that the error of our scheme displays an optimal scaling. Finally, we provide a simple decomposition of the multipartite collision model into elementary quantum gates, and show that it is efficiently simulable on a quantum computer according to the dissipative quantum Church-Turing theorem, i.e. it requires a polynomial number of resources.

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On the Alberti-Uhlmann Condition for Unital Channels

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We address the problem of existence of completely positive trace preserving (CPTP) maps between two sets of density matrices. We refine the result of Alberti and Uhlmann and derive a necessary and sufficient condition for the existence of a unital channel between two pairs of qubit states which ultimately boils down to three simple inequalities.

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Random Generators of Markovian Evolution: a Quantum to Classical Transition by Superdecoherence

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We consider ensembles of random generators of N-dimensional Markovian evolution, quantum and classical ones, and evaluate their universal spectral properties. We then show how the two types of generators can be related by superdecoherence. In analogy with the mechanism of decoherence, which transforms a quantum state into a a classical one, superdecoherence can be used to transform a Lindblad operator (generator of quantum evolution) into a Kolmogorov operator (generator of classical evolution). We study the spectra of random Lindblad operators undergoing superdecoherence and demonstrate that, in the limit of complete superdecoherence, the resulting operators exhibit universal spectral properties typical to random Kolmogorov operators.

By gradually increasing strength of superdecoherence, we observe a sharp quantum-to-classical transition. Finally, we define the inverse procedure of supercoherification, that is a generalization of the scheme used to construct a quantum state out of a classical one.

Thermodynamic Quantities in Quantum Speed Limit for Non-Markovian Dynamics

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Quantum speed limit (QSL) for open quantum systems in the non-Markovian regime is analyzed. We provide the lower bound for the time required to transform an initial state to a final state in terms of thermodynamic quantities such as the energy fluctuation, entropy production rate and dynamical activity. Such bound was already analyzed for Markovian evolution satisfying detailed balance condition. Here we generalize this approach to deal with arbitrary evolution governed by time-local generator. Our analysis is illustrated by three paradigmatic examples of qubit evolution: amplitude damping, pure dephasing, and the eternally non-Markovian evolution.

W–Meromorphic Solutions of Autonomous Ordinary Differential Equations and Related Topics

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Transcendental meromorphic functions that are elliptic or rational with an exponential argument are commonly called W–meromorphic functions in honor of Karl Weierstrass. We shall discuss the problem of finding all W–meromorphic solutions for a given autonomous ordinary differential equation. It is well known that any W–meromorphic function satisfies an autonomous algebraic first–order ordinary differential equation [1]. Consequently, if it is possible to obtain all algebraic first–order ordinary differential equations compatible with the original equation, then one can perform the classification of its W–meromorphic solutions. The main difficulty in finding compatible equations is that degrees of bivariate polynomials producing these equations are not known in advance. The aim of the talk is to introduce a method enabling one to find all first–order compatible differential equations [2]. A number of other applications of the method will be discussed.

Let us note that several strong results of the Nevanlinna theory can be used to prove that all transcendental meromorphic solutions of certain families of autonomous algebraic ordinary differential equations are W–meromorphic functions [3, 4].

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On Solutions of the Matrix Nonlinear Schrödinger Equation

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Let M_{nk} be the set of all complex $n \times k$-matrices. Consider the equation

$$iu_t = u_{xx} + uAu^*Bu$$

on an unknown M_{nk}-valued function $u(x,t)$, where $A \in M_{kk}$ and $B \in M_{nn}$ are given non-degenerate Hermitian matrices and the star stands for Hermitian conjugate. We shall prove that every real-analytic solution extends to a globally meromorphic function of x for every fixed value of t.

The case when all eigenvalues of A and B are of the same sign is referred to as totally focusing. In this case one can also assert that every local real-analytic solution extends to a real-analytic solution on an infinite strip parallel to the x-axis. This strip depends on the solution and can be a half-plane or the whole plane for some solutions. Moreover, every such strip carries a solution inextensible anywhere beyond it.

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Operator Error Estimates for Homogenization of the Nonstationary Schrödinger-Type Equations: Sharpness of the Results

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In $L_2(\mathbb{R}^d;\mathbb{C}^n)$,we consider a selfadjoint strongly elliptic opera-tor A_ε, $\varepsilon > 0$,given by the differential expression $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$. Here $g(\mathbf{x})$ is a periodic bounded and positive definite $(m \times m)$-matrix-valued function, and $b(\mathbf{D}) = \sum_{l=1}^{d} b_l D_l$ is a first order differential operator. It is assumed that b_l, $l = 1, \dots, d$, are constant $(m \times n)$-matrices, $m \geq n$, and the symbol $b(\boldsymbol{\xi})$ has maximal rank.

We study the behavior of the operator $\exp(-itA_\varepsilon)$, $t \in \mathbb{R}$, for small ε. It is proved that, as $\varepsilon \to 0$, $\exp(-itA_\varepsilon)$ converges to $\exp(-itA^0)$ in the norm of operators acting from the Sobolev space $H^s(\mathbb{R}^d;\mathbb{C}^n)$ (with a suitable s) to $L_2(\mathbb{R}^d;\mathbb{C}^n)$. Here $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is the effective operator. In [1], the following sharp order error estimate was obtained:

$$\| \exp(-itA_\varepsilon) - \exp(-itA^0) \|_{H^3(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1 (1 + |t|) \varepsilon. \quad (1)$$

Next, in [2] it was shown that, in the general case, this estimate is sharp with respect to the type of the operator norm. On the other hand, under some additional assumptions (formu-lated in the spectral terms near the lower edge of the spectrum) this result was improved:

$$\| \exp(-itA_\varepsilon) - \exp(-itA^0) \|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_2 (1 + |t|) \varepsilon. \quad (2)$$

Now we study the question about the sharpness of the results with respect to the dependence on t. We show that, in the general case, the factor $(1+|t|)$ in (1) cannot be replaced by $(1+|t|^\alpha)$ with $\alpha<1$. On the other hand, we prove that estimate (2) can be improved (under the same additional assumptions):

$$\|\exp(-itA_\varepsilon)-\exp(-itA^0)\|_{H^2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)} \le C_3(1+|t|^{1/2})\varepsilon.$$

This result allows us to obtain qualified estimates for large time $t=O(\varepsilon^{-\alpha})$ with $\alpha<2$. The results are applied to study the behavior of the solution $\mathbf{u}_\varepsilon$ of the Cauchy problem for the Schrödinger-type equation $i\partial_t\mathbf{u}_\varepsilon = A_\varepsilon\mathbf{u}_\varepsilon+\mathbf{F}$.

The results are published in [3]. The work was supported by Young Russian Mathematics award and Ministry of Science and Higher Education of the Russian Federation, agreement № 075-15-2019-1619.

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Nonlocal Gravity and its Cosmology

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General relativity (GR), i.e. Einstein theory of gravity, is recognized as one of the best physical theories – with nice theoretical properties and significant phenomenological confirmations. Nevertheless, GR is not a complete theory of gravity and there are many attempts to modify it. One of the actual approaches towards more complete theory is nonlocal modified gravity. Non-local gravity model, which I consider here without matter, is given by the action $S = \frac{1}{16\pi G}\int\sqrt{-g}(R - 2\Lambda + P(R)\mathcal{F}(\Box)Q(R))d^4x$, where R is scalar curvature and Λ – cosmological constant. $P(R)$ and $Q(R)$ are some differentiable functions of R. $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$ is an analytic function of the corresponding d'Alambertian $\Box$. I plan to present a brief review of general properties, and then consider cosmological solutions for two concrete functions $P(R)$ and $Q(R)$. Derivation of equations of motion is presented in [1].

The first case is $P(Q) = R(Q) = \sqrt{R - 2\Lambda}$. One of the exact cosmological solutions is $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}\, t^2}$, which mimics the dark matter and the dark energy, see [2]. In the second case $P(Q) = R(Q) = R - 4\Lambda$. It contains exact cosmological solution $a(t) = A\sqrt{t}\, e^{\frac{\Lambda}{4}\, t^2}$ that mimics properties similar to the radiation and the dark energy, see [3].

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On One Problem of the Tauberian Theory

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A new Tauberian theorem fo holomorphic function of bounded argument will be presented. Some application of this theorem in mathematical physics will be considered.

On the Space of Smooth Geometrically Integrable Maps in the Plane

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In [1] the concept of the geometric integrability for maps in the plane $\mathbf{R^2}$ is introduced.

Definition 1 [1]. We say that a self-map G of a connected set $\Pi \subset \mathbf{R^2}$ is *geometrically integrable on* Π if there exists a self-map ψ of an interval J of the real line $\mathbf{R^1}$ such that G is semiconjugate with ψ by means of a continuous surjection $H : \Pi \to J$, i.e. the following equality holds: $H \circ G = \psi \circ H$. The map $\psi : J \to J$ is said to be *the quotient* of G.

The self-similar groups is the source of integrable maps in the (projective) plane [2].

In [3] the criteria for geometric integrability of maps in the plane are proved.

Theorem 1 [3]. *Let* Π *be a connected compact in the plane* $\mathbf{R^2}$, *G be a self-map of* Π. *Let* J *be a segment of the real line* $\mathbf{R^1}$, ψ *be a self-map of* J.

Then G *is the geometrically integrable map with the quotient* ψ *by means of a continuous surjection* $H : \Pi \to J$ *such that for every* $y \in pr_2(\Pi)$ *(here* pr_2 *is the natural projection of the plane* $\mathbf{R^2}$ *on the second coordinate axis* Oy*) the map* H *is an injection with respect to* x, *if and only if* Π *is the support of a continuous invariant foliation for* Π *with fibres* $\{\gamma_{x'}\}_{x' \in J}$ *that are pairwise disjoint graphs of continuous functions* $x = x_{x'}(y)$ *for every* $y \in pr_2(\Pi)$. *Moreover, the inclusion* $G(\gamma_{x'}) \subseteq \gamma_{\psi(x')}$ *holds.*

Theorem 2 [3]. *Let* Π *be a connected compact in the plane* $\mathbf{R^2}$*,* G *be a self-map of* Π*, where*

$$G(x,\, y) = (g^1(x,\, y), g^2(x,\, y)).$$

Let J *be a segment of the line* $\mathbf{R^1}$*,* ψ *be a self-map of* J*.*

Then G *is the geometrically integrable with the quotient* ψ *by means of a continuous surjection* $H : \Pi \to J$ *such that for every* $y \in pr_2(\Pi)$ *the map* H *is injection on* x*, if and only if there is a homeomorphism* $\tilde{H}$ *that maps* Π *on the rectangle* $J^2 = J \times pr_2(\Pi)$ *and reduces* $G : \Pi \to \Pi$ *to the skew product* $F : J^2 \to J^2$ *satisfying*

$$F(u,\, v) = (\psi(u),\, g^2(x',\, v)),\ \ \textit{where}\ x' = pr_1 \circ \tilde{H}^{-1}(u,\, v),$$

where $\tilde{H}^{-1} : J^2 \to \Pi$ *is the inverse homeomorphism for* $\tilde{H}$*.*

Let $\Omega(G)$ be the nonwandering set of a continuous integrable map $G : \Pi \to \Pi$. Then by Theorems 1 - 2 the inclusion $\Omega(G) \subset \{\gamma_{x'}\}_{x' \in \Omega(\psi)}$ holds. We consider slices $(A)(\gamma_{x'})$ of a set $A \subset \Pi$ by graphs $\gamma_{x'}$ for all $x' \in \Omega(\psi)$, i.e.

$$(A)(\gamma_{x'}) = \{y \in J' : (x_{x'}(y),\, y) \in A\},$$

where $J' = pr_2(\Pi)$.

Definition 2. Let $G : \Pi \to \Pi$ be a continuous integrable map. The multifunction $\Omega^G : \Omega(\psi) \to 2^{J'}$, where $2^{J'}$ is the space of all closed subsets of the closed interval J' endowed with the exponential topology, is said to be *the* Ω*-function of* G if its value in every point $x' \in \Omega(\psi)$ satisfies the equality

$$\Omega^G(x') = (\Omega(G))(\gamma_{x'}).$$

Definition 2 generalizes the concept of the Ω-function for skew products of interval maps (for example, see [4]) on the case of integrable maps in the plane.

We consider C^1-smooth integrable maps with Ω-stable quotients and introduce multifunctions that approximate the Ω-function of a C^1-smooth integrable map. Using all these multifunctions we present the set of C^1-smooth integrable maps with a complicated dynamics of their quotients as the union of four nonempty pairwise disjoint subsets.

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Initial-Boundary Value Problems on a Half-Strip for the Zakharov–Kuznetsov Equation and its Modification

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The two-dimensional Zakharov–Kuznetsov equation

$$u_t + bu_x + u_{xxx} + u_{xyy} + uu_x = 0 \tag{1}$$

and its modified analogue

$$u_t + bu_x + u_{xxx} + u_{xyy} + au^2u_x = 0, \quad a = \pm 1, \tag{2}$$

are considered. Such equations model various aspects of nonlinear wave propagation in dispersive media, when the waves move in the x-direction with deformations in the transversal y-direction.

Initial-boundary value problems for these equations posed on a half-strip $\Sigma_{+,L} = \{(x, y) : x > 0, 0 < y < L\}$ are studied. Besides the initial condition $u\big|_{t=0} = u_0(x, y)$ and homogeneous Dirichlet boundary condition on the left boundary $u\big|_{x=0} = 0$ various homogeneous boundary conditions on the horizontal boundaries $y = 0$ and $y = L$: Dirichlet, Neumann, periodic, are set.

Problems of global existence, uniqueness, regularity and large-time behavior of solutions are discussed.

The initial function u_0 is considered, in particular, in the weighted L_2-spaces and H^1-spaces with power and exponential weights when $x \to +\infty$. Then the results on existence and uniqueness of global weak solutions are established for equation (1), similar for all types of boundary conditions on the horizontal part of the boundary. For equation (2) such results are obtained in the H^1-spaces and, moreover, if $a = 1$ only for small initial data.

The constructed solutions possess additional internal regularity for $t > 0$, $x > 0$, depending on the decay rate of the initial function when $x \to +\infty$ and here the results for equation (1) are different for different types of horizontal boundary conditions. For example, if the initial function decays exponentially at $+\infty$, in the cases of Neumann and periodic boundary conditions the solution is infinitely smooth up to the boundaries $y = 0$ and $y = L$, while in the case of Dirichlet boundary condition it is infinitely smooth strictly inside the half-strip. On the contrary, for equation (2) similar results on the gain of internal regularity are the same for all types of boundary conditions.

The same difference occurs in the case of more smooth initial data in weighted Sobolev spaces.

On the contrary, results on large-time decay of solutions for exponentially decaying small initial data are obtained only in the case of Dirichlet boundary condition both for equation (1) and equation (2).

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Uniform Approximation by Polynomial Solutions of Second-Order Elliptic Equations and Systems on Plane Compact Sets

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We plan to discuss the problem on uniform approximation of functions on compact sets in the complex plane by polynomial solutions of homogeneous second-order elliptic PDE with constant complex coefficients, and by solutions of systems of such equations. The question we are interested in is as follows: to describe compact sets $X \subset \mathbb{C}$ such that for every function f continuous on X and satisfying on $\mathrm{Int}(X)$ the equation $\mathcal{L}f = 0$ (where $\mathcal{L}$ is a given elliptic operator of the mentioned above type), there exists a sequence of polynomials $\{P_n\}$ (each P_n is a complex valued polynomial in two real variables) such that $\mathcal{L}P_n \equiv 0$ and $\|f - P_n\|_X \to 0$ as $n \to \infty$, where $\|\cdot\|_X$ stands for the uniform norm on X. We will present several recent results in this problem, and explain how they are related to the regularity property of a given bounded simply connected domain with respect to the Dirichlet problem for solutions to the equation or system under consideration, and to some special analytic properties of boundaries of such domains.

Information Properties of Trace Decreasing Quantum Operations

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We revisit the physically relevant scenario of both classical and quantum information transmission through lossy quantum communication lines, in which the loss probability is state dependent [1]. A typical example of state-dependent losses is the polarization dependent loss in optical fibers [2]. Let $\Lambda : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$ be a completely positive and trace nonincreasing map also referred to as a quantum operation. In the paper [1], we introduce the generalized erasure channel of the form

$$\Gamma_\Lambda[\rho] = \begin{pmatrix} \Lambda[\rho] & \mathbf{0} \\ \mathbf{0}^\top & \mathrm{tr}\left[\rho(I - \Lambda^\dagger[I])\right] \end{pmatrix}, \tag{1}$$

where $\Lambda^\dagger$ is a dual map and I is the $d \times d$ identity matrix. If ρ is a density operator, then $\mathrm{tr}\left[\rho(I - \Lambda^\dagger[I])\right]$ is nothing else but the loss probability. If $\Lambda = p\mathrm{Id}$, $0 \leq p \leq 1$, then $\Gamma_{p\mathrm{Id}}$ is the conventional erasure channel [3, 4]. If $\Lambda = p\Phi$, where $\Phi : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$ is a dephasure channel, then $\Gamma_{p\Phi}$ is a so-called dephrasure channel, for which the authors of Ref. [5] discovered that for a specific dephasure parameter and a specific loss probability the two-letter quantum capacity exceeded the single-letter quantum capacity by about $2.5 \cdot 10^{-3}$ bit per qubit sent. In contrast to these specific cases, Λ does not have to be unbiased, so the erasure probability $\mathrm{tr}\left[\varrho(I - \Lambda^\dagger[I])\right]$ is state-dependent in general. This is the reason we refer to the channel (1) as a generalized erasure channel.

Suppose the information is encoded into polarization degrees of freedom of single photons that can be potentially entangled among themselves. Let p_H and p_V be the attenuation factors for the horizontal and vertical polarizations, respectively. Then the effect of the optical fiber with polarization dependent losses is described by the following quantum operation with one Kraus operator A [2]:

$$\Lambda[\varrho] = A\varrho A^\dagger, \quad A = \sqrt{p_H}|H\rangle\langle H| + \sqrt{p_V}|V\rangle\langle V|. \tag{2}$$

Substituting Eq. (2) in Eq. gec, we obtain a quantum channel with interesting properties. In Ref. [1] we derive lower and upper bounds for the classical and quantum capacities of the generalized erasure channel as well as characterize its degradability and antidegradability. Γ_Λ is degradable if and only if $\min(p_H, p_V) \geq \frac{1}{2}$ or $p_H = 1$ or $p_V = 1$. Γ_Λ is antidegradable if and only if $\max(p_H, p_V) \leq \frac{1}{2}$ or $p_H = 0$ or $p_V = 0$. If either $\frac{1}{2} < p_H < 1$ and $0 < p_V < 1 - p_H$ or $\frac{1}{2} < p_V < 1$ and $0 < p_H < 1 - p_V$, then we analytically prove the superadditivity of coherent information [1], i.e., $\frac{1}{2}Q_1(\Gamma_\Lambda^{\otimes 2}) > Q_1(\Gamma_\Lambda)$, where

$$Q_1(\Psi) = \sup_{\rho \in \mathcal{D}(\mathcal{H})} \{S(\Psi[\rho]) - S(\widetilde{\Psi}[\rho])\},$$

$\widetilde{\Psi}$ is a complementary channel to the channel Ψ, $S(\rho) = -\mathrm{tr}[\rho \log \rho]$ is the von Neumann entropy, and $\mathcal{D}(\mathcal{H})$ is a set of density matrices. The maximum achievable difference $\frac{1}{2}Q_1(\Gamma_\Lambda^{\otimes 2}) - Q_1(\Gamma_\Lambda)$ approximately equals $7.197 \cdot 10^{-3}$ and is achieved in the vicinity of parameters $p_H = 0.7$ and $p_V = 0.19$ (or vice versa).

In the report, we revisit the divisibility properties of trace decreasing operations too.

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Normal Form of a Slow-Fast System with an Equilibrium Near Folded Slow Manifold

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We study a slow-fast system with one fast and two slow variables:

$$\begin{aligned} \varepsilon\dot{x} &= S(x,y,z) + \varepsilon F_1(x,y,z) \\ \dot{y} &= F_2(x,y,z) \\ \dot{z} &= F_3(x,y,z), \end{aligned}$$

where ε is a small parameter describing the ratio of two time scales. It is assumed that the "slow" manifold defined by

$$S(x,y,z) = 0$$

possesses a fold. Besides we suppose that in sufficiently small neighborhood of the fold there exists also an equilibrium of the system. Under these assumptions we derive a normal form for the system in a vicinity of the pair "equilibrium-fold" and study the dynamics of the normal form. In particular, as the the parameter ε tends to zero we obtain an asymptotic formula for the Poincaré map. We showed that despite the closeness of the Poincaré map to identity, its derivative can be large what leads to a cascade of the period-doubling bifurcations. We calculate the parameter values of the normal form which correspond to the first period-doubling bifurcation. The results are applied to the "generalized" FitzHugh-Nagumo equation:

$$\begin{aligned} \varepsilon\dot{x} &= x - x^3/3 - y - z \\ \dot{y} &= a + x \\ \dot{z} &= a + x - z, \end{aligned}$$

where a is a parameter assumed to be slightly less than one. The existence of the period-doubling cascade for this system was numerically discovered in [1].

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Dynamical Systems on $\mathbb{T}^2$ Modeling Josephson Junction, Isomonodromic Deformations and Painlevé 3 Equations

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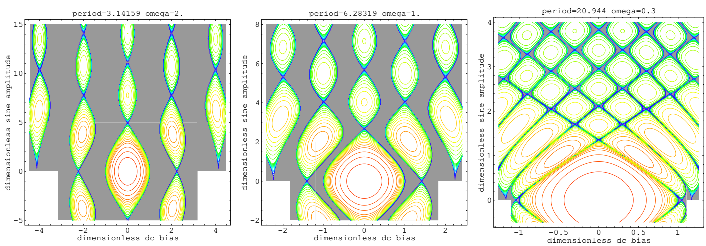
The tunneling effect predicted by B.Josephson (Nobel Prize, 1973) concerns the Josephson junction: two superconductors separated by a narrow dielectric. It states existence of a supercurrent through it and equations governing it. The *overdamped Josephson junction* is modeled by a family of differential equations on 2-torus depending on 3 parameters: B (abscissa), A (ordinate), ω (frequency). We study its *rotation number* $\rho(B, A; \omega)$ as a function of (B, A) with fixed ω. The *phase-lock areas* are the level sets $L_r := \{\rho = r\}$ with nonempty interiors; they exist for $r \in \mathbb{Z}$ [1]. They are analogues of the famous Arnold tongues. Each L_r is an infinite chain of domains going vertically to infinity and separated by separation points, which are called *constrictions* (expect for those with $A = 0$). See the figures for $\omega = 2,\ 1,\ 0.3$:



We show [2] that: 1) *all constrictions in L_r lie in the vertical line* $\{B = \omega r\}$; 2) *each constriction* is *positive*: some its punctured neighborhood in the vertical line lies in $\mathrm{Int}(L_r)$. These results confirm experiences of physicists (pictures from physical books of 1970-th) and two mathematical conjectures. The proof uses an equivalent description of model by linear systems of differential equations on $\overline{\mathbb{C}}$ [1], their isomonodromic deformations described by Painlevé 3 equations and methods of theory of slow-fast systems.

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Klibanov's Method of Ill-Posed Problem for the Black-Scholes Equation Solution and Machine Learning

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Our work presents a new empirical, mathematical model for generating more accurate multi-day option trading strategy using new intervals, initial conditions, and boundary conditions for the underlying stock, while foregoing the standard maturity time and strike price notions. The idea was proposed by professor Michael Klibanov. The basis for this model is the Black-Scholes Equation, solved as an ill-posed inverse problem using regularization methods.This equation is solved forwards in time to forecast prices of stock options. While manual calculations using this method can already be used to forecast profitable option purchases more accurately than the currently-used extrapolation-based prediction techniques, accuracy has been improved by applying machine learning and including larger datasets. Uniqueness, stability and convergence theorems for this method are formulated. For each individual option, historical data is used for input.

The latter is done for two hundred thousand stock options selected from the Bloomberg terminal of University of Washington. It used the index Russell 2000. The main observation is that it was demonstrated that technique, combined with a new trading strategy, results in a significant profit on those options. On the other hand, it was demonstrated the trivial extrapolation techniques results in much lesser profit on those options. This was an experimental work. The minimization process was performed by Hyak Next Generation Supercomputer of the research computing club of University of Washington. As a result, it obtained about 177,000 minimizers. The

code is parallelized in order to maximize the performance on supercomputer clusters. Python with the SciPy module was used for implementation. The work is also dedicated to application of binary classification *machine learning*. We were able to improve our results of profitability using minimizers as new data.

Dynamical Methods in Spectral Theory of Quasicrystals

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Methods of modern theory of dynamical systems turned out to be very useful and powerful in the study of different spectral problems. We will discuss several models of one-dimensional quasicrystals (such as Fibonacci Hamiltonian and Schrodinger operators with Sturmian potentials) and show how the trace map formalism and methods of hyperbolic dynamics provide new results on structure of the spectrum and the density of states measure for these models. Most of the results we plan to discuss were obtained in a series of joint projects with D.Damanik.

Feedback, Fractional Linear Transformations & Quantum Open Systems

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We consider the general problem of connecting open quantum stochastic systems into a network. In the Markov case, this turns out to be implemented by Möbius transformations acting on the category of quantum stochastic models: this is shown to be naturally related to the fractional linear transformations in block design in engineering. We will discuss both quantum measurement-based feedback (i.e., using filtering) and quantum coherent feedback.

Inductive Limits for Sequences of C^*-Algebras

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The report is concerned with the inductive limits for inductive sequences consisting of the Toeplitz-Cuntz algebras and their homomorphisms. We are interested in the structure of these inductive limits. This study is closely connected with the results on the reduced semigroup C^*-algebras in [1–2].

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Some Old and New Problems in Thermodynamic Formalism of Countable State Symbolic Markov Chains

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Thermodynamic formalism is a domain of the theory of dynamical systems related to some ideas of statistical physics. A starting material for developing thermodynamic formalism is a map $T : X \to X$ of some measurable space X and a measurable function $f : X \to \mathbf{R}$. In the case under consideration, T is a Markov shift in the space X of the sequences $x = (x_i, -\infty < i < \infty)$, $x_i \in V$, where V is a countable set and $f(x)$ depends on x_0 and x_1. One can describe thermodynamic formalism of countable state symbolic Markov chains either in terms of infinite matrices with nonnegative entries or in terms of infinite loaded graphs. Here the latter way is used. A loaded graph (LG) is the pair $(G, \mathcal{W})$, where $G = (V, E)$ is a directed graph with vertex set $V = V(G)$ and edge set $E = E(G)$, while $\mathcal{W}$ is a positive function on E. There are four classes of LGs: stable positive, unstable positive, null-recurrent, and transient. In the talk, a motivation for such a terminology will be presented and alternative characterization of some classes of LGs will be given. For stable and unstable positive LGs, there exists a translation invariant probability measure on the space of infinite paths of G that maximizes the difference between the Kolmogorov–Sinai entropy of the shift transformation and the mean energy determined by $\mathcal{W}$. The points of maximum of this functional are said to be equilibrium measures (or equilibrium states). In our situation there is only one equilibrium measure. The problem on asymptotic behavior of the equilibrium measures corresponding to an incre-asing sequence of finite subgraphs of G is rather old. It is solved for all classes of LGs with one exception: when the LG of interest is unstable positive.

Characteristic Lie Algebras of Integrable Differential-Difference Equations in 3D

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The purpose of the talk is to discuss an algebraic approach to the problem of integrable classification of differential-difference equations with one continuous and two discrete variables. As a classification criterion, we put forward the following hypothesis [1], [2]. Any integrable equation of the type under consideration admits an infinite sequence of finite-field Darboux-integrable reductions. The property of Darboux integrability of a finite-field system is formalized as finite-dimensionality condition of its characteristic Lie-Rinehart algebras. That allows one to derive effective integrability conditions in the form of differential equ-ations on the right hand side of the equation under study. To test the hypothesis, we use known integrable equations from the class under consideration. In this talk we show that all known examples do have this property [3].

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Chaotic Motions of Cardiac Electrophysiology

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The heart muscle work is controlled by a complex system that works based on the transmission of electrical signals between individual cells. Each cell must work in perfect harmony with its surroundings so that the whole communication across the whole heart does not break down. This essential synchronization between individual cells can be lost when only one cell produces chaotic electrical signals. In this work, the conditions under which cardiac cells can function or respond with irregular respon-ses and drive into chaos are investigated. For this purpose, a mathematical model of the heart cell is used, the regularity of the output is examined depending on the individual parameters of the excitation signal. For this purpose, classical and modern dynamical systems methods are used, such as frequency analysis, entropy, or the 0-1 test for chaos. Using these methods, the dependence of the dynamic properties of the heart cell on its excitation parameters is described.

Event Structures and Ultraproducts

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Definition [see, for example, S. Gudder, [1]] Let $\mathcal{E}$ be a nonempty set and let S be a set of functions from $\mathcal{E}$ into interval $[0,1]$. We call $(\mathcal{E}, S)$ an event structure if the following two axioms hold:

1. If $s(a) = s(b)$ for every $s \in S$, then $a = b$;

2. Let $a_1, a_2, \dots \in \mathcal{E}$ satisfy $s(a_i) + s(a_j) \leq 1$, $i \neq j$, for every $s \in S$. Then there exists a element $b \in \mathcal{E}$ such that $s(b) + s(a_1) + s(a_2) + ... = 1$ for every $s \in S$.

If $(\mathcal{E}, S)$ is an event structure, we call the elements of $\mathcal{E}$ events and the elements of S states. The events correspond to physical phenomena which occur or do not occur. If $a \in S$, then the event a occurs or does not occur depending upon the state of the system. In this case, $s(a)$ is interpreted as the probability that the event a occurs when the system is in the state s.

Within the framework of this model, we can consider ultraproducts of general probability spaces, of various operator algebras with a sets of states, and of quantum logics. The report will discuss ultraproducts of quantum logics and their properties.

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Further Results on the Theory of Galton-Watson Branching Processes Allowing Immigration without Finite Variances

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Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N} = \{1, 2, \ldots\}$. We consider the Galton-Watson Branching Process allowing Immigration (GWPI). An evolution of the process will occur by following scheme. An initial state is empty and the process starts owing to possible stream of immigrants. Each individual at any time $n \in \mathbb{N}$ produces j progeny with probability p_j, $j \in \mathbb{N}_0$ independently of each other so that $p_0 > 0$. Simultaneously into the population i immigrants possibly arrive with probability h_i, $i \in \mathbb{N}_0$ in each moment $n \in \mathbb{N}$. These individuals undergo further transformation obeying the offspring probability $\{p_j\}$. So denoting X_n the population size of GWPI in time n, we can write the following sum:

$$X_0 = 0, \quad X_{n+1} = \xi_1 + \xi_2 + \cdots + \xi_{X_n} + \eta_{n+1}$$

for any $n \in \mathbb{N}$, where ξ_k and η_k are i.i.d. random variables with $\mathbb{P}\{\xi_k = j\} = p_j$ and $\mathbb{P}\{\eta_k = j\} = h_j$ for all $k \in \mathbb{N}$. Throughout the paper be assumed that $\sum_{j \in \mathbb{N}_0} h_j = 1$. The process $\{X_n\}$ is an homogeneous Markov chain with state space $\mathcal{S} \subset \mathbb{N}_0$ and its n-step transition probabilities

$$p_{ij}^{(n)} := \mathbb{P}\left\{X_{n+k} = j \middle| X_k = i\right\} \qquad \text{for any} \qquad k \in \mathbb{N}$$

are given by

$$\mathcal{P}_n^{(i)}(s) := \sum_{j \in \mathcal{S}} p_{ij}^{(n)} s^j = \big(f_n(s)\big)^i \prod_{k=0}^{n-1} h\big(f_k(s)\big) \qquad \text{for any} \qquad i \in \mathcal{S} \tag{1}$$

where $f_n(s)$ is n-fold iteration of GF $f(s)$ for $s \in [0,1)$; see [1]. Thus the transition probabilities $\left\{p_{ij}^{(n)}\right\}$ are completely defined by the probabilities $\{p_j\}$ and $\{h_j\}$.

The population process $\{X_n\}$ described above was first considered by Heathcote [2] in 1965. Further long-term properties of $\mathcal{S}$ and a problem of existence and uniqueness of invariant measures of GWPI were investigated in papers of Seneta [3], [4], Pakes [5], [6] and by many other authors. Therein high order moment conditions for GF $f(s)$ and $h(s)$ was required to be satisfied.

Zolotarev [7] was one of the first who demonstrated the encouraging prospect of application the SV conception in the theory of Markov, branching processes with continuous-time. Afterwards due to the SV theory principally new results were proved, for the simple Galton-Watson Process without immig-ration by Slack [8] and Seneta [9], for GWPI by Pakes [10], [11]; see, also [12], [13] and [14].

In the paper we keep on the critical case, i.e. mean per-capita offspring number $\sum_{j\in\mathbb{N}} jp_j = f'(1-) = 1$. Discussing this case, we assume that the offspring GF $f(s)$ for $s \in [0,1)$ has the following form:

$$f(s) = s + (1-s)^{1+\nu}\mathcal{L}\left(\frac{1}{1-s}\right), \qquad [f_\nu]$$

where $0 < \nu < 1$ and $\mathcal{L}(x)$ is SV at infinity.

And also we everywhere will consider the case that immigration GF $h(s)$ for $s \in [0,1)$ has the following form:

$$1 - h(s) = (1-s)^{\delta}\ell\left(\frac{1}{1-s}\right), \qquad [h_\delta]$$

where $0 < \delta < 1$ and $\ell(x)$ is SV at infinity.

Main results of the paper appear provided that conditions $[f_\nu]$ and $[h_\delta]$ hold. We make also some extra assumptions for $\mathcal{L}(x)$

and $\ell(x)$. Namely, we assume that

$$\frac{\mathcal{L}(\lambda x)}{\mathcal{L}(x)} = 1 + o\left(\frac{\mathcal{L}(x)}{x^{\nu}}\right) \qquad \text{as} \qquad x \to \infty \qquad [\mathcal{L}_\alpha]$$

and

$$\frac{\ell(\lambda x)}{\ell(x)} = 1 + o\left(\frac{\ell(x)}{x^{\delta}}\right) \qquad \text{as} \qquad x \to \infty \qquad [\ell_\beta]$$

for each $\lambda > 0$; see [15, p. 185, condition SR3].

We need also the following result which is an improved analogue of the Basic Lemma of simple Galton-Watson Process without immigration.

Lemma A [13]. *Let conditions $[f_\nu]$ and $[\mathcal{L}_\alpha]$ hold. Then*

$$\frac{1}{\Lambda\big(R_n(s)\big)} - \frac{1}{\Lambda(1-s)} = \nu n + \frac{1+\nu}{2} \cdot \ln\big[\Lambda(1-s)\nu n + 1\big] + \rho_n(s),$$

where $\rho_n(s) = o\big(\ln n\big) + \sigma_n(s)$ and, $\sigma_n(s)$ is bounded uniformly for $s \in [0,1)$ and converges to the limit $\sigma(s)$ as $n \to \infty$ which is a bounded function for all $s \in [0,1)$.

In our previous work [16] we considered the case $\gamma := \delta - \nu > 0$, in which $\mathcal{S}$ is ergodic. In particular, we established the following theorem.

Theorem A. *Let conditions $[f_\nu]$, $[h_\delta]$ hold and $\delta > \nu$. Then $\mathcal{P}_n(s)$ converges to a limit function $\pi(s)$ which generates the invariant measures $\{\pi_j\}$ for GWPI. The convergence is uniform over compact subsets of the open unit disc. If in addition, the conditions $[\mathcal{L}_\alpha]$ and $[\ell_\beta]$ are fulfilled then*

$$\mathcal{P}_n^{(0)}(s) = \pi(s)\left(1 + \Delta_n(s)\mathcal{N}_\delta\left(\frac{(\nu n)^{1/\nu}}{\mathcal{N}(n)}\right)\right),$$

where $\mathcal{N}_\delta(x) = \mathcal{N}^\delta(x)\ell(x)$ and $\mathcal{N}(x)$ is known SV at infinity, herein

$$\Delta_n(s) = \frac{1}{\delta - \nu}\,\frac{1}{\big(\nu_n(s)\big)^{\delta/\nu - 1}} + \mathcal{O}\left(\frac{\ln n}{n^{\delta/\nu}}\right)$$

as $n \to \infty$ *and* $\nu_n(s) = \nu n + \Lambda^{-1}(1-s)$.

If $\delta = \nu$ then $\ell(u) \equiv (1+\nu)\mathcal{L}(u)(1+o(1))$, so that $L(u) \to 1+\nu$ as $u \to \infty$. Thus it lends us another Markov process called *Q-process* instead of GWPI; see [17, pp. 56–58] and [18].

The aim of this note is to obtain some information for a case not covered above, i.e. for the case $\gamma < 0$. We will establish a result similar to Theorem A.

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Limit Theorems for a Sum of Random Variables of a Special Form of Random Variables Linear Cocyles over Irrational Rotations

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Let $\mathbb{N}$ be the set of natural numbers and $\{\xi_n, n \in \mathbb{N}\}$ – a sequence of random variables. The solution of important problems in many areas of probability theory and mathematical statistics leads to the determination of the asymptotic state of the sum

$$S_n = \xi_1 + \xi_2 + \ldots + \xi_n$$

as $n \to \infty$. The asymptotic behaviour of S_n has been widely studied in probabilty theory, in which the law of large numbers, strong law of large numbers and central limit theorems play an important role. Along with this, in many cases it becomes necessary to study the asymptotic behaviours of random variables in the form of the sum $\sum_{k=1}^{n} f_{nk}(S_k)$, where f_{nk} is some functional. For example, if we consider the first-order auto-regression process ([1])

$$X_k = mX_{k-1} + \varepsilon_k, \qquad k \in \mathbb{N},$$

for some non-random variables m, its solution will be in the form of

$$X_n = \sum_{k=1}^{n} m^{n-k} \varepsilon_k,$$

where ε_k are independent random variables. Similarly consider a branching process allowing immigration, defined by a random sum

$$X_0 = 0, \quad X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k,$$

where $\xi_{k,j}$ and ε_k are number of "aborigines" and number of immigrants entered the system in the time k, respectively. Then the deviation of the random variable X_n from its mean will take the form

$$X_n - \mathsf{E}X_n = \sum_{k=1}^{n} m^{n-k} M_k,$$

where m is the per mean capita number of "aborigines" and M_k is martingale difference; see for instance [2]. In this model, it will be necessary to study the asymptotes of the random variables $\sum_{k=1}^{n} f_{nk}(M_k)$ where $f_{nk}(x) = m^{n-k}x$.

The Chebyshev inequality (also called the Biennem-Chebyshev inequality), the Bernoulli and Chebyshev theorems, the Moivre-Laplace integral theorem, the Lindeberg condition, and the Lindeberg theorem are important concepts of limit theorems. The study of limit theorems for a sum of random variables is one of the main problems of probability theory and mathematical statistics. In this field, significant results have been achieved by P.L.Chebyshev, A.N.Kolmogorov, B.V.Gnedenko, A.Ya.Khinchin, W.Feller, A.V.Prokhorov, Ya.V.Lindeberg, V.M.Zolotarev and etc.; see [3–5].

In the report, we define a condition for the applicability of the central limit theorem for sums of a special form that have not been studied previously. Our aim is to determine the conditions under which the central limit theorem is applicable to sums of a special form. Limit theorems for sums of random variables are studied in a special form and the main results obtained are described. The main assumptions of the work are the presence of second-order moments for the variables under consideration, and the fulfillment of the Lindeberg condition. The main results obtained are of theoretical importance and can be applied to determine the fluctuations of branching processes allowing immigration, as well as the asymptotic behavior of auto-regression processes.

Now let $\{\xi_n,\ n \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables with

$$\mathsf{E}\xi_1 = 0 \quad \text{and} \quad \mathsf{D}\xi_1 = \sigma^2 > 0.$$

Denote

$$F(x) = \mathsf{P}(\xi_1 < x);$$

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} \xi_i, \quad B_n^2 = \mathsf{D}S_n, \quad n \in \mathbb{N}.$$

It is well known that $\mathsf{E}S_n = 0,\ B_n^2 = \sigma^2 n$.

Let $\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_n^{(n)}$ be a sequence of independent and identically distributed random variables with

$$\mathsf{E}\xi_i^{(n)} = 0 \quad \text{and} \quad \mathsf{D}\xi_i^{(n)} = \sigma_n^2 > 0.$$

Introduce the notations:

$$F^{(n)}(x) = \mathsf{P}(\xi_1^{(n)} < x);$$

$$S_0^{(n)} = 0, \quad S_n^{(n)} = \sum_{i=1}^{n} \xi_i^{(n)}, \quad B_n^2 = \mathsf{D}S_n^{(n)}, \quad n \in \mathbb{N}.$$

Consider the following sums of random variables of a special form:

$$X_n = \sum_{k=1}^{n} m^k S_k, \qquad Y_n = \sum_{k=1}^{n} m^{n-k} S_k$$

for some $m > 0$ and

$$Z_n = \sum_{k=1}^{n} m_n^k S_k, \qquad T_n = \sum_{k=1}^{n} m_n^{n-k} S_k,$$

$$J_n^{(n)} = \sum_{k=1}^{n} m_n^k S_k^{(n)}, \qquad K_n^{(n)} = \sum_{k=1}^{n} m_n^{n-k} S_k^{(n)},$$

where

$$m_n = 1 + \frac{\alpha}{n} + o\left(\frac{1}{n}\right) \qquad \text{as} \qquad n \to \infty$$

for some $\alpha \in \mathbb{R}$. The theorems below reveal the asymptotic properties of the sums X_n, Y_n, Z_n, T_n, $J_n^{(n)}$ and $K_n^{(n)}$ as $n \to \infty$.

Throughout the paper we use the symbol "$\xrightarrow{d}$" to denote the convergence of random variables in distribution, and $\mathcal{N}(0,1)$ is a normal random variable with mean 0 and variance 1.

Theorem 1. *The following assertions hold:*

1) *for $m < 1$ the sequence of random variables X_n will converge with probability one;*

2) *for $m > 1$*

$$\frac{(m-1) \cdot X_n}{\sigma m^{n+1} \sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \qquad \text{as} \qquad n \to \infty;$$

3) *for $m = 1$*

$$\frac{\sqrt{3} \cdot X_n}{\sigma n \sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \qquad \text{as} \qquad n \to \infty.$$

Theorem 2. *The following assertions hold for the random variable Y_n:*

1) *for $m < 1$*

$$\frac{(m-1) \cdot Y_n}{\sigma \sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \qquad \text{as} \qquad n \to \infty;$$

2) *for* $m > 1$

$$\frac{(m-1)\cdot Y_n}{\sigma m^{n+1}\sqrt{m^2-1}} \xrightarrow{d} \mathcal{N}(0,1) \qquad as \qquad n \to \infty.$$

Comment. *We omit the case* $m = 1$*, since* $Y_n = X_n$ *in this case, and Part 3) of Theorem 1 holds identically.*

Theorem 3. *The following assertions hold:*

$$\frac{\alpha\sqrt{2\alpha}\cdot Z_n}{\sigma n\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \qquad as \qquad n \to \infty;$$

$$\frac{\alpha\sqrt{2\alpha}\cdot T_n}{\sigma n\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \qquad as \qquad n \to \infty.$$

Teorema 4. *The following assertions hold:*

$$\frac{\alpha\sqrt{2\alpha}\cdot J_n^{(n)}}{\sigma_n n\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \qquad as \qquad n \to \infty;$$

$$\frac{\alpha\sqrt{2\alpha}\cdot K_n^{(n)}}{\sigma_n n\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \qquad as \qquad n \to \infty.$$

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Primary and Secondary Collisions of Linear Cocyles over Irrational Rotations

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We consider a skew-product map

$$F_A : \mathbb{T}^1 \times \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}^2$$

defined for any $(x, v) \in \mathbb{T}^1 \times \mathbb{R}^2$ by

$$(x, v) \mapsto (\sigma_\omega(x), A(x)v),$$

where $\sigma_\omega(x) = x+\omega$ is a rotation of a circle $\mathbb{T}^1$ with irrational rotation number ω and

$$A : \mathbb{T}^1 \to SL(2, \mathbb{R})$$

is a differentiable function. We suppose the transformation A has a special form

$$A(x) = R(\varphi(x)) \cdot Z(\lambda(x)),$$

where

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad Z(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

and $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, $\lambda : \mathbb{T}^1 \to [\lambda_0, \infty)$ are C^1-functions such that φ has index 0 and λ_0 is sufficiently large positive constant.

It is also assumed that the critical set $\mathcal{C}_0 = \{x \in \mathbb{T}^1 : \cos\varphi(x) = 0\}$ consists of N points and the function φ satisfies the following conditions

$$|\varphi'(x)| \geq C_1\varepsilon^{-1}, \quad x \in \mathcal{C}_0; \quad |\cos\varphi(x)| \geq C_2, \quad \forall\, x \in \mathbb{T}^1 \backslash U_\varepsilon(\mathcal{C}_0),$$

where $U_\varepsilon(\mathcal{M})$ denotes a ε-neighborhood of a set $\mathcal{M}$, C_1, C_2 are some positive constants and $\varepsilon \ll 1$.

One notes that the hyperbolic properties of the cocycle $M(x,n) = A(\sigma_\omega^n(x)) \dots A(x)$ generated by F_A may become weaker as a trajectory of the point $x \in \mathbb{T}^1$ hits $U_\varepsilon(\mathcal{C}_0)$. On the other hand the irrationality of ω implies that trajectory of any point intersects $U_\varepsilon(\mathcal{C}_0)$ and, particularly, each point $x_0 \in \mathcal{C}_0$ interacts both with $U_\varepsilon(x_0)$ (primary collision) and $U_\varepsilon(x_1), x_1 \in \mathcal{C}_0, x_1 \neq x_0$ (secondary collision).

Using approach developed in [1] we show that for sufficiently small values of the parameter ε and under some additional require-ments on the function λ the secondary collisions may compensate weakening of the hyperbolicity due to primary collisions and the cocycle generated by F_A becomes hyperbolic in contrary to the case when secondary collisions are partially eliminated [1].

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Approximative Properties of Weakly and Strongly Convex Sets

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In the report some new approximative properties of weakly and strongly convex sets as well as a short review of known results on this topic are presented.

Application of Order Structures in the Study of Some Classes of Nonlinear Problems of Mathematical Physics

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In the study of various problems in mathematical physics order structures naturally arise. Ordered functional spaces with the order relation generated by the cone of nonnegative functions [1-3] can be used to study problems of mathematical physics, solutions of which are nonnegative in their physical meaning (the theory of diffusion processes, the theory of heat transfer, the theory of neutron and radiation transfer, the kinetic theory of gases). Here we should note, in particular, the classical results of the mathematical theory of nuclear reactors [4, 5].

The work considers some classes of nonlinear boundary value and initial boundary value problems for systems of integro-differential equations of the theory of radiation transfer, the theory of neutrino transfer, the theory of diffusion and heat transfer. For these problems, correctness issues are studied using Tarski theorem on fixed points of isotone operators acting in complete lattices [6]. The problems of order and metric stabilization of solutions of the corresponding nonstationary problems are investigated on the basis of general approaches related to the properties of semigroups of isotone operators acting in complete and conditionally complete lattices [7-10].

Linearizing algorithms for solving the considered nonlinear problems are proposed, the justification of which essentially uses the order properties of the operators arising in the investigation of problems. The numerical study of some nonlinear problems of mathematical physics are discussed.

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Asymptotics of Dynamic Andronov-Hopf Bifurcation

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The main object under consideration is a system of two differential equations with a small parameter $0 < \varepsilon \ll 1$

$$\varepsilon \frac{d\mathbf{x}}{d\tau} = \mathbf{f}(\mathbf{x};\mathbf{a}); \quad \mathbf{x} \in \mathbf{R^2}, \ \tau \in \mathbf{R^1}.$$

The right-hand side depends on a parameter $\mathbf{a} \in \mathbf{R^n}$. There is an equilibrium state $x \equiv \mathbf{p}(\mathbf{a})$ taken from functional equation

$$\mathbf{f}(\mathbf{x};\mathbf{a}) = 0.$$

The Andronov-Hopf bifurcation takes place [1] on the surface $S \subset \mathbf{R}^n$. Let be $\mathbf{a} = \mathbf{A}(\tau)\}$ a smooth line, which crosses the bifurcation surface at some moment: $\mathbf{A}(0) \in S$.

Problem for the non autonomous system with parameter depending on the slow time $\mathbf{a} = \mathbf{A}(\tau)$ is considered. We study the solutions, which in the leading order term of the asymptotics in small parameter coincide with zero of the right-hand side: $\mathbf{x} \equiv \mathbf{p}(\mathbf{A}(\tau))$ for $\tau < 0$. An asymptotic solution as $\varepsilon \to 0$ on a time interval $\tau \in (-\delta, \delta)$, including the bifurcation moment $\tau = 0$, is the purpose of the report.

Delay of the loss of stability occurs in the case $\delta = \text{const} > 0$. This problem was investigated by A.Neishtadt in [2]. We consider the case, when sturting point is close to bifurcation moment $\delta = \mathcal{O}(\sqrt{\varepsilon})$. The leading order term of the dynamic bifurcation is described by a solution of non autonomous Bernoulli equation.

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The Set of Zeros of the Riemann Zeta Function as the Spectrum of an Operator

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We construct a Sturm—Liouville operator on a semiaxis and a small perturbation of it so that the spectrum of the resulting operator coincides with the set of non-trivial zeros of the Riemann zeta function after a simple transformation of the complex plane.

Shilnikov Attractors in Three-Dimensional Non-Orientable Maps

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In this paper, we study peculiarities of Shilnikov homoclinic attractors of three-dimensional non-orientable diffeomorphisms. Recall that an attractor is called homoclinic if it contains a unique fixed point (an equilibrium in the case of flow systems) together with its unstable manifold, see Refs. [1,2]. As was shown in Ref. [3] such attractors can appear as a result of simple bifurcation scenarios and, thus, they are often found in applications. Among homoclinic attractors of three-dimensional maps the so-called discrete Shilnikov attractor containing a saddle-focus fixed point with the two-dimensional unstable manifold is of special interest by two reasons. First, it can appear as a result of a quite simple scenarios and, second, it can be hyperchaotic (with two positive Lyapunov exponents).

We consider the three-dimensional Hénon-like map $\bar{x} = y,\ \bar{y} = z,\ \bar{z} = Bx + Cy + Az - y^2$ with the negative Jacobian ($B < 0$) as a basic model demonstrating various types of Shilnikov attractors. Note that if $B = 0$ this map is efficiently a two-dimensional endomorphism introduced by C. Mirá in [4]. Thus, following Ref. [5], we also call this map *three-dimensional Mirá map*.

We show that depending on values of parameters A, C, and B such attractors can be of three possible types:

- hyperchaotic - with two positive Lyapunov exponents (LE));
- flow-like - with one positive and one very close to zero LE;

- and simply chaotic - with one positive and two negative LE.

For all types of attractors we study scenarios of their appearance in one- parameter families.

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The Global Conformal Gauge in String Theory

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The global conformal gauge is playing the crucial role in string theory providing the basis for quantization. Its existence for two-dimensional Lorentzian metric is known locally for a long time. We prove that if a Lorentzian metric is given on a plain then the conformal gauge exists globally on the whole $\mathbb{R}^2$. Moreover, we prove the existence of the conformal gauge globally on the whole worldsheets represented by infinite strips with straight boundaries for open and closed bosonic strings.

On Bifurcations of Lorenz Attractors in the Lyubimov-Zaks Model

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We provide a numerical evidence for the existence of the Lorenz and Rovella (contracting Lorenz) attractors in the extended Lorenz model proposed by Lyubimov and Zaks. Recall that the Lorenz attractor is robustly chaotic (pseudohyperbolic) in contrast to the Rovella attractor which is only persistent (it exists on a set of parameter values, which is nowhere dense and has a positive Lebesgue measure). It is well known that in this model, for a certain values of parameters, there exists a homoclinic butterfly bifurcation with the symmetric saddle equilibrium $O(0, 0, 0)$, which is neutral, i.e. its eigenvalues $\lambda_2 < \lambda_1 < 0 < \gamma$ are such that the saddle index $\nu = -\lambda_1/\gamma$ is equal to 1. The birth of the Lorenz attractor under this codimension-two bifurcation is established by means of numerical verification of the well-known Shilnikov criterion. For the birth of the Rovella attractor, we propose a new criterion that is also verified numerically.

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Generalized Invariant Manifolds for Integrable Equations and Their Applications

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In the talk will discuss the notion of the generalized invariant manifold introduced in our previous study (see [1]-[6]). In the literature, the method of the differential constraints is well known as a tool for constructing particular solutions for the nonlinear partial differential equations. Its essence is in adding to a given nonlinear PDE, another much simpler, as a rule ordinary, differen-tial equation, consistent with the given one. Then any solution of the ODE is a particular solution of the PDE as well. However the main problem is to find this consistent ODE. Our generalization is that we look for an ordinary differential equation that is consistent not with the nonlinear partial differential equation itself, but with its linearization. Such generalized invariant mani-fold is effectively sought. Moreover, it allows one to construct such important attributes of integrability theory as Lax pairs and recursion operators for integrable nonlinear equations. In the talk will show that they provide a way to construct particular solutions to the equation as well [7].

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On the Stationary Measure Conservation Laws for the Stochastic System of the Lorenz Model Describing a Baroclinic Atmosphere

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We consider the system of equations for the quasi-solenoidal Lorenz model for a baroclinic atmosphere

$$\frac{\partial}{\partial t}A_1 u + \nu A_2 u + A_3 u + B(u) = g, \qquad t > 0, \tag{1}$$

on the two-dimensional unit sphere S centered at the origin of the spherical polar coordinates (λ, φ), $\lambda \in [0, 2\pi)$, $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\mu = \sin\varphi$. Here $\nu > 0$ is the kinematic viscosity, $u(t,x,\omega) = (u_1(t,x,\omega), u_2(t,x,\omega))^{\mathrm{T}}$ is an unknown vector function and

$$g(t,x,\omega) = (g_1(t,x,\omega), g_2(t,x,\omega))^{\mathrm{T}}$$

is a given vector function, $x = (\lambda, \mu)$, $\omega \in \Omega$, (Ω, P, F) is a complete probability space,

$$A_1 = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + \gamma I \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \Delta^2 & 0 \\ 0 & \Delta^2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -k_0\Delta & 2k_0\Delta \\ k_0\Delta & -(2k_0 + k_1 + \nu\gamma)\Delta + \rho I \end{pmatrix},$$

$$B(u) = \big(J(\Delta u_1 + 2\mu, u_1) + J(\Delta u_2, u_2),$$
$$J(\Delta u_2 - \gamma u_2, u_1) + J(\Delta u_1 + 2\mu, u_2)\big)^{\mathrm{T}}.$$

Also, $\gamma, \rho, k_0, k_1 \geq 0$ are numerical parameters, I is the identity operator, $J(\psi, \theta) = \psi_\lambda \theta_\mu - \psi_\mu \theta_\lambda$ is the Jacobi operator and $\Delta\psi = ((1-\mu^2)\psi_\mu)_\mu + (1-\mu^2)^{-1}\psi_{\lambda\lambda}$ is the Laplace-Beltrami operator on the sphere S. A random vector function $g = f + \eta$ is taken as the right-hand side of (1); here $f(x) = (f_1(x), f_2(x))^{\mathrm{T}}$ and $\eta(t,x,\omega) = (\eta_1(t,x,\omega), \eta_2(t,x,\omega))^{\mathrm{T}}$ is a white noise in t. In [1] it was obtained for existence of a unique stationary measure of Markov semigroup defined by solutions of the Cauchy problem for (1) and for the exponential convergence of the distributions of solutions to the stationary measure as $t \to +\infty$ the sufficient conditions on the right-hand side of (1) and the parameters ν, γ, ρ, k_0, k_1: $Zk_0 < F(\nu,\gamma,\rho,k_1)$, (2) where $F > 0$ is some real function of the arguments ν, γ, ρ, k_1.

In the present work it was proven the theorem (for definitions of $\{b_i\}_{i=1}^{\infty}$, $\{E_i\}_{i=1}^{\infty}$, l_2^+, H^p, $|||\cdot|||_p$, $p \in Z$, $\langle\cdot,\cdot\rangle$, $P(X)$ see the third paragraph of [1]).

Theorem.

Let $\nu > 0$, $\gamma, \rho, k_0, k_1 \geq 0$ be such that inequality

$$k_0 \leq \min\left\{4k_1, \frac{4(2+\gamma)}{(2-\gamma)^2}(2k_1+\rho)\right\} \tag{3}$$

is satisfied, let $\{b_i\}_{i=1}^{\infty} \in l_2^+$ and let $f \in H^{-1}$. Then any stationary measure $\mu \in P(H^2)$ of system (1) is satisfied the two conservation laws:

$$\int_{H^2} \langle(\nu A_2 + A_3)\vartheta, A_1\vartheta\rangle\, \mu(d\vartheta) = \frac{\mathrm{b}^2}{2} + \int_{H^2} \langle f, A_1\vartheta\rangle\, \mu(d\vartheta),$$

$$\int_{H^2} \langle(\nu A_2 + A_3)\vartheta, \vartheta\rangle\, \mu(d\vartheta) = \frac{1}{2}\sum_{i=1}^{\infty} b_i^2 \langle A_1^{-1}E_i, E_i\rangle + \int_{H^2} \langle f, \vartheta\rangle\, \mu(d\vartheta),$$

here

$$\langle A_3\vartheta, A_1\vartheta\rangle \geq 0 \quad \textit{and} \quad \langle A_3\vartheta, \vartheta\rangle \geq 0 \quad \textit{for all } \vartheta \in H^2.$$

Note that the right-hand side of inequality (3) is less than the right-hand side of inequality (2).

A similar result is obtained for the equation of a barotropic atmosphere and the two-dimensional Navier-Stokes equation. A comparative analysis with some of the available related results is given for the latter.

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Carleman Estimates for Globally Convergent Numerical Methods for Coefficient Inverse Problems

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Since the field of Inverse Problems is an applied one, it is insufficient just to prove some theorems. Rather it is necessary to develop reliable numerical methods. However, conventional numerical methods for Coefficient Inverse Problems (CIPs) are unreliable. The reason is that they are based on the minimization of least squares cost functionals. These functionals are non convex. Therefore, as a rule, they have many local minima and ravines. Since any minimization procedure can stop at any local minimum, which can be far from the true solution, then these methods are unreliable and unstable.

In the past several years Klibanov and his research team have successfully developed a radically new and very effective numerical method of solving CIPs. Furthermore, this method is verified on a variety of microwave experimental data. This is the so-called "convexification" method. In the convexification one constructs a globally strictly convex weighted Tikhonov-like functional. Therefore, the problem of local minima is avoided. Furtheremore, the convexification generates globally convergent numerical methods. The key to this functional is the presence in it of the so-called Carleman Weight Function. This is the function which is involved as the weight in the Carleman estimate for the corresponding Partial Differential Operator.

The convexification will be presented for a broad variety of CIPs. Numerical results will also be presented for both compu-tationally simulated and experimental data.

The Hausdorff Dimension of the Besicovitch Set and the Velocity Tending to Infinity.

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Let $T_\rho\colon \mathbb{T} \to \mathbb{T}$ be an irrational circle rotation: $T_\rho x = x + \rho \pmod 1$, $f\colon \mathbb{T} \to \mathbb{R}$ be a continuous function with a zero mean. We consider a cylindrical transformation $T_{\rho,f}\colon \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ with a cocycle f: $T_{\rho,f}(x,y) = (T_\rho x, y + f(x))$. The iterations of a point (x,y) are described by

$$T^n_{\rho,f}(x,y) = (T^n_\rho x, y + f^n(x)), \quad n \in \mathbb{Z},$$

where $f^n(x)$ is the n-th Birkhoff sum.

A.S. Besicovitch [1] showed that for any irrational circle rotation T_ρ, there exists a continuous f such that $T_{\rho,f}$ is topologically transitive and has orbits tending to infinity (discrete orbits).

The *Besicovitch set* of points in the circle $\mathbb{T}\times\{0\}$ having discrete orbits, has the null Lebesgue measure, but may have the positive, and even full, Hausdorff dimension [2], [3].

The question of the velocity of the trajectory's tendency to infinity was formulated by A. B. Antonevich. This question is related to the spectral properties of the weighted shift operator.

By the ergodicity of T_ρ, the velocity of escaping to infinity tends to zero.

It is shown that the decreasing of this velocity may be arbitrarily slow.

As a consequence, for the weighted shift operator, it is shown (A.A.) that *the norm of the resolvent may grow arbitrarily fast as the spectral parameter approaches the spectrum.*

Additionally it is proved (A.K.), that *the subset of the points in* $\mathbb{T} \times \{0\}$ *with a slow decrease in the escape velocity may have the Hausdorff dimension equal 1.*

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Berezin-Toeplitz Quantizations Associated with Landau Levels of the Bochner Laplacian on a Symplectic Manifold

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We consider the Bochner Laplacian on high tensor powers of a positive line bundle on a compact symplectic manifold. First, we give a rough asymptotic description of its spectrum. Under certain assumption on the Riemannian metric, it allows us to prove clustering of the spectrum near some points, which can be naturally called Landau levels. Then we fix an arbitrary cluster and develop the Toeplitz operator calculus with the quantum space, which is the eigenspace of the Bochner Laplacian with eigenvalues from this cluster. We show that such a calculus provides a Berezin-Toeplitz type quantization of the symplectic manifold.

Attractors of the Nonlocal Ginzburg-Landau Equation

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Many branches of physics use the corresponding partial differen-tial equation:

$$u_t = u - (1 + ic)u|u|^2 + (a + ib)u_{xx}, \qquad (1)$$

which is known as the complex Ginzburg-Landau equation [1]. In this case $a, b, c \in \mathbb{R}$,
$a \geq 0$ [1]. In connection with problems arising in the theory of ferromagnetism, the following special version of equation (1) is used (for instance see [2])

$$u_t = u - (1 + ic)u(\frac{1}{2\pi}\int_0^{2\pi} |u|^2 dx) + (a + ib)u_{xx}. \qquad (2)$$

Next we will consider equation (2) together with periodic boundary conditions. We can rewrite periodic boundary conditions after a normalization of variable x in the form

$$u(t, x + 2\pi) = u(t, x). \qquad (3)$$

Next we will study the boundary value problem (2), (3) with the cognate initial condition

$$u(0, x) = f(x). \qquad (4)$$

We will distinguish the differences between two cases when $a > 0$ and $a = 0$. When $a > 0$, the following theorem holds true.

Theorem 1. *Suppose* $f(x) \in \mathbb{L}_2(0, 2\pi)$ *if* $x \in (0, 2\pi)$. *Then for all* $t > 0$, *the initial value problem (2), (3), (4) has a solution* $u(t, x)$, *where the function* $u(t, x)$ *has the corresponding properties:*

1) $\lim\limits_{t \to 0+} u(t, x) = f(x)$, *where the limit is interpreted in the sense of the norm of the space* $\mathbb{L}_0(0, 2\pi)$;

2) when $t > 0$ *and* $x \in \mathbb{R}$ *the function* $u(t, x)$ *has partial derivatives of any order.*

Next we denote C_0 the cycle of the boundary value problem (2), (3) formed by spatially homogeneous periodic solutions $u_0(t, x) = u_0(t) = \exp(-ict + i\varphi_0), \varphi_0 \in \mathbb{R}$. C_k denotes the three-dimensional invariant manifold of the phase space formed by the periodic solutions

$$u_k(t, x) = \eta_k \exp(i\sigma_k t + ikx + i\varphi_k) + \eta_{-k} \exp(i\sigma_{-k} t - ikx + i\varphi_{-k}),$$

where $\varphi_k, \varphi_{-k} \in \mathbb{R}, k = 1, \ldots, m, \eta_k, \eta_{-k} \in \mathbb{R}$ è $\eta_k + \eta_{-k} = a_k = 1 - ak^2 > 0, \sigma_k = \sigma_{-k} = -bk^2 - ca_k$. In this case $m = [1/\sqrt{a}]$, if $1/\sqrt{a} \notin \mathbb{N}$ and $m = [1/\sqrt{a}] - 1$, if $1/\sqrt{a} \in \mathbb{N}$ (the set of natural numbers). Now let $A_m = \{0\} \cup_{k=0}^{m} C_k$.

Theorem 2. A_m *– the global attractor of boundary value problem (2), (3).*

Suppose that $a = 0$. Then the initial value boundary value problem (2), (3), (4) has a solution if $f(x) \in \mathbb{H}_2^2$, those $f(x)$ is within the Sobolev space $\mathbb{W}_2^2[0, 2\pi]$. By V we denote the collection of those $f(x) \in \mathbb{H}_2^2$, for which $\frac{1}{2\pi} \int\limits_0^{2\pi} |f(x)|^2 dx = 1$.

Theorem 3. $A_\infty = \{0\} \cup V$ *is a global attractor of the initial value problem* $a = 0$.

Note that all the solutions for such a, except for zero, approach V in the norm of the space $\mathbb{W}_2^2[0, 2\pi]$. Some of the results are published in [3-5].

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Classification of Integrable Two-Dimensional Lattices

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In our papers [1, 2] the algorithm for classification of integrable equations with three independent variables was proposed. This method is based on the requirement of the existence of an infinite set of Darboux integrable reductions and on the notion of the characteristic Lie-Rinehart algebras. The method was applied for the classification of integrable cases of different subclasses of equations

$$u_{n,xy} = f(u_{n+1}, u_n, u_{n-1}, u_{n,x}, u_{n,y}), \quad -\infty < n < \infty$$

of special forms (see, for example, [3, 4]). Under this approach the novel integrable chain

$$u_{n,xy} = \alpha_n(u_{n,x} - u_n^2 - 1)(u_{n,y} - u_n^2 - 1) + 2u_n(u_{n,x} + u_{n,y} - u_n^2 - 1)$$

was obtained. Here $\alpha_n = \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1} - u_n}$, the sought function $u_n = u_n(x, y)$ depends on the real x, y and the integer n. In paper [5] we constructed Lax pair for this chain. We verified that the found Lax pair is not "fake" by studying the periodic reduction of the chain and the Lax pair. For the system of the hyperbolic type equations obtained from the chain we constructed generalized symmetry of the second order which has unusual structure.

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Dynamics of the Impact Body Colliding with a Moving Belt

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In the field of mechanical engineering, conveyors and moving belts are frequently used machine parts. In a lot of working regimes they are subjected to sudden loading, which can be a source of irregular motion of the impacting bodies and undesirable behaviour of the working machine. This contribution deals with a mechanical model where colisions between an impact body and a moving belt takes place. The impact body is constrained by a flexible rope, the upper end of which is excited by a slider in the vertical direction. The behaviour of the system was investigated in dependence of amplitude and frequency of excitation given by the movement of the slider, belt speed, and eccentricity of the centre of gravity of the impact body (see [1-3]). Outputs of the computations indicate that different combinations of the analysed parameters lead to a high complexity of the system movement. The bifurcation analysis shows multiple periodic areas changed by chaotic regions. The carried out research provides more details about the behaviour and properties of strongly nonlinear mechanical systems coming from impacts and dry friction. The obtained information will enable designers to propose parameters for industrial machines that make it possible to avoid their working at undesirable operating levels.

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Graded Semigroup C^*-Algebras and Banach Modules

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Let S be a cancellative semigroup with the identity. The object of our investigation is the reduced semigroup C^*-algebra $C_r^*(S)$ generated by the left regular representation of the semigroup S. We study the structure of $C_r^*(S)$ which is considered as a Banach module over its subalgebra.

Let G be a group and $\sigma : S \longrightarrow G$ be a surjective homomorphism of semigroups. Using the homomorphism σ, we introduce the notion of the σ-index for an operator monomial. The σ-index is the main tool for the construction of topological grading over the group G for the C^*-algebra $C_r^*(S)$. Namely, for $g \in G$, let $\mathfrak{A}_g$ be the closed linear hull for the set of all operator monomials with the σ-index g in $C_r^*(S)$. The family of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ constitutes a topological G-grading for the reduced semigroup C^*-algebra $C_r^*(S)$. Note that the Banach space $\mathfrak{A}_e$, where e is the identity of the group G, is a C^*-subalgebra of the C^*-algebra $C_r^*(S)$.

For each $g \in G$, the space $\mathfrak{A}_g$ is a cyclic Banach $\mathfrak{A}_e$-module. In order to get a generator of this module we take an arbitrary element x_g from the preimage set $\sigma^{-1}(g)$. Then one has the equality

$$\mathfrak{A}_g = \mathfrak{A}_e \cdot T_{x_g}.$$

Let us consider a set $X \subset S$ such that for every $g \in G$ there exists a unique element $x \in X$ satisfying the condition $X \cap \sigma^{-1}(g) = \{x\}$. We call X a set of representatives for the preimages $\sigma^{-1}(g)$, where g runs over the group G. Then the C^*-algebra $C_r^*(S)$ is a Banach $\mathfrak{A}_e$-module with the generating set $\{T_x \mid x \in X\}$.

In what follows, G is a finite group. In this case the underlying linear space of the C^*-algebra $C_r^*(S)$ is represented as the finite direct sum of its subspaces:

$$C_r^*(S) = \bigoplus_{g\in G} \mathfrak{A}_g.$$

If there exists a set X of representatives for the preimages $\sigma^{-1}(g)$, $g \in G$, which is contained in a subgroup of the semigroup S, then the $\mathfrak{A}_e$-module $C_r^*(S)$ is topologically isomorphic to the free Banach $\mathfrak{A}_e$-module

$$C_r^*(S) \cong \bigoplus_{1} \mathfrak{A}_e.$$

Here the number of summands in the direct l_1-sum is equal to the order of the group G.

We recall that for the topologically G-graded C^*-algebra $C_r^*(S)$ there is a conditional expectation $F : C_r^*(S) \longrightarrow \mathfrak{A}_e$. In the case of a finite group G, the conditional expectation F is faithful. Therefore, one can define the $\mathfrak{A}_e$-valued inner product on the C^*-algebra $C_r^*(S)$ by the formula

$$\langle A, B\rangle = F(AB^*),$$

where $A, B \in C_r^*(S)$. Then the C^*-algebra $C_r^*(S)$ becomes a finitely generated projective Hilbert $\mathfrak{A}_e$–module.

In the talk we shall discuss some results from [1-3].

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On Ergodic Theorems for Flows and Generalized Lyapunov Exponents

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The report considers asymptotic properties of measurable cocycles over measure preserving flows. The classical Oseledets multiplicative ergodic theorem (MET) describes the asymptotic behaviour of $GL(n,\mathbb{R})$-valued cocycles $A(t,x)$ under the condition

$$\sup_{0\leqslant t\leqslant 1} \ln^{+} \|A(t,\cdot)^{\pm 1}\| \in L^1.$$

In his work, V.I. Oseledets formulated the question of convergence in Birkhoff's theorem and MET for cocycles integrable for each individual t. In the joint work of the speaker with A.M. Stepin (2016) it was shown that, in general, such convergence in all t does not take place but it holds along time subsets of density 1. In particular, almost everywhere with respect to invariant measure, there exist exact Lyapunov exponents in the sense of convergence in density. Such generalized Lyapunov exponents seem to be natural from the applied point of view since they are not affected by large but rare measurement errors.

It turns out that the convergence in MET holds if one just neglects subsets of time axis of finite measure. The same is true for infinite-dimensional versions of MET and other generalizations of the Birkhoff theorem.

To the Question on the Existence of Periodic Points of Continuous Maps on Dendrites

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By continuum we mean a compact connected metric space. Dendrite is a locally connected continuum without subsets homeomorphic to a circle. Let X be a dendrite with a metric d. Dendrites have the next properties (see, e.g., [1] — [2]):

1) a dendrite is one-dimensional continuum;

2) for any points $x, y \in X$ there is a unique arc containing these points;

3) the set of branch points of X is at most countable;

4) the number of connected components of the set $X \setminus \{p\}$ is at most countable for any point $p \in X$.

Let $f : X \to X$ be a continuous map of a dendrite X. Then f has a fixed point in X [3]. Dendrites are not a linear-ordered set, therefore many authors studied the conditions of the existence of periodic points of f on subcontinua in X [4] – [10]. In the report we study both the properties of a given map f and the structure of a dendrite X under which a subcontinuum $A \subset X$ with $A \subset f(A)$ contains a periodic point of f. Obtained results allowed to prove the Mendez's conjecture for such maps [11].

Let 2^X denote the set of all nonempty closed subsets of a dendrite X endowed with the Hausdorff metric d_H induced by d. Let $2^f : 2^X \to 2^X$ be the induced map by f in the hyperspace 2^X. Mendez's conjecture: for each continuous map $f : X \to X$ of a dendrite X the density of $Per(2^f)$ in 2^X is equivalent to the density of $Per(f)$ in X [11].

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Minimizing Properties of the Viscosity Solution of the Eikonal equation in a singularly perturbed variational problem

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We are interested in the asymptotic behavior as $\varepsilon \to 0^+$ of the following functionals introduced by Aviles and Giga [**?**]:

$$F_\varepsilon(u,\Omega) := \int_\Omega \left(\varepsilon |\nabla^2 u| + \frac{1}{\varepsilon} \left| 1 - |\nabla u|^2 \right|^2 \right) dx, \qquad \text{where } \Omega \subset \mathbf{R}^2.$$

It is conjectured in [**?**] that under appropriate boundary conditions if Ω is smooth and convex, then the minimizers u_ε of $F_\varepsilon(\cdot,\Omega)$ converge to

$$\bar{u} := \mathrm{dist}(\cdot, \partial\Omega).$$

We answer positively to this conjecture when Ω is an ellipse and for some other special domains.

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Decay of Solutions to Damped Kawahara Qquation

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We will consider an initial-boundary value problem for the damped Kawahara equation:

$$u_t - u_{xxxxx} + bu_{xxx} + au_x + g(x)u + \gamma uu_x = 0, \qquad (1)$$

$$u(0,x) = u_0(x),\ x \geq 0, \qquad u(t,0) = u_x(t,0) = 0,\ t \geq 0, \quad (2)$$

where $a, b \in \mathbb{R}$ and $\gamma = \pm 1$, is considered. Here $g(x)$ is a positive function, so $g(x)u$ is a damping term. The result on global existence and large-time decay of solutions is established.

Define special weighted spaces: let for $\alpha \in \mathbb{R}$

$$L_2^\alpha(\mathbb{R}_+) = \{\varphi(x) : (1+x)^\alpha \varphi \in L_2(\mathbb{R}_+)\}.$$

For $T > 0$ let $\Pi_T^+ = (0,T) \times \mathbb{R}_+$. Solutions to the considered problem are constructed in a space

$$X_2^\alpha(\Pi_T^+) = \big\{u \in C([0,T]; H_0^2(\mathbb{R}_+) \cap L_2^\alpha(\mathbb{R}_+)),$$
$$\partial_x^j u \in C_b(\overline{\mathbb{R}}_+; H^{(4-j)/5}(0,T)),\ 0 \leq j \leq 4,$$
$$u \in L_8(0,T; C_b^2(\overline{\mathbb{R}}_+)), \quad u \in L_2(\mathbb{R}_+; C[0,T]),$$
$$u_{xx} \in L_2(0,T; L_2^{\alpha-1/2}(\mathbb{R}_+))\big\}$$

for $\alpha > 0$.

The main result is the following theorem.

Theorem 1. *Let* $u_0 \in H_0^2(\mathbb{R}_+) \cap L_2^\alpha(\mathbb{R}_+)$ *for certain* $\alpha > 0$, $g \in W_\infty^2(\mathbb{R}_+)$ *and there exist positive constants* M *and* c_0 *such that for a.e.* $x > 0$

$$g(x) \geq \frac{c_0}{1+x},$$

$$|g'(x)| \leq Mg(x), |g''(x)| \leq Mg(x).$$

Then there exists a unique weak solution $u(t,x)$ *to problem (1), (2), such that* $u \in X_2^\alpha(\Pi_T^+)$ $\forall T > 0$. *Moreover,*

$$\lim_{t \to +\infty} \|u(t,\cdot)\|_{L_2(\mathbb{R}_+)} = 0.$$

The weak solution is understood in the Sobolev sense. Note that the damping effect vanishes at $+\infty$.

Previously for the damped Korteweg–de Vries equation similar results were established in [1], but only in the case of small initial data. For the damped Kawahara equation itself (with the nonlinearity uu_x) such rusults were obtained in [2].

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Attractors of Direct Products

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For Milnor, statistical, and minimal attractors, we construct examples of smooth flows φ on S^2 for which the attractor of the Cartesian square of φ is smaller than the Cartesian square of the attractor of φ. In the example for the minimal attractors, the flow φ also has a global physical measure such that its square does not coincide with a global physical measure of the square of φ.

We are interested in attractors definitions of which rely on a natural (in our case, Lebesgue) measure on the phase space, which allows these attractors to capture asymptotic behavior of most points while possibly neglecting what happens with a set of orbits of zero measure. One type of such attractors was introduced by J. Milnor in [2] under the name "the likely limit set". We refer to it as the Milnor attractor.

Milnor attractor, [2] The Milnor attractor $A_{Mil}(\varphi)$ of a dynamical system φ is the smallest closed set that contains the ω-limit sets of μ-almost all orbits.

Another way to define an attractor is via an "attracting" invariant measure: the attractor is its support. Most suitable are the notions of *physical* and *natural* measures, which may be viewed as analogues of SRB-measures for general, non-hyperbolic dynamical systems (see, e.g., [1]; we adapt the definitions from [1] to the case of flows). Physical measures describe the distribution of μ-a.e. orbit, while natural measures capture the limit behaviour of the reference measure.

For a flow φ on a compact manifold X with measure μ, a probability measure ν is called *physical* if there is a set B with $\mu(B) > 0$ such that for any $x \in B$ and any continuous function $f \in C(X, \mathcal{R})$ the Birkhoff time average over the orbit of x is equal to the space average w.r.t. ν:

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\varphi^t(x))dt = \int_X f \, d\nu.$$

A measure ν is called *natural* for a flow φ if there exists an open subset $U \subset X$ such that for any probability measure $\tilde{\mu}$ absolutely continuous with respect to μ and with $\operatorname{supp}(\tilde{\mu}) \subset U$ one has weak-$*$ convergence

$$\frac{1}{T} \int_0^T \varphi_*^t \tilde{\mu} \, dt \to \nu, T \to +\infty.$$

Statistical and minimal attractors are supports of these measures, respectively.

When constructing examples of dynamical systems with required properties, it is not uncommon to utilize, at least as a piece of the construction, direct products of systems in lower dimensions. It is tempting to think that the attractor of the direct product of two systems always coincides with the direct product of their attractors. Although this holds, indeed, for so-called maximal attractors, this is not true for several other types of attractors, namely, for Milnor, statistical, and minimal attractors, and also for the supports of physical measures, when the latter exist. We present examples of smooth flows on S^2 that exhibit such non-coincidence. Our examples are mostly Cartesian squares of flows. Although the square of a flow is always a flow of infinite codimension, it is interesting to find, for every type of attractor, the least codimension for the flow itself in which one can have non-coincidence between the attractor of the square and the square of the attractor.

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Topological Entropy of Bunimovich Stadium Billiards

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We estimate from below the topological entropy of the Bunimovich stadium billiards. We do it for long billiard tables, and find the limit of estimates as the length goes to infinity.

On Non-Commutative Operator Graphs Defined via Various Unitary Groups

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A non-commutative operator graph is a linear subspace $\mathcal{V}$ in the space of bounded linear operators on a Hilbert space with the property $A \in \mathcal{V} \Rightarrow A^* \in \mathcal{V};\ I \in \mathcal{V}$. Those objects help to determine the ability to encode quantum information in such a way that it can be transmitted with zero error. In the context of this problem, we are interested in graphs satisfying the Knill-Laflamme condition, the graph $\mathcal{V}$ and the orthogonal projection P satisfy that condition iff $\dim P\mathcal{V}P = 1$. The projection P is called a quantum anticlique.

We will discuss that problem for non-commutative operator graphs generated by positive operator-valued measures (POVMs). It is possible to construct several examples of the non-commutative operator graphs generated POVMs defined via the groups determining the dynamics of quantum systems, for that examples possible to show the existence of anticliques. That problem is solved for the two-mode quantum oscillator, Jaynes-Cummings model, and in the bosonic Fock space.

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Estimation of Reachable and Controllability Sets for an Open Two-Level Quantum System Driven by Coherent and Incoherent Controls

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Control of individual quantum systems is an important direction in modern quantum technologies [1]. In laboratory quantum systems often interact with the environment which can hinder the control. However, it can also be used for actively controlling the system. A powerful method of incoherent control was proposed and studied in [2]. In this case, spectral density of the environment, i.e., distribution of its particles in momenta and internal degrees of freedom, is the control. General method using combinations of coherent and incoherent controls, either subsequent of simultaneous, was proposed for any multilevel quantum systems [2]. Numerical simulations were performed for an explicit example of four level systems using global optimization by genetic algorithm. In [3], a significant advance was made. It was shown that coherent and incoherent controls combined together allow to approximately steer *any* initial density matrix to *any* target density matrix. This result has several important features. (1) It allows to approximately realize strongest possible degree of quantum state control — controllability of open quantum systems in the set of all density matrices. (2) It is obtained with a physical class of GKSL master equations well known in quantum optics. (3) It was obtained in the generic case — for n-level quantum systems of arbitrary dimension and for almost all values of system parameters. (4) This scheme provides an explicit analytic solution for incoherent control. (5) The control method is robust to variations of the initial state — the optimal control steers simultaneously *all* initial states into the target state, thereby physically realizing all-to-one Kraus maps previously theoretically proposed for quantum control in [4].

In [5–9], this method was applied to analyze control in a two-level open quantum system governed by the Gorini-Kossakowski-Sudarshan-Lindblad master equation [2],

$$d\rho(t)/dt = -i\left[\mathbf{H}_0 + \mathbf{V}v, \rho(t)\right] + \mathcal{L}_{n(t)}(\rho(t)), \rho(0) = \rho_0$$

for the case of $N = 2$ levels, with various objective criteria, classes of coherent v and incoherent n controls, constraints on controls' magnitudes and variations, in combination with various methods for optimization of controls and for estimating reachable and controllability sets, and using machine learning. The constraint $n(t) \geq 0$ is obligatory. Time-minimal control problems (steering from ρ_0 to ρ_{target} in minimal time) were considered in [5, 6]. In [6], we used machine learning (kNN, neural networks) for generating suboptimal final time and controls. The training dataset was formed using the differential evolution and dual annealing optimization methods. In [7], the goal was to maximize $\langle \rho(T), \rho_{\mathrm{target}} \rangle$ in minimal time. In [8, 9], various tools (support hyperplanes, sections, etc.) were used to numerical estimate reachable and controllability sets. An exact description of such sets for some class of controls was obtained. An "underwater rock" was illustrated by the example where the controls $v = 0$ and $n = 0$ satisfy the Pontryagin maximum principle but they are not optimal. Comparative results were obtained using the dual annealing method, Krotov method, and Gradient Ascent Pulse Engineering with one- and- two-step gradient projection methods to illustrate (1) how zero controls v and n are far from the optimized controls and (2) the importance of adjusting the parameters of the methods. This talk presents the work partially

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A Quasilinear Differential Game of Neutral Type with Integral Constraints in a Hilbert Space

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The paper considers the problem of pursuit in the sense of L.S. Pontryagin for a quasilinear differential game of neutral type in a Hilbert space. Two main theorems on the solvability of the pursuit problem are proved.

Dirichlet Problem in Half-Spaces for Elliptic Differential-Difference Equations

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The Dirichlet problem in the half-space $\left\{(x,y)\middle|x\in\mathbf{R}^n, y\geq 0\right\}$ is considered for the equation

$$\frac{\partial^2 u}{\partial x_1^2}(x,y)+a\frac{\partial^2 u}{\partial x_1^2}(x_1+h,x_2,\dots,x_n,y)+ \\ +\sum_{j=2}^{n}\frac{\partial^2 u}{\partial x_j^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y)=0 \qquad (1)$$

under the assumption that $|a|<1$, h is an arbitrary real parameter, and the boundary-value data is summable.

The Poisson kernel is constructed such that its convolution with the boundary-value function satisfies the considered Dirichlet problem in the sense of generalized functions and satisfies Eq. (1) in $\left\{(x,y)\middle|x\in\mathbf{R}^n, y>0\right\}$ in the classical sense.

Efficient Semiclassical Asymptotics

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We give an overview of modern versions and modifications of Maslov's canonical operator permitting one to construct efficient asymptotics easy to implement on technical computing systems like *Wolfram Mathematica* or *MatLab* in various problems. The talk is based on the results of many years of joint work by S. Yu. Dobrokhotov, A. I. Shafarevich, and the author.

Sparks of Dictionaries in Sparse Representations

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Let $\mathbf{\Phi}$ be $d \times n$-matrix with real or complex numbers, and the columns of $\mathbf{\Phi}$ are ℓ_2-normalized. Consider a linear underdetermined set of equations

$$\mathbf{\Phi}\alpha = \mathbf{x}.$$

We shall refer hereafter to $\mathbf{x}$ as a signal to be processed, and α will stand for its *representation*. The matrix $\mathbf{\Phi}$ will be referred to as the *dictionary*, and its columns $\{\varphi_i\}_{i=1}^n$ will be called *atoms* [1].

Assuming $\mathbf{\Phi}$ is full rank, we have received that:

(i) the set $\{\varphi_i\}_{i=1}^n$ becomes a *unit-norm frame* for the $\mathbf{H}_d$, the matrix $\mathbf{\Phi}$ becomes the matrix of *the synthesis operator* of the frame $\{\varphi_i\}_{i=1}^n$ [2];

(ii) there is an infinite number of representations.

Such a situation is unacceptable, since we aim to get a single representation to our problems. In order to be able to choose the best single solution, we require some quality measure $J(\alpha)$ that basically ranks the solutions based on some measure, this way choosing the solution that solves the problem

$$\dot{\alpha} = \arg\min_{\alpha} J(\alpha) \text{ s.t. } \mathbf{\Phi}\alpha = \mathbf{x}.$$

The theory of sparse representations proposes ℓ_0-norm instead of $J(\alpha)$, as a function counting the number of non-zeros in α.

[1]The work of the author was carried out as part of the implementation development programs of the Scientific and Educational Mathematical Center Volga Federal District, Agreement No. 075-02-2020-1488/1

So we have the following minimization problem:

$$\dot{\alpha} = \arg\min_{\alpha} \|\alpha\|_0 \text{ s.t. } \mathbf{\Phi}\alpha = \mathbf{x}. \qquad (P_0)$$

In many cases our system might contain noise, and we cannot expect a perfect solution to the linear system $\mathbf{\Phi}\alpha = \mathbf{x}$. Instead, we require some proximity between $\mathbf{\Phi}(\dot{\alpha})$ and $\mathbf{x}$, arriving at the following alternative problem:

$$\dot{\alpha} = \arg\min_{\alpha} \|\alpha\|_0 \text{ s.t. } \|\mathbf{\Phi}\alpha - \mathbf{x}\|_2 \leq \epsilon, \qquad (P_0^{\epsilon})$$

where ϵ is closely related to the properties of the noise.

By the substitution ℓ_0-norm by the ℓ_1-norm we receive the following problem, which brings us to the Basis Pursuit (BP) algorithm:

$$\dot{\alpha} = \arg\min_{\alpha} \|\alpha\|_1 \text{ s.t. } \|\mathbf{\Phi}\alpha - \mathbf{x}\|_2 \leq \epsilon, \qquad (P_1^{\epsilon})$$

Definition 1. Given a matrix $\mathbf{\Phi}$, $\sigma(\mathbf{\Phi}) = \text{Spark}(\mathbf{\Phi})$ is the smallest number of atoms from $\mathbf{\Phi}$ that are linearly dependent.

The spark of a matrix $\mathbf{\Phi}$ is the size of the smallest linearly dependent subset of columns, i. e.,

$$\text{Spark}(\mathbf{\Phi}) = \min\{\|\mathbf{x}\|_0 : \mathbf{\Phi}\mathbf{x} = \mathbf{0},\ \mathbf{x} \neq \mathbf{0}\}.$$

There are two differences between the definitions: while the rank is the largest number of *independent* atoms, the Spark is the smallest number of *dependent* atoms. The example follows where these two numbers differ. For a $d \times n$ dictionary $\mathbf{\Phi}$ the spark satisfies $1 \leq \sigma \leq d + 1$.

Definition 2. For a matrix $\mathbf{\Phi}$ (with normalized columns) the mutual coherence $\mu(\mathbf{\Phi})$ is the largest (in magnitude) off-diagonal element of the Gram matrix $G = \mathbf{\Phi}^*\mathbf{\Phi}$, i.e.,

$$\mu(\mathbf{\Phi}) := \max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle| \,.$$

The mutual coherence measures the maximal similarity (or anti-similarity) between two columns on the dictionary. Therefore, its value ranges between 0 (an orthonormal basis) and 1 (when there are two identical atoms).

Theorem. [3, 2] The inequality

$$\sigma(\mathbf{\Phi}) \geq \frac{1}{\mu(\mathbf{\Phi})} + 1$$

is valid for each dictionary $\mathbf{\Phi}$.

The Welch bound on μ in $\mathbf{H}_d$ is well-known [3, 2]:

$$\mu(\mathbf{\Phi}) \geq \mu_{\min}(\mathbf{\Phi}) = \left[\frac{n-d}{d(n-1)}\right]^{1/2}.$$

The equality $\mu(\mathbf{\Phi}) = \mu_{\min}(\mathbf{\Phi})$ takes place iff:

(i) $\mathbf{\Phi}\mathbf{\Phi}^* = \frac{n}{d}\mathbf{I}_{\mathbf{H}_d}$;

(ii) for all $i \neq j$,

$$|\langle \varphi_i, \varphi_j \rangle|^2 = \frac{n-d}{d(n-1)},$$

i. e., the atoms of $\mathbf{\Phi}$ form the so-called ETF (equiangular tight frame).

The importance of the spark will be clear from the following observations.

Suppose Alice holds a dictionary $\mathbf{\Phi}$, and generates some sparse vector of coefficients α. Multiplying the two, Alice generates the signal $\mathbf{x} = \mathbf{\Phi}\alpha$. Then, Alice contacts her friend, Bob, and gives him the dictionary $\mathbf{\Phi}$ and the vector $\mathbf{x}$, and asks him to find the vector α she had used. Bob knows that α is sparse, and so he sets out to seek the sparsest possible solution of the system $\mathbf{\Phi}\alpha = \mathbf{x}$, namely, solve the optimization problem

$$\dot{\alpha} = \arg\min_{\alpha} \|\alpha\|_0 \; s.t. \; \mathbf{\Phi}\alpha = \mathbf{x}.$$

We shall assume that Bob can actually solve this problem exactly (really it is proved that this is generally NP-hard, and therefore impossible in practice). The question is if (or when) can we expect that $\dot{\alpha} = \alpha$? It might turn our that $\dot{\alpha}$ is even sparser than the original α. This is essentially the question of uniqueness what properties must α and the dictionary $\mathbf{\Phi}$ have so that α is the sparsest vector to create $\mathbf{x}$? The answer is in the

Theorem (Uniqueness Requirements of (P_0))[1]. The following two claims define the uniqueness condition and implication:

(1) Suppose we have found a sparse solution $\dot{\alpha}$ that satisfies $\mathbf{\Phi}\dot{\alpha} = \mathbf{x}$. If this solution satisfies $\|\dot{\alpha}\|_0 < \sigma(\mathbf{\Phi})/2$, then this solution is unique (i.e., it is the sparsest solution possible).

(2) If Alice generated a signal with $\|\alpha\|_0 < \sigma(\mathbf{\Phi})/2$, than Bob is able to recover α exactly: $\dot{\alpha} = \alpha$.

Definition. Let s, d be positive integers, and $s < d$. The sequence $\{\varphi_j\}_{j=1}^{s+1} \subset \mathbf{H}_d$ is a *regular s-simplex*, if it is an ETF for the span $\left(\{\varphi_j\}_{j=1}^{s+1}\right)$ and $\dim \operatorname{span}\left(\{\varphi_j\}_{j=1}^{s+1}\right) = s$.

Let's take a look at the example of the dictionary formed by the ETF [3, 2]. We see the 6×10 matrix

$$\mathbf{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

Really we see, that any φ_j lies in 5-dimensional orthogonal complement to the vector $(1,1,1,1,1,1)^t$ in $\mathbf{R}^6$. So we have the ETF with 10 vectors for its 5-dimensional span, and

$\mu\left(\{\varphi_j\}_{j=1}^{10}\right) = 1/3$. Moreover, $\text{rank}(\mathbf{\Phi}) = 5$,

$$\sigma(\mathbf{\Phi}) = 4 = \frac{1}{\mu(\mathbf{\Phi})} + 1,$$

and there is a regular 4-simplex within $\mathbf{\Phi}$ [2].

This example illustrates a general fact [2]:

Let $\mathbf{\Phi}$ be an ETF for $\mathbf{H}_d$. Equality

$$\text{spark}\,(\mathbf{\Phi}) = \frac{1}{\mu(\mathbf{\Phi})} + 1$$

is valid if and only if the set of atoms $\{\varphi_j\}_{j=1}^{n}$ contains a regular simplex.

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On Typical Properties of Lebesgue Measure Preserving Maps in Dimension One

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(joint work with J. Bobok, J. Činč and S. Troubetzkoy)

In this talk I will discuss selected properties of generic continuous maps of the interval and circle which preserve the Lebesgue measure. Among others, I will focus on the structure of the set of periodic points and the shadowing property.

The Garsia Entropy and Related Questions

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The Garcia entropy is calculated for the Garcia numbers. A theorem is proved, a special case of which is the Garcia theorem on the absolute continuity of the infinite Bernoulli convolution for the Garcia numbers. The proof uses the connection between the multidimensional Erdos problem and the one-dimensional Erdos problem.

It is a joint work with Kulikov V.L and Olekhova E.F.

Steady State Non-Newtonian Flow with Strain Rate Dependent Viscosity in thin Tube Structure

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Thin tube structures are finite unions of thin cylinders depending on the small parameter, ratio of the diameter of the cross section to the length of the cylinder. Flows in such domains model blood flow in a network of vessels. The asymptotic expansion of the solution of the steady Stokes and Navier-Stokes equations in these domains with no slip boundary condition was constructed in the papers [1], [2] and the book [3]. However, the blood exhibits a non-Newtonian rheology, when the viscosity depends on the strain rate. In the present talk we consider such rheology. Applying the Banach fixed point theorem we prove the existence and uniqueness of a solution and its regularity. An asymptotic approximation is constructed and justified by an error estimate.

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Spherically Symmetric Flows of a Rarefied Two-Phase Fluid

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Consider a system of differential equations which describes two-phase fluid dynamics

$$\begin{gathered}
\frac{d\rho_1}{dt_1} + \rho_1 \mathrm{div}\ u_1 = 0, \\
\frac{d\rho_2}{dt_2} + \rho_2 \mathrm{div}\ u_2 = 0, \\
\rho_1 \frac{du_1}{dt_1} + \nabla P(\rho_1) = -\frac{\rho_2}{\tau}(u_1 - u_2), \\
\rho_2 \frac{du_2}{dt_2} = \frac{\rho_2}{\tau}(u_1 - u_2),
\end{gathered} \tag{1}$$

here u_1, u_2 are the velocity vectors of the first and the second phase, ρ_1, ρ_2 are the densities of phases, $P(\rho_1) = a^2\rho_1$ is the pressure in the first phase, $\frac{d}{dt_1} = \frac{\partial}{\partial t} + u_1 \cdot \nabla$, $\frac{d}{dt_2} = \frac{\partial}{\partial t} + u_2 \cdot \nabla$. This model of a two-phase fluid is called a rarefied gas suspension [1].

An invariant submodel of spherically symmetric stationary motions is derived from this system. The submodel can be reduced to a dynamical system in R^3

$$\begin{gathered}
q_1' = q_1(-\mu(q_1 - q_2)q_1 r + 2a^2 q_2 \tau), \\
q_2' = (q_1 - q_2) r (q_1^2 - a^2), \\
r' = q_2 \tau r (q_1^2 - a^2),
\end{gathered}$$

here the derivative is taken by some parameter, q_1, q_2 is the velocities of the phases, r is the distance to the origin, τ, a, μ are the relaxation time, the speed of sound in the first phase and the ratio of the initial mass flow rates of the phases through the initial sphere $\mu = \frac{\rho_{20} q_{20} r_0^2}{\rho_{10} q_{10} r_0^2}$.

In the report, we will tell about a phase portrait of this dynamical system at different values of the parameter μ. Note that this dynamical system has a different curves which consist of singular points and when $\mu = 8$ there is a bifurcation one of singular curve.

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Resolvent Approximations in Homogenization of High Order Operators

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In the whole space R^d $(d \geq 2)$, we study homogenization of a divergence form elliptic operator A_ε of an arbitrary even order $2m \geq 4$ with measurable ε-periodic coefficients, where ε is a small parameter. For the resolvent $(A_\varepsilon + 1)^{-1}$, we construct an approximation with the remainder term of order ε^2 in the energy operator norm, that is $(L^2 \to H^m)$-norm. To find such kind of approximations, we use the resolvent of the homogenized operator, solutions of several auxiliary problems on the unit cell of periodicity, and smoothing operators. The homogenized operator here differs from the one commonly employed in homogenization: it has, as usually, constant coefficients, but is of order $2m + 1$.

The case $m = 2$ was considered in [1] where the specifics of fourth order operators was taken in an essential way into account. Now, we elaborate the general approach to treat the operators of arbitrary order $2m \geq 4$.

Similar approximations with the remainder term of order ε in the energy operator norm are known, e.g., from [2].

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Dynamical Properties of Volterra Integro-Differential Equations and Lyapunov Direct Method in the Stability Study

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Consider a nonlinear Volterra integro-differential equation such as

$$\dot{x} = f(t, x(t)) + \int_{t_0}^{t} g(t, s, x(t), x(s))ds, \tag{1}$$

where $x \in R^p$, R^p is a p-dimensional real linear space with some norm, $f : R \times R^p \to R^p$ and $g : R^2 \times R^{2p} \to R^p$ are functions that satisfy the Lipschitz conditions in all their variables with the Lipschitz constants depending on compact sets $K_1 \subset R^p$ and $K_2 \subset R^{2p}$ respectively.

It is also assumed that on each compact set $K \subset R^{2p}$ the function g satisfies the condition

$$||g(t, s, x, y)|| \leq g_0(s - t, K), \quad \int_{-\infty}^{0} g_0(\nu, K)d\nu \leq m_g(K) < \infty \tag{2}$$

Under the conditions (2), the solution $x = x(t)$ of (1) with the initial data $x(t_0) = x_0$ is unique and continuous in (t_0, x_0).

Let F be the space of continuous functions $f : R \times R^p \to R^p$, G be the space of continuous functions $g : R^2 \times R^{2p} \to R^p$ satisfying the condition (2) with a given function g_0.

Define convergence in F and G in accordance with the open-compact topology [1].

For the functions f and g included in the equation (1), construct the families of translates $F_\tau = \{f_\tau(t,x) = f(\tau+t,x)\}$ and $G_\tau = \{g_\tau(t,s,x,y) = g(\tau+t,\tau+s,x,y)\}$ and find the corresponding compact hulls $H(F_\tau) = \{f \in F : \exists t_n \to \infty, f_\tau^{(n)}(t,x) \to f^*(t,x)\}$, $H(G_\tau) = \{g \in F : \exists t_n \to \infty, g_\tau^{(n)}(t,s,x,y) \to g^*(t,s,x,y)\}$.

Equation (1) can be associated with a family of limiting equations of the form

$$\dot{x} = f^*(t,x(t)) + \int_{-\infty}^{t} g^*(t,s,x(t),x(s))ds,$$

which are integro-differential equations with infinite delay [2].

Introduce C_∞ the countably normed space of all continuous functions $\varphi : R^- \to R^p$ with seminorms $|||\varphi||| = max(||\varphi(s)||, -l \le s \le 0)$ $l = 1,2,\dots$ and a metric

$$\rho(\varphi^{(2)},\varphi^{(1)}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|||\varphi^{(2)} - \varphi^{(1)}|||}{1 + |||\varphi^{(2)} - \varphi^{(1)}|||} \qquad (3)$$

For the domain of definition of the equation (2) one can choose the domain $R \times C_\infty$.

Definition 1. it Let $x = x(t,\alpha,\varphi)$ be a solution of (1) bounded for all $t \in R$, $\|x(t,\alpha,\varphi)\| \le H$ $\forall t \in R$. The function $\varphi^* \in C_\infty$ is called a limit one for this solution if $\exists t_n \to \infty$ such that the corresponding sequence

$$x_t^{(n)}(\alpha,\varphi) = x(t_n+s,\alpha,\varphi)$$

converges to φ^* in C_∞ as $n \to \infty$, or $\rho(x_t^{(n)}(\alpha,\varphi),\varphi^*) \to 0$ as $n \to \infty$.

The set $\Omega^+(\alpha,\varphi)$ of all such functions forms in C_∞ a positive limit set of the given solution $x = x(t,\alpha,\varphi)$.

The following property of the quasi-invariance type is derived.

Theorem 1. it Let $x = x(t, \alpha, \varphi)$ be a solution of (1) bounded for all $t \in R$, $\|x(t, \alpha, \varphi)\| \leq H_1$ $\forall t \in R$. Then, the set $\Omega^+(\alpha, \varphi)$ is nonempty, compact, connected, and quasi-invariant.

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Solvability of Fractional-Order Differential Inclusions with an Almost Lower Semicontinuous Multioperator

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We are considering the Cauchy problem for a semilinear fractional differential inclusion in a Banach space E of the following form:

$$D^q x(t) \in Ax(t) + F(t, x(t)),\ t \in [0, a], \tag{1}$$

$$x(0) = x_0, \tag{2}$$

where D^q, $0 < q < 1$, is the Caputo fractional derivative, $F : [0, a] \times E \to P(E)$ is a multivalued map with nonempty compact convex values($P(E)$ is the collection of all non empty subset of E) , $A : E \to E$ is a linear closed, not necessarily bounded operator in E, and $x_0 \in E$.

Let a multimap F be such that:

($F1$) for each $x \in E$ the multifunction $F(\cdot, x) : [0, a] \to Kv(E)$ admits a measurable selection;

($F2$) for a.e. $t \in [0, a]$ the multimap $F(t, \cdot) : E \to Kv(E)$ is almost lower semicontinuity;

($F3$) there exists a function $\alpha \in L^\infty_+([0, a])$ such that

$$\|F(t, x)\|_E \le \alpha(t)(1 + \|x(t)\|_E) \text{ for a.e. } t \in [0, a];$$

($F4$) there exists a function $\mu \in L^\infty([0, a])$ such that for each bounded set $Q \subset E$ we have:

$$\chi(F(t, Q)) \le \mu(t)\chi(Q),$$

for a.e. $t \in [0, a]$, where χ is the Hausdorff MNC in E.

On a linear operator A we pose the following condition:

(A) $A : D(A) \to E$ is a linear closed operator in E generating a C_0–semigroup$\{T(t)\}_{t\geq 0}$.

Theorem. Under conditions (A), $(F1)$, $(F2)$, $(F3)$, $(F4)$, the set of all solutions to Cauchy problem (1) - (2) is a non empty compact subset of the space $C([0, a]; E)$.

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On Gradient of the Control Objective for a Qubit Driven by Coherent and Incoherent Controls

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Quantum control, that is control of individual quantum systems, is an important tool necessary for development of modern quantum technologies [1]. Often in experimental circumstances controlled systems can not be isolated from the environment, i.e., they are open quantum systems. Moreover, the environment can be used for actively controlling quantum systems, for example via incoherent control [2,3]. While in some cases the solution for the optimal shape of the control can be obtained analytically, often it is not the case and various numerical optimization methods are needed. Large class of methods are gradient-based numerical optimization algorithms [4–6].

In this talk, we consider the problem of computing gradient of the target functional for open quantum systems with respect to the controls. The functional depends on the terminal density matrix of a two-level open quantum system driven by coherent and incoherent controls [3,7]. An expression for gradient of the target functional was found under the assumption that the control is piecewise constant. This computed analytical expression then was implemented for numerical optimization for the problem of creating a given target density matrix starting from initial density matrix. This problem was formulated as minimization of some target functional. As an explicit example, two-level calcium atom was considered. For certain parameters of this system the result of the optimization algorithm can be easily visualized.

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Continuity of the IDS for Some Discrete Schrödinger Operators

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A one-by-one exhaustion is a combinatorial/geometric condition which excludes eigenvalues from the spectra of Laplace and Schrödinger operators on graphs. Isoperimetric inequalities in graphs with a cocompact automorphism group provide an upper bound on the von Neumann dimension of the space of eigenfunctions. Any finitely generated amenable group with a non-trivial homomorphism to the infinite cyclic group has a Cayley graph with continuous integrated density of states.

The Problem of Quantum Markovianity for Noninvertible Dynamical Maps

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I will analyze the problem of quantum Markovianity, in particular, the relation between completely positive (CP) divisibility and the lack of information backflow for an arbitrary –not necessarily invertible– dynamical map. It is well known that CP divisibility always implies a lack of information backflow. Moreover, these two notions are equivalent for invertible maps. I will explain that for a map which is not invertible the lack of information backflow always implies the existence of a completely positive propagator which, however, needs not be trace preserving. Interestingly, for a wide class of image nonincreasing dynamical maps, this propagator becomes trace preserving as well, and hence, the lack of information backflow implies CP divisibility. This result sheds new light into the structure of the time-local generators giving rise to CP-divisible evolutions as, if the map is not invertible, it turns out that positivity of dissipation/decoherence rates is no longer necessary for CP divisibility.

Existence of Optimal Shapes in Linear Acoustics

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In the aim to find the most efficient shape of a noise absorbing wall to dissipate the acoustical energy of a sound wave, we consider a frequency model described by the Helmholtz equation with a damping on the boundary of a bounded (ε, ∞) domain. We introduce a class of admissible Lipschitz boundaries, in which an optimal shape of the wall exists in the following sense: We prove the existence of a Radon measure on this shape, greater than or equal to the usual Lebesgue measure, for which the corresponding solution of the Helmholtz problem realizes the infimum of the acoustic energy defined with the Lebesgue measure on the boundary. If this Radon measure coincides with the Lebesgue measure, the corresponding solution realizes the minimum of the energy. To be able to ensure the minimum of the acoustical energy, We introduce new parametrized classes of shape admissible domains in R^n, $n \geq 2$, and prove that they are compact with respect to the convergence in the sense of characteristic functions, the Hausdorff sense, the sense of compacts, and the weak convergence of their boundary volumes. The domains in these classes are bounded (ε, ∞)-domains with possibly fractal boundaries that can have parts of any nonuniform Hausdorff dimension greater than or equal to $n-1$ and less than n. We prove the existence of optimal shapes in such classes for maximum energy dissipation in the framework of linear acoustics. A by-product of our proof is the result that the class of bounded (ε, ∞)-domains with fixed ε is stable under Hausdorff convergence. An additional and related result is the Mosco convergence of Robin-type energy functionals on converging domains.

New Bifurcation Diagram in One Generalized Model of the Vortex Dynamics

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The report is a continuation of the study of the phase topology of the parametric family of a completely Liouville integrable Hamiltonian system with two degrees of freedom with a Hamiltonian

$$H = \frac{1}{2}\Big[\Gamma_1^2 \ln(1-|z_1|^2) + \Gamma_2^2 \ln(1-|z_2|^2) + \\ +\Gamma_1\Gamma_2 \ln\left(\frac{[|z_1-z_2|^2 + (1-|z_1|^2)(1-|z_2|^2)]^{\varepsilon}}{|z_1-z_2|^{2(c+\varepsilon)}}\right)\Big]. \tag{1}$$

The Hamiltonian (1) describes the generalized mathematical model of the vortex dynamics, which covers such two limiting cases, namely, the model of two point vortices enclosed in a harmonic trap in a Bose-Einstein condensate ($\varepsilon = 0$) [1], [2], [3] and the model of two point vortices bounded by a circular region in an ideal fluid ($c = 0$, $\varepsilon = 1$) [4], [5]. Here, the Cartesian coordinates of k-th vortex ($k = 1, 2$) with intensities Γ_k are denoted by $z_k = x_k + \mathrm{i}y_k$. Physical parameter "$c$" expresses the extent of the vortices interaction, ε is a parameter of deformation. The phase space $\mathcal{P}$ is defined as a direct product of two open disks of radius 1 with the exception of vortices' collision points: $\mathcal{P} = \{(z_1, z_2) : |z_1| < 1, |z_2| < 1, z_1 \neq z_2\}$. The Poisson structure on the phase space $\mathcal{P}$ is given in the standard form: $\{z_k, \bar{z}_j\} = -\frac{2\mathrm{i}}{\Gamma_k}\delta_{kj}$, where δ_{kj} is the Kronecker delta. The system with a Hamiltonian (1) admits an additional first integral of motion, *the angular momentum of vorticity*, $F = \Gamma_1|z_1|^2 + \Gamma_2|z_2|^2$.

The main role in the study of such system is played by the bifurcation diagram Σ, which for this model coincides with the set of critical values of the integral map $\mathcal{F} : \mathcal{P} \to \mathbb{R}^2$, $\mathcal{F}(\boldsymbol{x}) = (F(\boldsymbol{x}), H(\boldsymbol{x}))$.

In this talk, we define the new type of the bifurcation diagram, which was not observed in any of the works [6], [7], [8]. Such a diagram contains two “loops” with the cusped points, and one of the loops has only one vertical asymptote.

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Entropy Invariants of Generic Measure-Preserving Actions

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We fix a standard probability space (X, μ) and consider the group Aut of all measure-preserving transformaitions of X. Aut is equipped by the Halmos complete metric. The sets that contain a dense G_δ-set are called comeager.

Now we consider modified Kushnirenko's entropy. Let $P = \{P_j\}$ be a sequence of finite subsets of a coutable infinite group G. We suppose that $|P_j| \to \infty$. For a measure-preserving G-action $T = \{T_g\}$ we define

$$h_j(T,\xi) = \frac{1}{|P_j|} H\left(\bigvee_{p \in P_j} T_p \xi\right),$$

$$h_P(T,\xi) = \limsup_j \, h_j(T,\xi),$$

$$h_P(T) = \sup_\xi h_P(T,\xi),$$

where ξ runs over all finite measurable partitions of X.

Theorem 1. *Given coutable infinite group G, let P be a sequence of the collections $P_j = \{p_j(1), p_j(2), \ldots, p_j(L(j))\} \subset G$, $L(j) \to \infty$, such that for any finite $F \subset G$ for all large j and all m, n, $m \neq n \leq L(j)$ the poduct $q_j(m)^{-1} q_j(n)$ is outside F. Then the set of actions $\{T : h_P(T) = \infty\}$ is comeager in the space of all G-actions.*

The following statement shows that many natural classes of dynamical systems are atypical. For example, rearrangements of a finite set of geometric shapes form atypical sets.

Theorem 2. *If K is a compact subset of Aut, and all $S \in K$ have zero classical entropy, then the set $\{J^{-1}SJ : S \in K, J \in Aut\}$ is meager in Aut.*

Monotonicity Criterion of Topological Entropy and Kneading Invariants for Lorenz Families

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In this talk, we consider one-dimensional factor maps for the geometric model of Lorenz-like attractors (see [1]) in the form of two-parameter family of Lorenz maps on the interval $I = [-1, 1]$ given by $T_{c,\nu}(x) = (-1 + c \cdot |x|^{\nu}) sign(x)$. This family is the normal form for splitting the homoclinic loop under additional assumption on degeneracy for smooth symmetric flows with saddle equilibrium having one-dimensional unstable manifold (see [2]). Due to L. P. Shilnikov' results, such a bifurcation (under certain conditions) corresponds to the birth of the Lorenz attractor. We indicate those regions in the parameter plane where the topological entropy and the kneading invariant depend monotonically on the parameter c, as well as those for which the monotonicity does not take place. Also, we indicate the corresponding bifurcations for the Lorenz attractors.

While studying the monotonicity problem of the family under consideration, we investigate more general situation as follows. Let $T_\alpha : I \to I$, $I = [-1, 1]$, be a family of symmetric Lorenz maps, i.e., maps that are C^1 (in x and α) monotone increasing on both $[-1, 0)$ and $(0, 1]$ and have infinite one-sided derivatives at 0: $\lim_{x \to 0\pm} DT_\alpha(0) = +\infty$. We also assume an expanding condition: there are $q > 1$ and a natural number N such that $DT^n(x) > q^n$ for all $n > N$ and $x \in I$. Note that this condition is weaker that usually explored one for geometric models of the Lorenz attractor (namely, the l.e.o. condition: $DT > q \geq \sqrt{2}$). Consider the space of continuous symmetric functions $f \in C_{sym}(I)$ defined on the interval I with the uniform topology and define the following operator $L : C_{sym}(I) \to C_{sym}(I)$

$$L(f)(x) = \frac{\partial T_\alpha(x) + f(T_\alpha(x))}{DT_\alpha(x)}$$

Theorem 1. *Let T_α be a symmetric Lorenz family as above. Then the following holds.*
1. The space $C_{sym}(I)$ is invariant under the action of L, and moreover, L has a unique fixed point (say F_α) in $C_{sym}(I)$.
2. If the inequality $F_\alpha(T_\alpha(0-)) > 0$ is satisfied for all $\alpha \in [\alpha_0, \tilde{\alpha}_0]$ (where $[\alpha_0, \tilde{\alpha}_0]$ is some subinterval) then the kneading invariant $K^-_{T_\alpha}$ is strictly monotone increasing at α, while the topological entropy is monotone increasing (not necessarily strictly). The same result is valid if $F_\alpha(T_\alpha(0-)) < 0$ with replacing the increasing property by decreasing one.

We apply this criterion to the initial two-parameter family of Lorenz maps $T_{c,\nu}(x) = (-1 + c|x|^\nu) \cdot sign(x)$.

Theorem 2. *In the parameter region of (c, ν)-plane with $1 < c < 2$, $0 < \nu < 1$ and $c\nu - 1 > (c-1)^{1-\nu} - (c-1) > 0$, the kneading invariant $K^-_{T_{c,\nu}}$ is a strictly monotone increasing function of the parameter c, while the topological entropy is monotone increasing (not necessarily strictly).*

We also describe the regions in the parameter plane where the monotonicity property doesn't take place and show the curve in the parameter plane which separate the monotonicity and non-monotonicity regions.

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Random Hamiltonian Flows on Infinite Dimensional Phase Space

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To study Hamiltonian systems with infinite dimensional phase space the measure on the phase space which is invariant with respect to Hamiltonian flows should be investigated. But according to A. Weyl's theorem there is no Lebesgue measure on a topological vector space which is not locally compact. Therefore the studying of the shift-invariant measures on the Hilbert space deal with additive function of a set without some properties of Lebesgue measure such as countable additivity [1,2] or σ-finiteness [3,4]. The generalized shift-invariant measure on Hilbert space is constructed as the shift-invariant functional on the propper space of test functions [5].

The existence of an analog of the Lebesgue measure on an infinite-dimensional real Hilbert space is established in the following sense. We construct a finite-additive measure on an infinite-dimensional real separable Hilbert space E which is invariant with respect to shifts and orthogonal transformations.

We study Hamiltonian systems in phase space which is the Hilbert space E endowed with the shift-invariant symplectic form ω. We construct the measure on the phase space which is invariant to the group of symplectomorphisms. Any of these measures is called symplectic-invariant measure. Any symplectic-invariant measure λ_ω is the continuation of a shift-invariant measure.

On the Hilbert space E with the shift-invariant symplectic structure ω the random Hamilton function is defined as the random variable with values in space of smooth functions $E \to R$ on the phase space.

The constructed symplectic-invariant measure λ_ω on the phase space E can be used for studying of random Hamiltonian flows g_t, $t \in R$, generated by the family of independent identically distributed random Hamiltonians h. The symplectic-invariant measure gives the Koopman representation of the Hamiltonian flow g in the phase space by means of unitary group $\mathbf{U}_g$ in the space $\mathcal{H}$ of square integrable with respect to symplectic-invariant measure λ_ω functions.

The conditions of strong continuity of Koopman unitary group $\mathbf{U}_g$ in terms of the Hamilton function are obtained. The random operator valued processes generated by the composition of the independent identically distributed Hamiltonian flows is studied.

The analog of law of large numbers for the sequence of compositions $\{g_n(\frac{t}{n}) \circ ... \circ g_1(\frac{t}{n}),\ t \in R\}$ is obtained.

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Existence and Stability of Equilibrium Solutions of the Vlasov Equation with a Modified Gravitational Potential

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We consider the gravitating particles that can collide. Collisions can be described in various ways. We can use the theory of inelastic interaction of solids with Newton's recovery coefficient for the relative velocity of colliding particles. In numerical implementation, the main difficulty of this approach is to track and refine a huge number of time moments of particle collisions.

Another approach is to add to the gravitational potential the potential of repulsive forces, similar to the intermolecular Lennard-Jones forces. Numerical experiments show that when the Jacobi stability condition is satisfied, both models lead to a qualitatively identical character of evolution with the possible formation of stable configurations.

As it is known, when pair collisions of an infinitely large number of gravitating particles are taken into account, the probability density function evolves in accordance with the Vlasov-Boltzmann-Poisson system of equations. We suggest a research method using the Vlasov equation with the Lennard-Jones type potential. This allows to take into account the size of the interacting particles, and also take into account not only paired, but also triple or more collisions of the particles. For this dynamical system the existence of a large class of nonlinearly stable equilibrium solutions is proved by the Energy-Casimir method.

Partial Control and Beyond: Forcing Escapes and Controlling Chaotic Transients with the Safety Function

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A new control algorithm based on the partial control method has been developed. The general situation we are considering is an orbit starting in a certain phase space region Q having a chaotic transient behavior affected by noise, so that the orbit will definitely escape from Q in an unpredictable number of iterations. Thus, the goal of the algorithm is to control in a predictable manner when to escape. While partial control has been used as a way to avoid escapes, here we want to adapt it to force the escape in a controlled manner. We have introduced new tools such as escape functions and escape sets that once computed makes the control of the orbit straightforward. The partial control method aims to avoid the escape of orbits from a phase space region Q where the transient chaotic dynamics takes place. The technique is based on finding a special subset of Q called the safe set. The chaotic orbit can be sustained in the safe set with a minimum amount of control. We have developed a control strategy to gradually lead any chaotic orbit in Q to the safe set by using the safety function. With the technique proposed here, the safe set can be converted into a global attractor of Q. This is joint work with Gaspar Alfaro and Ruben Capeans, URJC, Spain.

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Geometric Interpretation of the Properties of Space-Time

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Every day in life we use such concepts as time, distance, weight. And sometimes we ask ourselves questions, what is it, how and why it works like that. But it is precisely these questions, we simply cannot answer, they are so complex, it would seem, where the answers should be simple. In the above work, we examined the modeling of a one-dimensional structure, where, within the framework of reasoning and simple analogies, we tried to create a geometric interpretation of the physical phenomena available in our world. As an option, the mechanics of this constructed model is considered, where one can consider the analogies of the two basic concepts of our space, time and mass, and their interaction between them - gravity, through the option of applying geometric constructions, with the restrictions that we can observe in our world. Through this simple model, we tried to give them formulations that can be partly applicable to our world. What is time?

First, I would like to build the simplest geometric model in which there would be one coordinate, and the concept of time would be present. As a variant of the analogy, we could consider the approach of transition to considering space from another point and the coordinate system is not connected with the space itself. So in due time one of the most difficult mathematical problems was solved, the model of planetary motion in the geocentric system. The concrete embodiment of this principle was the theory of homocentric spheres of Eudoxus-Callippus supported by Aristotle and the theory of epicycles by Apollonius of Perga, Hipparchus and Ptolemy. But after the transition from the geocentric system to the heliocentric system of the Copernican world, the explanation of the motion of the planets and the construction of a model of the solar system became available even for teaching in the lower grades, so the model was simplified.

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Topological Insulators on Mathematical Problems in the Theory of Topological Insulators

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The role of topology in the theory of condensed matter first became clear in the study of quantum Hall effect. From the physical point of view topological invariance is equivalent to adiabatic stability. A key role in the classification of topological objects in the theory of solid states is played by the study of their symmetry groups. The description of possible symmetry types goes back to Kitaev who proposed a classification of topological insulators based on the investigation of their symmetry groups and their representations.

In this talk we pay main attention to the topological insulators invariant under time reversion. Such systems are characterized by having the wide energy gap stable under small deformations. An example of these systems is provided by the quantum spin Hall insulator. It has a non-trivial topological $\mathbb{Z}_2$-invariant introduced by Kane and Mele.

Methods of Calculation of the Schwarzian Integrals

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We construct the series of functional integrals in the Schwarzian theory as the integrals on the group of diffeomorphisms. The Schwarzian theory is behind various physical models including the SYK model and the two-dimensional dilaton gravity.

We consider the ways of calculation of the Schwarzian integrals. We derive the explicit form of the polar decomposition of the Wiener measure, and obtain the equation connecting functional integrals in conformal quantum mechanics to those in the Schwarzian theory.

Using this connection we evaluate some nontrivial functional integrals in the Schwarzian theory and also find the fundamental solution of the Schroedinger equation in imaginary time in the model of conformal quantum mechanics.

The Integrable Abel Equation

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The first integral of Abel-type equation arising in Korteweg – De Vries equation asymptotic solution studying is presented. We have put forward a hypothesis that similar integrable ordinary differential equations arise in studying large time asymptotics of symmetry solutions to other evolutionaty equations that may be solved by the inverse scattering method.

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On Constant Solutions of the Yang-Mills-Dirac Equations

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We present a complete classification and an explicit form of all constant solutions of the Yang-Mills-Dirac equations with SU(2) gauge symmetry in Minkowski space $\mathbb{R}^{1,3}$. Nonconstant solutions are considered in the form of series of perturbation theory using all constant solutions as a zeroth approximation. We use our previous results on all constant solutions of the SU(2) Yang-Mills equations with arbitrary current [1, 2] and results on the hyperbolic singular value decomposition [3] of an arbitrary real or complex matrix.

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Reconstruction Function by its Weighted Spherical Mean

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In this talk we present an inversion formula for the weighted spherical mean.

Suppose that R^{n+1} is the $n+1$–dimensional Euclidean space,

$$R^{n+1}_+=\{(t,x)=(t,x_1,\ldots,x_n)\in R^{n+1},\ \ x_1>0,\ldots,x_n>0\},$$

$\gamma=(\gamma_1,...,\gamma_n)$ is a multiindex consisting of fixed real numbers $\gamma_i\geq 0$, $i=1,...,n$, and $|\gamma|=\gamma_1+\ldots+\gamma_n$.

Let ${}^\gamma T^y_x$ be a multidimensional generalized translation

$$({}^\gamma T^y_x f)(t,x)=({}^{\gamma_1}T^{y_1}_{x_1}...{}^{\gamma_n}T^{y_n}_{x_n} f)(t,x).$$

Each of one–dimensional generalized translations ${}^{\gamma_i}T^{y_i}_{x_i}$ is defined for $i=1,...,n$ by the next formula

$$({}^{\gamma_i}T^{y_i}_{x_i} f)(t,x)=\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^\pi \sin^{\gamma_i-1}\varphi_i\times$$

$$\times f(t,x_1,...,x_{i-1},\sqrt{x_i^2+y_i^2-2x_iy_i\cos\varphi_i},x_{i+1},...,x_n)\;d\varphi_i,$$

$\gamma_i>0$, $i=1,...,n$ and for $\gamma_i=0$ generalized translation ${}^{\gamma_i}T^{y_i}_{x_i}$ is

$${}^0T^{y_i}_{x_i}=\frac{f(x+y)-f(x-y)}{2}.$$

Weighted spherical mean of function $f(x)$, $x\in\overline{R}{}^n_+$ for $n\geq 2$ is

$$(M^\gamma_t f)(x)=(M^\gamma_t)_x[f(x)]=$$

$$= \frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} {}^\gamma T_x^{t\theta} f(x) \theta^\gamma dS, \qquad (1)$$

where $\theta^\gamma = \prod\limits_{i=1}^{n} \theta_i^{\gamma_i}$, $S_1^+(n) = \{\theta{:}|\theta|=1, \theta \in R_+^n\}$ is a part of a sphere in R_+^n, and $|S_1^+(n)|_\gamma$ is given by

$$|S_1^+(n)|_\gamma = \int\limits_{S_1^+(n)} x^\gamma dS = \frac{\prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}.$$

For $n = 1$ let $M_t^\gamma[f(x)] = {}^\gamma T_x^t f(x)$.

Theorem. Let $f = f(x) \in C^2(R_+^n)$, such that $\left.\frac{\partial f}{\partial x_i}\right|_{x_i=0} = 0$, $i = 1, ..., n$ and

$$(M_t^{\gamma,k} f)(x) =$$

$$= \frac{t^{k-n-|\gamma|}}{2C(k-n-|\gamma|+1)} \left(\mathcal{P}_t^{k-n-|\gamma|+1} t^{n+|\gamma|-1} (M_\rho^\gamma f)(x)\right)(t),$$

where $M_\rho^\gamma f$ is the weighted spherical mean (1) of the function f, is $\mathcal{P}_t^\nu$ the one-dimensional Poisson operator

$$\mathcal{P}_x^\nu f(x) = \frac{2C(\nu)}{x^{\nu-1}} \int\limits_0^x \left(x^2 - t^2\right)^{\frac{\nu}{2}-1} f(t)\, dt, \qquad C(\nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}.$$

Then the function f can be reconstructed by its weighted spherical mean by the formula

$$h(t)f(x) = \frac{|S_1^+(n)|_\gamma}{N(2m,\gamma,n)} \left(\frac{\partial^2}{\partial t^2} - \Delta_\gamma\right)^m \int\limits_0^\infty h(t-\tau)(M_\tau^{\gamma,2m} f)(x) d\tau,$$

where $2m \in (n + |\gamma| - 1, n + |\gamma| + 1)$, $m \in N$, function $h(t)$ is arbitrary such that the function $h(t-\tau)(M_\tau^{\gamma,k} f)(x)$ is an integrable by τ by the interval from 0 to ∞, $N(2m, \gamma, n)$ is given by

$$N(\alpha, \gamma, n) = \frac{2^{\alpha-n-1}}{\sqrt{\pi}} \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i + 1}{2}\right) \Gamma\left(\frac{\alpha - n - |\gamma| + 1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right).$$

Interpolating Between Positive and Completely Positive Maps: a New Hierarchy of Entangled States

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A new class of positive maps is introduced. It interpolates between positive and completely positive maps. It is shown that this class gives rise to a new characterization of entangled states. Additionally, it provides a refinement of the well-known classes of entangled states characterized in term of the Schmidt number. The analysis is illustrated with examples of qubit maps.

Stationary Solutions of Vlasov-Poisson System and Plasma Confinement in Tokamak

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In this lecture we consider the Vlasov-Poisson system for two-component high-temperature plasma with external magnetic field in a three-dimensional torus. The Vlasov-Poisson system of equations regarding to density distribution functions of charged particles and electric potential describes the kinetics of high-temperature plasma in a fusion reactor. If a considerable part of particles reaches the boundary, this can lead either to destruction of the reactor, or to cooling the plasma due to its contact with the reactor wall. Therefore, it is necessary to provide plasma confinement at some distance from the vacuum container wall. In most models of thermonuclear fusion reactors an external magnetic field is used as a control ensuring plasma confinement. From the point of view of differential equations this means that one has to prove the existence of solutions of the Vlasov-Poisson system with external magnetic field for which the supports of density distribution functions do not intersect with the boundary.

In this lecture we consider the Vlasov-Poisson system for two-component high-temperature plasma with external magnetic field in three-dimensional torus, which corresponds to "tokamak". We prove the existence of stationary solutions of the Vlasov-Poisson system in the three -dimensional torus with compactly supported density distribution functions. Using symmetries of the domain, we reduce this problem to semilinear second order elliptic differential equation, which is studied with the help of sub- and supersolutions method. A construction of stationary solution is also based on the method of truncation functions.

For the first time, stationary solutions of the Vlasov-Poisson system with homogeneous external magnetic field having supports of density distribution functions strictly inside infinite cylinder were studied in [1]. The above mentioned results concerning stationary solutions with compact supports in a torus were obtained in [2].

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Homogenization of Fourth Order Periodic Elliptic Operator

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In $L_2(\mathbf{R}^d;\mathbf{C}^n)$, a strongly elliptic fourth order differential operator A_ε with periodic coefficients depending on $\mathbf{x}/\varepsilon$ is studied. The following approximation for the resolvent $(A_\varepsilon + I)^{-1}$ in the operator norm in $L_2(\mathbf{R}^d;\mathbf{C}^n)$ is obtained:

$$(A_\varepsilon + I)^{-1} = (A^0 + I)^{-1} + \varepsilon K_1 + \varepsilon^2 K_2(\varepsilon) + \varepsilon^3 K_3(\varepsilon) + O(\varepsilon^4).$$

Here A^0 is the effective operator with constant coefficients, and $K_1, K_2(\varepsilon), K_3(\varepsilon)$ are certain correctors. [3] The correctors allow estimates $K_i(\varepsilon) = O(1),\ \ \varepsilon \to 0,\ \ i = 2, 3$. The operator A^0 and the correctors K_i, $i = 1, 2, 3$, are calculated in terms of some auxiliary boundary value problems on the periodicity cell of the operator $A := A_\varepsilon,\ \ \varepsilon = 1$.

We use the operator-theoretic approach developed by M.Sh. Birman and T.A. Suslina in some works [2001 – 2006] for the second order differential operators. The strongly elliptic high order differential operators was studied in the works of N. A. Veniaminov [2010], A. A. Kukushkin and T. A. Suslina [2016], S. E. Pastukhova [2016]. They received an analogue of estimate (1) without correctors. Estimate (1) was obtained by the authors [2020]. Independently, estimate (1) was obtained by S. E. Pastukhova by the shift method [2021].

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[3]*Keywords:* periodic differential operators, homogenization, operator error estimates, effective operator, corrector.

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An Approximation of the Wiener Process Local Time

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Let $w(s)$, $s \geqslant 0$ be a standard Wiener process and $l(t,x)$ be a local time of $w(s)$. For each $x \in \mathbb{R}$ the local time $l(t,x)$ characterizes the portion of time the process $w(s)$ spends at x up to the time t. It is known (see[2]) that with probability 1 there exists a continuous two-parameter process $l(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}$, satisfying the relation

$$\int_0^t \mathbf{1}_A(w(s))\, ds = \int_A l(t,x)\, dx$$

for any $t > 0$ and any Borel subsets A of $\mathbb{R}$.

Further, let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of centered ($\mathbf{E}\xi_1 = 0$) i.i.d. random variables with a general distribution law $\mathcal{P}$ and $\eta(t)$, $t \geqslant 0$ be a standard ($\mathbf{E}\eta(t) = t$) Poisson process independent of $\{\xi_j\}$. Suppose that $\mathbf{E}\xi_1^4 < \infty$ and $\mathbf{D}\xi_1 = \mathbf{E}\xi_1^2 = 1$.

For every $n \in \mathbb{N}$ define a compound Poisson process $\zeta_n(t)$, $t \geqslant 0$, by

$$\zeta_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\eta(nt)} \xi_j.$$

The sequence $\zeta_n(t)$ weakly converges (in Skorokhod space) to the Wiener $w(t)$, this statement is called the invariance principle. From the invariance principle follows that the distribution law of every continuous (with respect to the Skorokhod topology) functional of the processes ζ_n converges to the distribution law of the same functional of the limit Wiener process as $n \to \infty$.

We consider some discontinuous functionals, such that the invariance principle can't be applied to them. Namely, we prove the convergence of some functionals of the processes $\zeta_n(t)$ to the local time of the Wiener process. More discontinuous functionals were considered in [2].

Since for every n the local time of the process $\zeta_n(t)$ does not exist, for the process $\zeta_n(t)$ we consider instead a generalised (see [1]) local time $m_n(t,x)$, where

$$m_n(t,x) = \int_0^t h_n^0(x - \zeta_n(\tau))d\tau$$

and a mollifying kernel h_n^0 is defined by

$$h_n^0(x) = n \int_{\mathbb{R}} (|x - \frac{y}{\sqrt{n}}| - |x| + \frac{y}{\sqrt{n}} \operatorname{sgn}(x))\mathcal{P}(dy).$$

Theorem 2. *Let $f \in W_2^3(\mathbb{R})$, $f(0) = 0$ and the function $g : [0,\infty) \to \mathbb{R}$ is given by $g(y) = \int_0^\infty e^{-\lambda y}\sigma(d\lambda)$, where σ is a signed measure such that $L = \int_0^\infty (1+\lambda^2)|\sigma|(d\lambda) < \infty$.*
Set

$$\psi(t,x) = \mathbf{E}[f(x - w(t))g(l(t,x))],$$
$$\psi_n(t,x) = \mathbf{E}[f(x - \zeta_n(t))g(m_n(t,x))].$$

Then there exists a constant $C > 0$ such that the following inequality holds

$$\|\psi_n(t,\cdot) - \psi(t,\cdot)\|_2 \leqslant \frac{CL}{n^{1/4}}(\|f'\|_{W_2^2(\mathbb{R})} + \|f\|_\infty).$$

Corollary 1. *For every function g satisfying theorem 1 conditions we have*

$$\lim_{n\to\infty} \|\mathbf{E}g(m_n(t,\cdot)) - \mathbf{E}g(l(t,\cdot))\|_2 = 0.$$

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On Periodic Solutions of Parabolic Problems with Nonlocal Boundary Conditions

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The time–periodic solutions of a parabolic equation with nonlocal boundary conditions of Bitsadze–Samarskii type are investigated.

We consider parabolic problems with Laplasian or p–Laplasian ($p > 2$) in the rectangular parallelepiped $\Omega_T = (0,T) \times (0,2) \times (0,1)$

$$\partial_t w(t,x) - \Delta_p w(t,x) = f(t,x) \qquad ((t,x) \in \Omega_T), \tag{1}$$

with nonlocal boundary conditions

$$\left.\begin{array}{ll} w(t,x_1,0) = w(t,x_1,1) = 0 & (t \in (0,T), 0 \leq x_1 \leq 2), \\ w(t,0,x_2) = \gamma_1 w(t,1,x_2) & (t \in (0,T), 0 < x_2 < 1), \\ w(t,2,x_2) = \gamma_2 w(t,1,x_2) & (t \in (0,T), 0 < x_2 < 1). \end{array}\right\} \tag{2}$$

Here $f \in L_q(\Omega_T)$ ($p \in [2,\infty)$, $1/p + 1/q = 1$).

Theorem 1. *Let* $p = 2$ *and* $|\gamma_1 + \gamma_2| < 2$. *Then for any* $f \in L_2(\Omega_T)$ *there exists a unique generalized solution of problem (1)–(2)* $w \in L_2(0,T; W_2^1(Q)$ *such that* $\partial_t w \in L_2(\Omega_T)$ *and* $w(0,x) = w(T,x)$.

Let $p \in (2,\infty)$ *and* $|\gamma_1 + \gamma_2| + p^{1/p} q^{1/q} |\gamma_1 - \gamma_2| < 2$. *Then for any* $f \in L_q(0,T; W_q^{-1}(Q))$ *there exists a generalized solution of problem (1)–(2)* $w \in L_p(0,T; W_p^1(Q))$ *such that* $\partial_t w \in L_q(0,T; W_q^{-1}(Q))$ *and* $w(0,x) = w(T,x)$.

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Two-Point Invertible Transformations and Darboux Integrability of Discrete Equations

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Let us consider a discrete equation of the form

$$u_{n+1,m+1} = F(u_{n,m}, u_{n+1,m}, u_{n,m+1}),$$
$$\frac{\partial F}{\partial u_{n+1,m}} \frac{\partial F}{\partial u_{n,m+1}} \frac{\partial F}{\partial u_{n,m}} \neq 0, \qquad (1)$$

where n, m are integers. If there exist functions

$$\Omega_{n,m}(u_{n,m}, u_{n+1,m}, \ldots, u_{n+p,m}),$$

$p > 0$, and $\bar{\Omega}_{n,m}(u_{n,m}, u_{n,m+1}, \ldots, u_{n,m+\bar{p}})$, $\bar{p} > 0$, such that they satisfy the relations

$$\Omega_{n,m+1}(u_{n,m+1}, u_{n+1,m+1}, \ldots, u_{n+p,m+1}) =$$
$$= \Omega_{n,m}(u_{n,m}, u_{n+1,m}, \ldots, u_{n+p,m}),$$

$$\bar{\Omega}_{n+1,m}(u_{n+1,m}, u_{n+1,m+1}, \ldots, u_{n+1,m+\bar{p}}) =$$
$$= \bar{\Omega}_{n,m}(u_{n,m}, u_{n,m+1}, \ldots, u_{n,m+\bar{p}})$$

for any n, m and any solution of (1), then (1) is called *Darboux integrable.* The functions $\Omega_{n,m}$ and $\bar{\Omega}_{n,m}$ are called an *n-integral of order p* and an *m-integral of order $\bar{p}$*, respectively.

We deal with the subclass of (1) for which there exist functionally independent $\varphi(x,y)$ and $\psi(x,y)$ such that

$$\varphi(u_{n,m+1}, u_{n+1,m+1}) = \psi(u_{n,m}, u_{n+1,m})$$

on solutions of (1). Under some additional assumptions, $v_{n,m} = \varphi(u_{n,m}, u_{n+1,m})$ maps solutions of (1) into solutions of an equation $v_{n+1,m+1} = G(v_{n,m}, v_{n+1,m}, v_{n,m+1})$. It is proved in [1] that the last equation is Darboux integrable, has n-integral of order $p-1$ and admits higher symmetries if (1) is Darboux integrable, has n-integral of order p and admits higher symmetries, respectively. In particular, this implies that (1) cannot have n-integrals if the transformation $v_{n,m} = \varphi(u_{n,m}, u_{n+1,m})$ maps solutions of (1) into solutions of (1) again.

For example, the generic Hietarinta equation is reduced to the equation

$$\begin{gathered} u_{n+1,m+1}(u_{n,m}+B)(u_{n,m+1}+A) = \\ = u_{n,m+1}(u_{n+1,m}+B)(u_{n,m}+A), \\ AB(A-B) \neq 0, \end{gathered} \tag{2}$$

by a point transformation. For (2) the corresponding functions φ and ψ are

$$\varphi(u_{n,m}, u_{n+1,m}) = u_{n+1,m}\frac{u_{n,m}+A}{u_{n,m}} - A,$$

$$\psi(u_{n,m}, u_{n+1,m}) = \frac{(u_{n+1,m}+B)(u_{n,m}+A)}{u_{n,m}+B} - A,$$

and $v_{n,m} = \varphi(u_{n,m}, u_{n+1,m})$ maps solutions of (2) into solutions of (2) again. Hence, (2) has no n-integrals. For more details see [2] and the references within.

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Dispersion Relationship and Spectrum in the Collisionless Plasma Kinetic Model

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Stability of collisionless plasma oscillations is studied within the approach specifying the spectrum of the generator for the corresponding dynamics by means of the associated Landau dispersion relationship. The spectrum of the problem is thus localized and the formula is obtained for the instability index supplementing the Nyquist criterion. Conditions for stability and instability are derived which can be effectively verified in terms of the unperturbed density distribution.

Some News from the Levi-Flat World

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Levi-flat hypersurfaces naturally arise on intersection of the Complex Geometry, the Foliation Theory and Holomorphic Dynamical Systems. We present a survey of recent results and open problems.

Singular Traces and The Density of States

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The density of states is a non-negative measure associated to a Schrodinger operator H which is supported on its essential spectrum. Theoretical questions concerning the existence and properties of the density of states are of interest in solid state physics. We have recently found that quite generally the density of states measure can be computed by a formula involving a Dixmier trace, a tool from quantised calculus of A.Connes. This is a surprising new application of singular traces and methods from quantised calculus to mathematical physics which also uses recently developed techniques in operator integration theory. Joint work with N. Azamov, E. McDonald and D. Zanin.

On the Spectra of Separable 2D almost Mathieu Operators

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We consider separable 2D discrete Schrödinger operators generated by 1D almost Mathieu operators. For fixed Diophantine frequencies we prove that for sufficiently small couplings the spectrum must be an interval. This complements a result by J. Bourgain establishing that for fixed couplings the spectrum has gaps for some (positive measure) Diophantine frequencies. Our result generalizes to separable multidimensional discrete Schrödinger operators generated by 1D quasiperiodic operators whose potential is analytic and whose frequency is Diophantine. The proof is based on the study of the thickness of the spectrum of the almost Mathieu operator, and utilizes the Newhouse Gap Lemma on sums of Cantor sets.

Free-Boundary Problem for the Relaxation Equation of Transfer

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In the study of potential transfer processes under high-intensity influences, the hypothesis of proportionality of the flux vector to the potential gradient vector leads to the paradox of the infinite propagation velocity of disturbances. In various fields of natural science, hypotheses have been put forward about the propagation of disturbances as a process that is simultaneously wave and diffusion. The development of Maxwell's ideas led to laws, where a term appeared that takes into account relaxation phenomena in media [1].

There are many other generalizations of the Fourier equation, in addition to the Maxwell-Catteneo-Vernotti (MCV) equation, such as the Guer-Krumhansl (GK) equation [2-5], the two-phase delay model [6]. The simplest generalization of the MCV equation is the GK model, which has the form

$$\tau \mathbf{q}_t + \mathbf{q} + k\nabla T - k^2 \Delta \mathbf{q} = 0,$$

where τ is the relaxation time, k^2 is the dissipation parameter, T is the temperature, $\mathbf{q}$ is the flow.

The case $k^2 = \tau a$ is called the Fourier resonance condition, for $k^2 < \tau a$, a wavy behavior is observed, and for $k^2 > \tau a$, diffusion behavior prevails.

It should be noted that problems with a free boundary are widely used in the mathematical modeling of various physical, biological, medico-biological and similar processes. Problems with a free boundary of Stefan type for relaxation transport equations have appeared (see, for example, [7]).

In this paper, in the case $a = \frac{k^2}{\tau}$, we will try to formulate and investigate a well-posed boundary value problem with a free boundary of Florin type for the equation

$$T_t + \tau T_{tt} = a(T_{xx} + \tau T_{xxt}).$$

in the region without initial condition

$$D_H = \{(x,t) : 0 < x < s(t), 0 < t \leq H\}, s(0) = 0.$$

Find on some segment $0 < t \leq H$ a continuously differentiable function $s(t), s(0) = 0, s(t) > 0, 0 < t < H$, and a solution to the equation (1), continuous in $\bar{D}_H \backslash (0,0)$, together with derivatives and satisfying the conditions

$$T_x(0,t) - \alpha T(0,t) = \psi(t), \quad 0 \leq t \leq H,$$

$$T_x(s(t),t) = 0, \quad 0 \leq t \leq H,$$

$$T_{xx}(s(t),t) = 0, \quad 0 \leq t \leq H,$$

$$T(s(t),t) = \int_0^t d\eta \int_0^\eta \bar{q}(s(\xi),\xi)d\xi, 0 \leq t \leq H.$$

Here a, τ, α, H are positive constants, $\psi(t), \bar{q}(x,t)$ are given functions, and $\psi(0) \neq 0, \bar{q}(x,t)$ - is defined and continuous in the half-strip $\{(x,t) : 0 \leq x \leq x_0, t \geq 0\}$ and $\bar{q}(x,t) < 0$. Apparently the Florin problem for an equation of the form (1) is considered for the first time. A peculiar approach to the study of the problem (1)-(5) is proposed. By introducing a new function

$$T(x,t) + \tau T_t(x,t) = u(x,t),$$

from (1) we obtain a parabolic equation for $u(x,t)$:

$$u_t = a u_{xx}(x,t).$$

If boundary conditions are given for (1), then with the help of (6) the necessary conditions for $u(x,t)$ are restored.

In our case, for $T(x,t)$ we get the problem

$$T(x,t) + \tau T_t(x,t) = u(x,t), \quad (x,t) \in D_H,$$
$$T(s(t),t) = \varphi(t), \quad 0 \le t \le H,$$

where $\varphi(t) = \int\limits_0^t d\eta \int\limits_0^\eta \bar{q}(s(\xi),\xi) d\xi$.

Research is carried out according to the following scheme. First, the original problem is reduced to a problem for the second order equation. Found a representation for the solution using the Green's function. We establish some initial a priori estimates for $u(x,t)$, and then prove a uniqueness theorem for the solution. Next, consider a problem with an initial condition, and reduce this problem to a Stefan-type problem. Their equivalence is proved. For the solution of the Stefan-type problem, a priori estimates of Schauder type are established and on their basis the existence theorem is proved. Moreover, for the unknown boundary, two-sided estimates are established using known curves, which give the behavior of the unknown boundary as $t \to 0$. Finally, it is proved that with an unlimited increase in time, the free boundary tends to some constant.

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Structure of Essential Spectra and Discrete Spectrum of the Four-Electron Quintet in the Impurity Hubbard Model

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The spectrum and wave functions of the system four electrons in a crystal described by the Hubbard Hamiltonian in the quintet and singlet states and triplet states were studied in [1,2]. Here, we consider the energy operator of four electron systems in the impurity Hubbard model and describe the structure of essential and discrete spectra of the system for quintet state. The Hamiltonian of the chosen model has the form

$$H = A\sum_{m,\gamma} a^{+}_{m,\gamma}a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^{+}_{m,\gamma}a_{m+\tau,\gamma}+$$

$$+U\sum_{m} a^{+}_{m,\uparrow}a_{m,\uparrow}a^{+}_{m,\downarrow}a_{m,\downarrow} + (A_0 - A)\sum_{\gamma} a^{+}_{0,\gamma}a_{0,\gamma}+$$

$$+(B_0 - B)\sum_{\tau,\gamma}(a^{+}_{0,\gamma}a_{\tau,\gamma} + a^{+}_{\tau,\gamma}a_{0,\gamma})+$$

$$+(U_0 - U)a^{+}_{0,\uparrow}a_{0,\uparrow}a^{+}_{0,\downarrow}a_{0,\downarrow}.$$

Here A (A_0) is the electron energy at a regular (impurity) lattice site, B (B_0) is the transfer integral between (between electron and impurities) neighboring sites (we assume that $B > 0$ ($B_0 > 0$) for convenience), and the summation over τ ranges the nearest neighbors, U (U_0) is the parameter of the on-site Coulomb interaction of two electrons in the regular (impurity), γ is the spin index, and $a^{+}_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^{\nu}$. The Hamiltonian H acts in the antisymmetric Fo'ck space $\mathcal{H}_{as}$. Let φ_0 by the vacuum vector in the space $\mathcal{H}_{as}$. The quintet state corresponds to the free motion of four electrons over the lattice with the basic functions $q^2_{m,n,k,l\in Z^\nu} = a^+_{m,\uparrow}a^+_{n,\uparrow}a^+_{k,\uparrow}a^+_{l,\uparrow}\varphi_0$. The subspace $\mathcal{H}^q_2$, corresponding to the quintet state is the set all vectors of the form $\psi^q_2 = \sum_{m,n,k,l\in Z^\nu} f(m,n,k,l)q^2_{m,n,k,l\in Z^\nu}$, $f \in l^{as}_2$, where l^{as}_2 is the subspace of antisymmetric functions in the space $l_2((Z^\nu)^4)$. Denote by H^q_2 the restriction of H to the subspace $\mathcal{H}^q_2$. We let $\varepsilon_1 = A_0 - A$, $\varepsilon_2 = B_0 - B$, and $\varepsilon_3 = U_0 - U$.

Theorem 1. *Let $\nu = 1$, and $\varepsilon_2 = -B$, and $\varepsilon_1 < -2B$ (respectively, $\varepsilon_2 = -B$, and $\varepsilon_1 > 2B$). Then the essential spectrum of the operator H^q_2 is consists of the union of four segments $\sigma_{ess}(H^q_2) = [4A-8B, 4A+8B] \cup [3A-6B+z, 3A-6B+z] \cup [2A-4B+2z, 2A+4B+2z] \cup [A-2B+3z, A+2B+3z]$ and discrete spectrum of the operator H^q_2 is consists of a single eigenvalue, $\sigma_{disc}(H^q_2) = \{4z\}$, where $z = A + \varepsilon_1$.*

Theorem 2. *Let $\nu = 1$, and $\varepsilon_2 > 0$, and $-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B} < \varepsilon_1 < \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, then the essential spectrum of the operator H^q_2 is consists of the union of the nine segment $\sigma_{ess}(H^q_2) = [4A-8B, 4A+8B] \cup [3A-6B+z_1, 3A-6B+z_1] \cup [3A-6B+z_2, 3A-6B+z_2] \cup [2A-4B+2z_1, 2A+4B+2z_1] \cup [2A-4B+2z_2, 2A+4B+2z_2] \cup [2A-4B+z_1+z_2, 2A+4B+z_1+z_2] \cup [A-2B+3z_1, A+2B+3z_1] \cup [A-2B+3z_2, A+2B+3z_2] \cup [A-2B+2z_1+z_2, A+2B+2z_1+z_2] \cup [A-2B+z_1+2z_2, A+2B+z_1+2z_2]$, and discrete spectrum of the operator H^q_2 is consists of five eigenvalues, $\sigma_{disc}(H^q_2) = \{4z_1, 4z_2, 3z_1+z_2, z_1+3z_2, 2z_1+2z_2.\}$*

Theorem 3. *If $-2B < \varepsilon_2 < 0$, then the essential spectrum of the operator $\widetilde{H}^t_2$ is consists of a single segment $\sigma_{ess}(\widetilde{H}^t_2) = [2A-4B, 2A+4B]$, and discrete spectrum of the operator $\widetilde{H}^t_2$ is empty set.*

Theorem 4. *If $\varepsilon_2 > 0$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$), then the essential spectrum of the operator $\widetilde{H}^t_2$ is consists of the union of two segments $\sigma_{ess}(\widetilde{H}^t_2) = [2A-4B, 2A+4B] \cup [A-2B+z_1, A+2B+z_1]$,* and discrete spectrum of the operator $\widetilde{H}_2^t$ is consists of a single point, $\sigma_{disc}(\widetilde{H}_2^t) = \{2z_1\}$, where z_1, is the eigenvalue of operator $\widetilde{H}_1$.

Theorem 5. *If* $\varepsilon_2 > 0$ *and* $\varepsilon_1 > \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ *(respectively,* $\varepsilon_2 < -2B$ *and* $\varepsilon_1 > \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$*), then the essential spectrum of the operator* $\widetilde{H}_2^t$ *is consists of the union of two segments* $\sigma_{ess}(\widetilde{H}_2^t) = [2A-4B, 2A+4B] \cup [A-2B+z_1, A+2B+z_1]$, *and discrete spectrum of the operator* $\widetilde{H}_2^t$ *is consists of a single point,* $\sigma_{disc}(\widetilde{H}_2^t) = \{2z_1\}$, *where* z_1, *is the eigenvalue of operator* $\widetilde{H}_1$.

Theorem 6. *If* $\varepsilon_1 = \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, *(respectively,* $\varepsilon_1 = -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$*), then the essential spectrum of the operator* $\widetilde{H}_2^t$ *is consists of the union of two segments* $\sigma_{ess}(\widetilde{H}_2^t) = [2A-4B, 2A+4B] \cup [A-2B+z, A+2B+z]$, *(respectively,* $\sigma_{ess}(\widetilde{H}_2^t) = [2A-4B, 2A+4B] \cup [A-2B+\widetilde{z}, A+2B+\widetilde{z}]$,*) and discrete spectrum of the operator* $\widetilde{H}_2^t$ *is consists of a single point,* $\sigma_{disc}(\widetilde{H}_2^t) = \{2z\}$, *(respectively,* $\sigma_{disc}(\widetilde{H}_2^t) = 2\widetilde{z}$,*) where* z *(respectively,* $\widetilde{z}$,*) is the eigenvalue of operator* $\widetilde{H}_1$.

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Quantum Markovian Dynamics after Bath Correlation Time

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We show that reduced density matrix dynamics $\rho_{SI}(t)$ in the interaction picture of the multi-level model interacting with a bosonic reservoir (with N excited and one ground states) discussed in [1,2] could be represented in the block form $\rho_{SI}(t)$:

$$\begin{pmatrix} (\rho_{SI}(0))_{gg} + \mathrm{Tr}\,((\rho_{SI}(0))_{ee} - V(t)(\rho_{SI}(0))_{ee}V^+(t)) & (\rho_{SI}(0))_{ge}V^+(t) \\ V(t)(\rho_{SI}(0))_{eg} & V(t)(\rho_{SI}(0))_{ee}V^+(t) \end{pmatrix},$$

where $V(t)$ is an $N \times N$ matrix which is a unique solution of

$$\frac{d}{dt}V(t) = -\int_0^t ds\; G(t-s)e^{iH_S(t-s)}V(s)$$

with the initial condition $V(0) = I$, where H_S is an $N \times N$ block of the system Hamiltonian describing the dynamics in the excited subspace of the system, $G(t)$ is the reservoir correlation function assumed to be continuous. This allows us to obtain the asymptotic correction (similar to [3]) to the dynamics described by a usual weak coupling master equation in this case. We introduce a small parameter λ into the coupling constant, which leads to the squared small parameter in the bath correlation function $G(t) \to \lambda^2 G(t)$ and we additionally assume that

$$H_S = H_S^{(0)} + \lambda^2 H_S^{(2)}.$$

So we denote the correspondent (λ-dependent) solution of the above integro-differential equation by $V_\lambda(t)$. We use the Bogolubov-van Hove scaling and define $W_\lambda(t) \equiv V_\lambda(\lambda^{-2}t)$, then we obtain the following integro-differential equation for $W_\lambda(t)$:

$$\frac{d}{dt}W_\lambda(t) = -\int_0^t ds\, \frac{1}{\lambda^2} G\left(\frac{t-s}{\lambda^2}\right) e^{i(\lambda^{-2}H_S^{(0)}+H_S^{(2)})(t-s)} W_\lambda(s).$$

Let us denote the Laplace transform by $\tilde{G}(p)$. Let $\tilde{G}(p)$ be twice continuously differentiable with respect to p. For fixed $t > 0$ at $\lambda \to 0$ one has $W_\lambda(t) = e^{Lt}r + O(\lambda^4)$ (pointwise, i.e. non-uniformly in t), where r and L are $N \times N$ matrices defined as $r = 1 - \lambda^2 \tilde{G}'(-iH_S^{(0)})$ and

$$L = -\sum_E \tilde{G}(-iE)\Pi_E + \lambda^2 \sum_E \left(\tilde{G}(-iE) + i\Pi_E H_S^{(2)} \Pi_E\right) \tilde{G}'(-iE)\Pi_E$$
$$-\lambda^2 \sum_{E \neq E'} \frac{\tilde{G}(-iE) - \tilde{G}(-iE')}{E - E'} \Pi_E H_S^{(2)} \Pi_{E'},$$

where $H_S^{(0)} = \sum_E E\,\Pi_E$ is spectral expansion of $H_S^{(0)}$ with the eigenvalues E and the eigenprojectors Π_E.

If one omits the terms of order λ^2 both for r and L, then one recovers usual weak coupling limit results from our ones.

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A New Method of Time Series Prediction and its Application to the Prediction of Supercomputer Power Consumption

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Accurate prediction methods are generally very computationally intensive, and therefore their calculation takes a long time. These are mostly machine learning methods where a lot of time is spent creating a mathematical model. Quick prediction methods, on the other hand, are not very accurate. These are methods based on some simple principle, such as the zeroth algorithm, exponential smoothing or a moving average. This paper introduces a new prediction method that incorporates the benefits of both of the above approaches. This method does not create any mathematical model, but uses the procedures of some machine learning methods to refine originally inaccurate simple prediction methods. Although these quick and simple methods lose some speed, they gain more accuracy. The level of speed sacrifice for accuracy can be tuned by changing several parameters of this new method. The comparison with several currently most used prediction methods was performed on time series of electricity consumption of the supercomputer infrastructure, which can be in terms of this consumption considered as a complex system.

Quantum Master Equation for a System with Strong Decoherence

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Based on a generalization of the Förster and modified Redfield approximations known in the excitation energy transfer theory, we derive a generic quantum master equation for an open quantum system that undergo strong decoherence in some basis and also may have other interactions with the environment, which are assumed to be weak. Thus, this master equation partially overcomes the weak system-bath coupling approximation. In some cases, our approximation describes an arbitrarily strong (and ultrastrong) interaction of a system with a bath.

Mach Disks and Caustic Reflections, Caustics, Application to Astrophysics

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The report will address issues related to the receipt of caustic reflection in, generally speaking, asymmetrical spaces. These investigations are applied to the substantiation of the occurrence of Mach disks (or rhombuses, or diamonds). The official version of the these disks origin is criticized and there is built a competing new model based on caustic reflection from environments boundary. The reasoning is illustrated by both real and the mathematical model pictures. In this report, we consider causthic reflections in the mirrors of a semi-ellipsoid and a paraboloid shape. These cases will model well-known examples from everyday life. The radiation of the waves from the respective surfaces will be carried out in a fairly small or large enough part of these surfaces near these shapes top. In this way, we will be able to emulate, for example, the behavior of plasma flow when it erupts from an engine nozzle. And then we will study strong flow compaction (i.e. caustic), arising as a result of multiple reflections inside such mirrors, which in some first approximation simulate the boundaries of separation of environments: air-plasma or air-liquid. The main conclusion of the report is that Mach disks are, in fact, the caustic reflections of the formed surface, which is the two environments boundary. If the surface dynamically changes its shape, then the corresponding caustic reflections change. In constructing the caustic reflection, we will use the law of reflection in the case when in the space there is considered, generally speaking, a non-euclidean structure, defined by an asymmetrical norm. The asymmetrical norm is determined by the functionality of the Minkovsky asymmetric convex body (ball). Choosing an asymmetrical ball can be useful, as shown in the introduction, when studying the situation with the light passage in an anisotropic environment (for example, in crystals). Note (to avoid misunderstanding) that symmetrical (normalized) spaces are the special case of asymmetrical spaces. Here asymmetrical norms are used to simulate reflections in conditions where the environment has its own speed.

An important part of the special set is caustic. Dynamically, the body of a certain form tends to change its shape to this form caustic. In fact, this effect can be observed for the liquid in weightlessness which continuously changes its shape trying to approach the caustic generated by this dynamically changing form. As a result, the form is constantly changing in the pursuit of caustic. This, of course, occurs due to the properties of fluid irrescomity. For gases, the same form eventually begins to take the form of caustic (or caustic lows or caustic highs). There is also an effect of gravity which keeps the clumps of gas near the caustic. This explains the evolution of elliptical (ellipsoid form) galaxies, both from younger to spiral and from spiral to spiral with a jumper ones. In short, caustic is a law of forms evolution.

In general we will always use for reflections only a part of the caustic, which is generated by radiation from a small area near the curve top. In this way, we simulate the eruption of the flow from a nozzle or from a neck of the bottle (in case of fluid leakage). Since at some point matter flow reflection will occur from most part of the curve, then we will study this situation and see what happens to Mach disks up to the destruction of the flow itself.

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On Some Elliptic Problems with an Integral Condition

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In 90th the author has developed a certain approach to the theory of elliptic pseudo-differential equations in non-smooth domains, it is based on a concept of special wave factorization for an elliptic symbol in a boundary singular point [1]. It was observed that besides standard local boundary conditions one can consider different integral conditions which permit to obtain unique solution for a model boundary value problem[2].

Latest studying are related to limit situations in which model conical domain transforms to a conical domain of lower dimension [3,4,5]. It was proved that tho corresponding boundary value problem can be solvable if the function from integral boundary condition will satisfy certain functional integral equation.

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Quantum Control Landscape for Phase Shift Quantum Gates

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Quantum control is an important tool necessary for development of modern quantum technologies [1]. Control is realized by applying to the system an external control action $u(t)$. e.g., shaped laser field. A typical quantum control problems can be formulated as maximization of control objective functional $J(u)$ which depends on the state of the system and hence on the control u. Globally optimal control is a control which maximizes the control objective functional, e.g., for the problem of controlled gate generation which produces the gate with maximal possible fidelity. Trap is a control which is a local but not global extremum of the objective functional. The analysis of quantum control landscapes, including proving either the absence or existence of traps, is an important topic in modern quantum control [2-4].

In this talk, we consider the problem of ultrafast controlled generation of single-qubit phase shift quantum gates. The absence of traps for controlled generation of arbitrary single-qubit quantum gates was previously proved for sufficiently long times [5-7]. The analysis of control landscape for ultrafast generation of phase-shift gates, which is interesting practically, was missed in the previous works. In [8], we have shown, combining analytical methods based on the analysis of spectrum of the Hessian of the objective functional and numerical optimization methods, that control landscape for ultrafast generation of phase shift single-qubit quantum gates is free of traps. Also we have found the minimal time required for providing exact generation of phase shift gate.

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On Nonexistence of Weak Solutions for Higher Order Nonlinear Singular Parabolic Inequalities with Nonlocal Source Term

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In this paper, we prove the nonexistence of solutions of higher order semilinear parabolic inequality and system with singular potential and nonlocal source terms. The proofs are based on the test function method developed by Mitidieri and Pohozaev.

Investigation of Stability of an Averaged Catalyst Oxidative Regeneration Model

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At the presents oil catalytic cracking plays an important role in the refining industry. One of the problems facing researchers is the development of methods to improve the efficiency of industrial processes. A decrease of catalyst activity is inevitable due to its carbonization during the processing of petroleum feedstock. The question of restoring the properties of the catalyst is actual. One of the simple methods for restoring catalytic activity is oxidative regeneration. It includes run of the oxygen-containing reaction mixture folw through the catalyst layer, which leads to the burning of coke sediments. The efficiency of the process is determined by the measure and degree of coke removal. In addition, it is necessary to monitor the temperature mode, since the accumulation of adsorbed oxygen in the coke layer can lead to uncontrolled fires and irreversible deterioration of the catalyst. Due to the high cost of field experiments, mathematical modeling is an effective means of researching of the oxidative regeneration process, and the problem of uncontrolled temperature rise can be solved using the methods of differential equations stability theory [1], [2].

The report contains an averaged mathematical model of the oxidative regeneration process described by a nonlinear system of ordinary differential equations [3].

Temperature stability of the process means the stability of the system with respect to the variable corresponding to the temperature of the catalyst grain. Due to the fact that the linear approximation of the system has several zero eigenvalues, the first Lyapunov method is inapplicable for solving this problem.

The second method is inapplicable due to the complex structure of the system right-hand side.

To solve the problem, the original model was transformed in such a way that the eigenvalues of the linear approximation matrix of the new system became negative, therefore, the equilibrium position is asymptotically stable.

Asymptotic stability with respect to all variables of the equilibrium position of the transformed system means that the equilibrium position of the original system is asymptotically stable and stable with respect to some variables [4].

To the practical application, the result obtained means the absence of uncontrolled temperature jumps, while oxygen in the reaction mixture and in the layer of coke deposits will be completely consumed.

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Characterization of the Extension of Lattice Linear Space of Continuous Bounded Functions, Generated by μ-Riemann-Integrable Functions, by means of Order Boundaries

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In 1867 Riemann for interval T introduced the Riemann integral and the notion of Riemann integrable function f on T. After that Lebesgue constructed the Lebesgue measure λ on T , and gave the famous functional description of Riemann integrable functions. Consider the factor-family R of equivalence classes of all Riemann integrable function f on T with respect to λ. Then we get the remarkable Riemann extension R of the lattice linear space C of all continuous bounded functions f on T. Long time any functionally-analytic inter-relation between the whole families C and R were very mysterious, because nothing was known about approximation of Riemann integrable functions by means of sequences continuous functions, in this relation the remarkable Lebesgue functional description is not informative. In 1995 the author gave the other functional description of Riemann integrable functions different from Lebesgue[U+0092]s one (see [2; 3.7.2, Theorem 3]). It gave to the author the opportunity to get the characterization of the Riemann extension R of C in some category of lattice linear extensions A of C. In this category the Riemann extension is characterized as some completion of C obtained by the adjunction to C some tight order boundaries of some tight countable cuts in C. The tightness is determined by some new functionally-analytic structure, called by the author the refinement [1]. The mentioned characterization is given for the general case of an arbitrary completely regular topological space T and for the family of functions on T, μ-Riemann-integrable with respect to an arbitrary positive bounded Radon measure μ with the support equal to T (see [2; 3.7]).

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A Model of Relativistic Dynamics

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The known difficulties in introducing universal time in relativistic dynamics (both in the classical and quantum cases) have given rise to quite natural attempts to describe relativistic dynamics in a parametrized form. A comprehensive study of the history of this approach is far beyond the scope of the present report, so we only mention the basic concepts. Traditionally, Stueckelberg [1, 2] is considered a pioneer of this direction. The core of the approach is that the proper time of each particle is assumed to be a dynamic variable. The Hamiltonian formalism is used, and trajectories in the Minkowski phase spacetime are parametrized by an independent variable, often called historical time [3, 4]. The development of this direction can be traced, for example, via the publications [5-7]. Here, we describe relativistic systems by their world lines in the Minkowski configuration spacetime with the use of the Lagrangian formalism. The construction proposed is purely geometrical and is independent of the parametrization of the world lines (in contrast to the fixed historical time in the Hamiltonian approach).

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Chaotic Behavior of Cartan Foliations

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This is the joint work with Yaroslav Bazaikin and Anton Galaev [1].

The notion of Devaney's chaos for dynamical systems is trajectory invariant and it is generalized to foliations, which may be considered as dynamical systems in the absence of time. The property of a foliation to be chaotic is transversal, i.e, depends on the structure of the leaf space of the foliation.

Examples of chaotic foliations are given by the Anosov flows on closed 3-dimensional manifolds. Such foliations with smooth stable and unstable distributions are transversally Lorentzian and are modeled either on the 2-dimensional torus with a flat Lorentzian metric, or on the de Sitter space. Cartan foliations admit holonomy-invariant Cartan geometry, where holonomy diffeomorphisms are higher-dimensional analogs of the Poincaré map that plays a prominent role in the theory of dynamical systems. Transversal Lorentzian, Riemannian, conformal, projective and many other foliations are classes of Cartan foliations [2]. We study properties of chaotic Cartan foliations of arbitrary codimension, and compactness of foliated manifolds are not assumed.

The problem of investigation of chaotic Cartan foliations is reduced to the corresponding problem for their holonomy pseudogroups of local automorphisms of transversal Cartan manifolds. For a Cartan foliation of a wide class this problem is reduced by us to the corresponding problem for its global holonomy group, which is a countable discrete subgroup of the Lie automorphism group of an associated simply connected Cartan manifold.

Several types of Cartan foliations that cannot be chaotic, are indicated. Examples of chaotic Cartan foliations are constructed.

The structure of chaotic Lorentzian foliations of codimension two on n-dimensional closed manifolds is described in [3].

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Two-Dimensional Attractors of A-flows and Fibered Links on 3-Manifolds

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Introduction. Dynamical systems satisfying an Axiom A (in short, A-systems) were introduced by S.Smale. By definition, a non-wandering set of A-system is the topological closure of periodic orbits endowed with a hyperbolic structure. Due to Smale's Spectral Decomposition Theorem, the non-wandering set of any A-system is a disjoint union of closed, invariant, and topologically transitive sets called *basic sets*. E.Zeeman proved that any n-manifold, $n \geq 3$, supporting nonsingular flows supports an A-flow with a one-dimensional nontrivial basic set. It is natural to consider the existence of two-dimensional (automatically non-trivial) basic sets on n-manifolds beginning with closed 3-manifolds M^3. We prove that any closed orientable 3-manifolds supports A-flows with two-dimensional attractors. Our main attention concerns to embedding of non-mixing attractors and its basins (stable manifolds) in M^3.

Main results. Let f^t be an A-flow on a closed orientable 3-manifold M^3 and Λ_a a two-dimensional non-mixing attractor of f^t. The stable manifold (in short, a basin) $W^s(\Lambda_a)$ of Λ_a is an open subset of M^3 consisting of the trajectories whose ω-limit sets belong to Λ_a. First, we construct a special compactification of $W^s(\Lambda_a)$ called a casing by a collection of circles that form a fiber link in the casing.

Theorem 1. Let f^t be an A-flow on an orientable closed 3-manifold M^3 such that the non-wandering set $NW(f^t)$ contains a 2-dimensional non-mixing attractor Λ_a. Then there is a compactification $M(\Lambda_a) = W^s(\Lambda_a) \cup_{i=1}^k l_i$ of the basin $W^s(\Lambda_a)$ by the family of circles $l_1, \ldots, l_k$ such that

- $M(\Lambda_a)$ is a closed orientable 3-manifold;

- the flow $f^t|_{W^s(\Lambda_a)}$ is extended continuously to the non-singular flow $\tilde{f}^t$ on $M(\Lambda_a)$ with the non-wandering set $NW(\tilde{f}^t) = \Lambda_a \cup_{i=1}^{k} l_i$ where $l_1, \ldots, l_k$ are repelling isolated periodic trajectories of $\tilde{f}^t$;
- the family $L = \{l_1, \ldots, l_k\} \subset M(\Lambda_a)$ is a fibered link in $M(\Lambda_a)$.

The second result of the paper, in a sense, is reverse to the first one.

Theorem 2. Let $\{l_1, \ldots, l_k\} \subset M^3$ be a fibered link in a closed orientable 3-manifold M^3. Then there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains a 2-dimensional non-mixing attractor and the repelling isolated periodic trajectories $l_1, \ldots, l_k$.

Corollary. Given any closed orientable 3-manifold M^3, there is a nonsingular A-flow f^t on M^3 such that the non-wandering set $NW(f^t)$ contains a two-dimensional attractor.

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Математическое Моделирование Колебаний Неограниченной Струны с Движущейся Границей в Нелинейной Постановке

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До настоящего времени задачи о продольно – поперечных колебаниях объектов с движущимися границами решались в основном при линейной постановке, не учитывался энергетический обмен через движущуюся границу и взаимодействие между продоль¬ными и поперечными колебаниями [1–5, 7, 8]. В редких случаях учитывалось действие сил сопротивления внешней среды [6]. Реальные же технические объекты намного сложнее, например, при увеличении интенсивности колебаний большое влияние на колебательный процесс оказывают геометрические нелинейности объекта.

В связи с интенсивным развитием численных методов появилась возможность более точного описания сложных математических моделей продольно–поперечных колебаний объектов с движущимися границами, учитывающих большое число факторов, влияющих на колебательный процесс.

В работе поставлена новая нелинейная математическая модель поперечных колебаний неограниченной струны с движущейся границей, в которой учтена геометрическая нелинейность. Получены граничные условия в случае наличия взаимодействия между частями объекта слева и справа от границы.

Произведена линеаризация полученной модели. При этом соблюдается принцип однородности: в частном случае малых колебаний полученные линейные модели совпали с классическими, что свидетельствует о корректности полученных результатов. Полученная математиче¬ская модель позволяет описывать колебания большой интенсивности струны с движущейся границей.

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